Well-Balanced Designs for Unreliable Data Placement

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AGCO Seminar, Santiago, 13 April 2016

Outline

Progress

Motivation: Data placement and replication

Origin of the problem: Distributed Storage and Download System

- many storage sites, each with limited capacity
- potentially unavailable
- existence of a permanent backup data storage

Questions

Among the design issues for such a system

- how much replication of the data?
- where to place it?
- what download strategy?

Question for this talk

Consider that some data must be replicated k times on unreliable servers: where to place the replications?

Progress

Model elements

Document model

The typical document is made of

- b distinct pieces of same size called “blocks”
- stored with a replication factor k

Server model

- v identical servers
- available with uniform probability δ, independently

Model elements, ctd.

Network model

The client:

- can identify instantly blocks on available servers
- downloads available blocks from the servers at data rate θ_c (blocks/s)
- then downloads missing blocks from central server at data rate θ_s (blocks/s)

Model elements, end

$\rightarrow$ model for the time to download one document

Let Λ be the number of available blocks in the document:

Individual response time

Response time for the document

$$R = \frac{\Lambda}{\theta_c} + \frac{b - \Lambda}{\theta_s} = \frac{b}{\theta_s} - \Lambda \left(\frac{1}{\theta_s} - \frac{1}{\theta_c} \right).$$

$\rightarrow$ the statistics of Λ determine those of R.

Progress

Availability

Let Λ, $\overline{\Lambda}$ the number of (un)available blocks.
One block is available if at least one of its replications is online:

$$P(\text{block available}) \;=\; 1 \;-\; (1-\delta)^k \;=\; 1 \;-\; \bar{\delta}^k \;.$$

Theorem

$$\mathbb{E}(z^{\overline{\Lambda}}) \;=\; \sum_{\mathcal{S}\subset\mathcal{B}} (z-1)^{|\mathcal{S}|}\; \bar{\delta}^{|\cup_{B\in\mathcal{B}}\; B|} \;.$$

The first moments of Λ and $\overline{\Lambda}$ are given by:

$$\begin{aligned}
\mathbb{E}(\overline{\Lambda}) &= b\;\bar{\delta}^k \\
\mathbb{E}(\Lambda) &= b\;(1-\bar{\delta}^k) \\
\mathbb{V}(\overline{\Lambda}) = \mathbb{V}(\Lambda) &= \sum_{B,B'} \left(\bar{\delta}^{|B\cup B'|} \;-\; \bar{\delta}^{2k} \right) \;.
\end{aligned}$$

Optimization

The expected value of Λ does not depend on the placement.

$\rightarrow$ minimize variance.

$$
\begin{aligned}
\mathbb{V}(\Lambda) &= \sum_{B,B'} \left(\bar{\delta}^{|B \cup B'|} - \bar{\delta}^{2k} \right) \\
&= \sum_{B \neq B'} \bar{\delta}^{|B|+|B'|-|B \cap B'|} + b\bar{\delta}^{k} - b^2\bar{\delta}^{2k} \\
&= \sum_{B,B'} \bar{\delta}^{2k}\, \bar{\delta}^{-|B \cap B'|} + b\bar{\delta}^{k} - b^2\bar{\delta}^{2k} \\
&= \bar{\delta}^{2k} \sum_{B \neq B'} \gamma^{|B \cap B'|} + b\bar{\delta}^{k} - b^2\bar{\delta}^{2k}
\end{aligned}
$$

with $\gamma := \bar{\delta}^{-1}$.

The MinVar Problem

Such a placement is a block design $\mathcal{B}$ of b "blocks", each being a k-subset of $[1..v]$.

MinVar Problem

Let γ be a real number, $\gamma \geq 1$, and b, v, k be integers.
Find one design $\mathcal{B}$ with $|\mathcal{B}| = b$, $|\mathcal{V}| = v$ and $|B| = k$ for all $B \in \mathcal{B}$, which minimizes the function:

$$J(\mathcal{B}, \gamma) \;:=\; \sum_{B \neq B' \in \mathcal{B}} \gamma^{|B \cap B'|} \;.$$

The MinVar Conjecture

Consider the set of all designs with these characteristics:

$$\mathcal{C}_{v,b,k} := \{\mathcal{B} \text{ with } |\mathcal{V}| = v,\ |\mathcal{B}| = b \text{ and } |B| = k \text{ for all } B \in \mathcal{B}\}.$$

MinVar Conjecture

For each integer v, b, k there exists one design $\mathcal{B}^* \in \mathcal{C}_{v,b,k}$ such that

$$J(\mathcal{B}^*, \gamma) \ \leq \ J(\mathcal{B}, \gamma)$$

for all $\mathcal{B} \in \mathcal{C}_{v,b,k}$ and all $\gamma \geq 1$.

Alternate representations

Define the coefficients:

$$\begin{aligned}
\nu_\ell &= \#\left\{(b, b') \text{ s.t. } b \neq b' \text{ and } |L(b) \cap L(b')| = \ell\right\} \\
\mu_p &= \sum_{b \neq b' \in \mathcal{B}} \binom{|L(b) \cap L(b')|}{p} \\
\lambda_{x_1, \ldots, x_j} &= \#\left\{B \in \mathcal{B}, \{x_1, \ldots, x_j\} \subset B\right\} \ .
\end{aligned}$$

By definition:

$$\begin{aligned}
J(\mathcal{B}, \gamma) &= \sum_{B \neq B' \in \mathcal{B}} \gamma^{|B \cap B'|} \\
&= \sum_{j=0}^{k} \nu_j \ \gamma^j
\end{aligned}$$

Representations, ctd.

Proposition

$$
\begin{aligned}
J(\mathcal{B}, \gamma) &= \sum_{j=0}^{k} \mu_j \, (\gamma - 1)^j \\
&= \sum_{j=0}^{k} \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)(\gamma - 1)^j \\
&= \sum_{j=1}^{k} \sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}(\gamma - 1)^j - b\gamma^k + b^2 \; .
\end{aligned}
$$

→ minimize the coefficients simultaneously?

Minimizing unbalance

Conservation of block counts

For every design $\mathcal{B} \in \mathcal{C}_{v,b,k}$, for all j,

$$\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j} = b\binom{k}{j}.$$

$\rightarrow$ minimizing the sum of squares when all elements are "almost equal".

Balanced families

A design $\mathcal{B}$ is j-balanced if the $\lambda_{x_1,\ldots,x_j}$ are all equal or almost equal, that is, if for any two j-element subsets $\{x_1,\ldots,x_j\}$ and $\{y_1,\ldots,y_j\}$, $|\lambda_{x_1,\ldots,x_j} - \lambda_{y_1,\ldots,y_j}| \leq 1$.

A design $\mathcal{B}$ is well balanced if it is j-balanced for $1 \leq j \leq k$.

Well-balanced families

Well balanced theorem

If $\mathcal{B}^*$ is well balanced, then $\mathcal{B}^*$ is optimal, that is, $P(\mathcal{B}^*, \gamma) \leq P(\mathcal{B}, \gamma)$ for any $\mathcal{B}$ and any $\gamma \geq 1$.

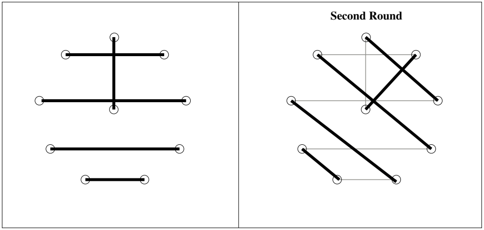
The MinVar conjecture holds for well-balanced families.

But: are there well-balanced families for all v, b, k?

Solutions

Solutions can be constructed:

- systematically for $k = 2$:



- for $k = 3$
- for Tactical Configurations
- for Steiner systems

Steiner Systems

Definition

A a t-Steiner system (or (v, k, λ) t-design) is a family of blocks such that each t-element subset appears in exactly λ blocks.

In that case it is well-known that also, for $1 \leq j \leq t$ each j-element subset appears in exactly λ_j blocks, where $\lambda_j = \lambda \frac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}}$. Therefore:

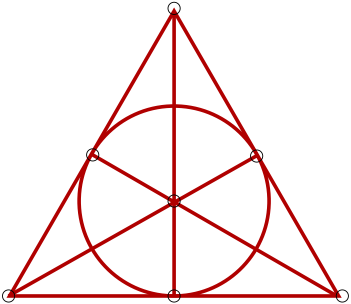
Property

A t-design is j-balanced for all j, $1 \leq j \leq t$.

In particular, if $t = k - 1$ and the blocks are repeated the same or almost the same number of times, then a k-Steiner System is also well balanced.

Steiner Systems, ctd.

Example: the Fano plane is a 2-(7,3,1)-design



$\lambda_x = 3 \qquad \lambda_{xy} = 1 \qquad \lambda_{xyz} = 0$ or 1

Tactical Configurations

Definition (Configuration and Quasi-Configuration Graphs)

A Configuration is a design $\mathcal{B}$ such that:

1) $\forall B \in \mathcal{B}$, $|B| = k$;
2) $\forall x$, $\lambda_x = r$;
3) $\forall B \neq B' \in \mathcal{B}$, $|B \cap B'| \leq 1$.

A Quasi-Configuration is defined with 1), 3) and 2'):
$\forall x$, $\lambda_x \in \{r, r+1\}$.

Optimality

Configurations and Quasi-Configurations are optimal.

The case $k = 3$

Well-balanced families do not always exist for $k = 3$.

Proposition: Non-existence, $k = 3$

There does not exist a WBF(v, b) whenever v is even, λ odd and $\lambda\frac{v(v-1)}{2} - \frac{v}{2} < 3b < \lambda\frac{v(v-1)}{2} + \frac{v}{2}$.
If $\lambda\frac{v(v-1)}{6}$ is not an integer, then there does not exist a well balanced family for $b = \lfloor\lambda\frac{v(v-1)}{6}\rfloor$ and $b' = \lceil\lambda\frac{v(v-1)}{6}\rceil$.

$k = 3$, the general solution

Theorem

There exists a WBF for all cases not excluded by the previous proposition.

Solution brought by Wei, Ge, and Colbourn:

- the balancing lemma
- families of 2-balanced 3-designs

Lemma (balancing lemma)

From every 2- and 3-balanced 3-design, it is possible to construct a design with the same number of blocks, which is 1-, 2- and 3-balanced

The case $k = 3$, direct constructions

Two disjoint Kirkman Triple Systems for $v = 9$:

$$\begin{array}{cccc} \{0,7,8\} & \{0,2,5\} & \{0,3,4\} & \{0,1,6\} \\ \{1,2,4\} & \{1,3,8\} & \{1,5,7\} & \{2,3,7\} \\ \{3,5,6\} & \{4,6,7\} & \{2,6,8\} & \{4,5,8\} \end{array}$$

$$\begin{array}{cccc} \{1,7,8\} & \{1,3,6\} & \{1,4,5\} & \{1,2,0\} \\ \{2,3,5\} & \{2,4,8\} & \{2,6,7\} & \{3,4,7\} \\ \{4,6,0\} & \{5,0,7\} & \{3,0,8\} & \{5,6,8\} \end{array}$$

These are 3-Steiner systems. Each one has $\lambda_x = 9, \lambda_{xy} = 1$.

$\rightarrow$ constructions for $v = 6t + 3$

$\rightarrow$ constructions for $v = 6t + 4$

Random Designs

Random Algorithm

Choose each block uniformly at random over $\binom{[1..v]}{k}$.

Define the function

$$\pi(\gamma) = \binom{v}{k}^{-1} \sum_{j=0}^{k} \binom{k}{j} \binom{v-k}{k-j} \gamma^j .$$

It is the generating function of $X_{BB'} = |B \cap B'|$, where B and B' are two uniformly chosen random blocks.

Theorem

If $\mathcal{B}$ is a design generated by the Random Algorithm, then

$$\begin{aligned} \mathbb{E}(J(\mathcal{B},\gamma)) &= b(b-1)\,\pi(\gamma) \\ \mathbb{V}(J(\mathcal{B},\gamma)) &= 2b(b-1)\left(\pi(\gamma^2) - \pi^2(\gamma)\right) . \end{aligned}$$

Bibliography

- A-E. Baert, V. Boudet, A. Jean-Marie, and X. Roche. Minimization of download time variance in a distributed VoD system. *Scalable Computing Practice and experience* , 10(1):75–86, 2009.
- A-E. Baert, V. Boudet, A. Jean-Marie, and X. Roche. Combinatorial Designs and Availability. INRIA Research report RR 7119, December 2009.
- Jean-Claude Bermond, Alain Jean-Marie, Dorian Mazauric and Joseph Yu. Well balanced designs for data placement. *Journal of Combinatorial Designs*, pp. 55–76, feb. 2016. doi:10.1002/jcd.21506
- Hengjia Wei, Gennian Ge and Charles J. Colbourn. The Existence of Well-Balanced Triple Systems *Journal of Combinatorial Designs*, pp. 77–100, feb. 2016. doi:10.1002/jcd.21508