Pricing Differentiated Services: A Game-Theoretic Approach

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***Abstract*— The goal of this paper is to study pricing of differentiated services and its impact on the choice of service priority at equilibrium. We consider both TCP connections as well as non controlled (real time) connections. The performance measures (such as throughput and loss rates) are determined according to the operational parameters of a RED buffer management. The latter is assumed to be able to give differentiated services to the applications according to their choice of service class. We consider a best effort type of service differentiation where the QoS of connections is not guaranteed, but by choosing a better (more expensive) service class, the QoS parameters of a session can improve (as long as the service class of other sessions are fixed). The choice of a service class of an application will depend both on the utility as well as on the cost it has to pay. We first study the performance of the system as a function of the connections' parameters and their choice of service classes. We then study the decision problem of how to choose the service classes. We model the problem as a noncooperative game. We establish conditions for an equilibrium to exist and to be uniquely defined. We further provide conditions for convergence to equilibrium from non equilibria initial states. We finally study the pricing problem of how to choose prices so that the resulting equilibrium would maximize the network benefit.**

Keywords: TCP, Buffer Management, RED/AQM, Nash equilibrium, Pricing, Mathematical programming/optimization, Economics

I. Introduction

We study in this paper the performance of competing connections that share a bottleneck link. Both TCP connections with controlled rate as well as CBR (Constant Bit Rate) connections are considered. A RED buffer management is used for early drop of packets. We allow for service differentiation between the connections through the rejection probability (as a function of the average queue size), which may depend on the connection (or on the connection class). More specifically, we consider a buffer management scheme that uses a single averaged queue length to determine the rejection probabilities (similar to the way it is done in the RIO-C (coupled RIO) buffer management, see [9]); for any given averaged queue size, packets belonging to connections with higher priority have smaller probability of being rejected than those belonging to lower priority classes. To obtain this differentiation in loss probabilities, we assume that the loss curve of RED is scaled by a factor that represents the priority level of the application. We obtain various performance measures of interest such as the throughput, the average queue size and the average drop probability.

We then address the question of the choice of priorities. Given utilities that depend on the performance measures on one hand and on the cost for a given priority on the other hand, the sessions at the system are faced with a non-cooperative game in which the choice of priority of each session has an impact on the quality of services of other sessions. For the case of CBR traffic, we establish conditions for an equilibrium to exist. We further provide conditions for convergence to equilibrium from non equilibria initial states.

We shall finally study numerically the pricing problem of how the network should choose prices so that the resulting equilibrium would maximize its benefit.

We briefly mention some recent work in that area. Reference [5] has considered a related problem where the traffic generated by each session was modeled as a Poisson process, and the service time was exponentially distributed. The decision variables were the input rates and the performance measure was the goodput (output rates). The paper restricted itself to symmetric users and symmetric equilibria and the pricing issue was not considered. In this framework, with a common RED buffer, it was shown that an equilibrium does not exist. An equilibrium was obtained and characterized for an

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alternative buffer management that was proposed, called VLRED. We note that in contrast to [5], since we also include in the utility of CBR traffic a penalty for losses (which is supported by studies of voice quality in packet-based telephony [6]), we do obtain an equilibrium when using RED. For other related papers, see for instance [8] (in which a priority game is considered for competing connections sharing a drop-tail buffer), [1] as well as the survey [2]. In [13], the authors present mechanisms (e.g., AIMD of TCP) to control end-user transmission rate into differentiated services Internet through potential functions and corresponding convergence to Nash equilibrium.

The approach of our pricing problem is related to the Stackelberg methodology for hierarchical optimization: for a fixed pricing strategy one seeks the equilibrium among the users (the optimization level corresponding to the "follower"), and then the network (considered as the "leader") optimizes the pricing strategy. This type of methodology has been used in other contexts of networking in [3], [7].

The structure of this paper is as follows. In Section II we describe the model of RED, then in Section III we compute the throughputs and the loss probabilities of TCP and of CBR connections for given priorities chosen by the connections. In Section IV we introduce the model for competition between connections at given prices. In section V we focus on the game in the case of only CBR connections or only TCP connections and provide properties of the equilibrium: existence, uniqueness and convergence. In section VI we provide an algorithm for computing Nash equilibrium for symmetric case. The optimal pricing is then discussed in Section VII. We present numerical examples in sectionVIII to validate the model.

II. The model

RED is based on the following idea: there are two thresholds $q_{\min}$ and $q_{\max}$ such that the drop probability is 0 if the average queue length q is less than $q_{\min}$, 1 if it is above $q_{\max}$, and $p(i)(x - q_{\min})/(q_{\max} - q_{\min})$ if it is x with $q_{\min} < x < q_{\max}$; the latter is the *congestion avoidance* mode of operation. This is illustrated in Figure 1.

We consider a set $\mathcal{N}$ containing N TCP flows (or aggregate of flows) and a set $\mathcal{I}$ containing I real time flows that can be differentiated by RED; they all share a common buffer yet RED treats them differently[1]. We assume that they all have common values of $q_{\min}$ and $q_{\max}$ but each flow i may have a different value of $p(i)$, which is the value of the drop probability as the average queue tends to $q_{\max}$ (from the left). In other words, the slope t_i of the linear part of the curve in Figure 1 depends the flow i:

$$t_i = \frac{p(i)}{q_{\max} - q_{\min}}$$

Denote $\mathbf{t} = (t_i, i \in \mathcal{I} \cup \mathcal{N})$. We identify t_i as the priority class of a connection. The service rate of the bottleneck router is given by μ.

[1]RED punishes aggressive flows more by dropping more packets from those flows

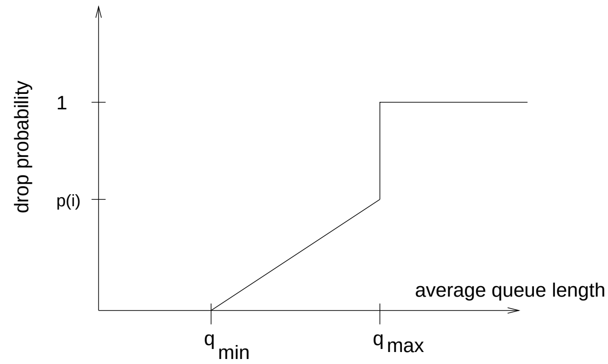


Fig. 1. Drop probability in RED as function q

III. Computing the throughputs

We use the well-known relation for TCP rate:

$$\lambda_i = \frac{1}{R_i}\sqrt{\frac{\alpha}{p_i}}, \quad i \in \mathcal{N}, \tag{1}$$

where R_i and p_i are TCP flow i's round trip time and drop probability, respectively. α is typically taken as $3/2$ (when the delayed ack option is disabled) or $3/4$ (when it is enabled). We shall assume throughout the paper that the queueing delay is negligible with respect to R_i for the TCP connections.

In contrast, the rates λ_i, for $i \in \mathcal{I}$, of real time flows are not controlled and are assumed to be fixed. If $\mathcal{N} = \emptyset$ we assume throughout the paper that $\sum_{j \in \mathcal{I}} \lambda_j > \mu$ (unless otherwise specified), otherwise the RED buffer is not a bottleneck. Similarly, if $\mathcal{I} = \emptyset$ we assume that TCP senders are not limited by the receiver window.

In general, since the bottleneck queue is seen as a fluid queue, we can write

$$\sum_{j \in \mathcal{I} \cup \mathcal{N}} \lambda_j (1 - p_j) = \mu$$

If we operate in the linear part of the RED curve then this leads to the system of equations:

$$\begin{cases} \sum_{j \in \mathcal{I} \cup \mathcal{N}} \lambda_j (1 - p_j) & = & \mu \\ p_i & = & t_i (q - q_{\min}), \\ & & \forall i \in \mathcal{I} \cup \mathcal{N} \end{cases}$$

with $(N+I+1)$ unknowns: q (average queue length), and $p_i,\ i \in \mathcal{I} \cup \mathcal{N}$, where $\lambda_i,\ i \in \mathcal{N}$ is given by (1). Substituting (1) and

$$p_i = t_i(q - q_{\min})\ \forall i, \tag{2}$$

into the first equation of the above set, we obtain a single equation for q:

$$\sum_{j\in\mathcal{N}} \frac{1}{R_j}\sqrt{\frac{\alpha}{t_j(q-q_{min})}}(1-t_j(q-q_{\min})) + \sum_{j\in\mathcal{I}} \lambda_j(1-t_j(q-q_{\min})) = \mu \tag{3}$$

If we write $x = \sqrt{q - q_{\min}}$, then (3) can be written as a cubic equation in x:

$$Z(x) = z_3x^3 + z_2x^2 + z_1x + z_0 = 0 \tag{4}$$

where

$$z_3 = \sum_{j\in\mathcal{I}} \lambda_j t_j,\ \ z_2 = \sum_{j\in\mathcal{N}} \frac{1}{R_j}\sqrt{\alpha t_j},\ \ z_1 = \mu - \sum_{j\in\mathcal{I}} \lambda_j,$$

$$z_0 = -\sum_{j\in\mathcal{N}} \frac{1}{R_j}\sqrt{\frac{\alpha}{t_j}}.$$

Note that this equation has a unique positive solution if there are only TCP or only real-time connections; in either case, it becomes a quadratic equation.

Proposition 1: Fix the values of $t_j,\ j \in \mathcal{I} \cup \mathcal{N}$. The cubic equation (4) has a unique real positive solution. Assume that the solution lies in the linear region of RED. Then the average queue size is given as $q_{\min} + x^2$ where x is the unique positive solution of (4) and the loss probability for session i is given by $p_i = t_i(q - q_{\min})$.

Proof. Assume first that $\mathcal{I}$ and $\mathcal{N}$ are both nonempty. Since the coefficients of the cubic equation are real, it has either a single real solution and two other conjugate complex solutions, or it has three real solutions [14]. Consider first the case in which all solutions are real. Then since the product of solutions is positive (it equals $-z_0$), there are either one or three positive solutions. But the latter is excluded since the sum of solutions is positive (it equals $-z_2$).

Next consider the case of a single real solution. Since the two other solutions are conjugate, their product is positive. Then since the product of all solutions is positive (it equals $-z_0$), the real solution is positive. ∎

Note that, in the case of only real-time connections ($\mathcal{N} = \emptyset$) operating in the linear region, we have

$$q = q_{\min} + \frac{\sum_{j\in\mathcal{I}} \lambda_j - \mu}{\sum_{j\in\mathcal{I}} \lambda_j t_j} \quad \text{and} \tag{5}$$

$$p_i = t_i \frac{\sum_{j\in\mathcal{I}} \lambda_j - \mu}{\sum_{j\in\mathcal{I}} \lambda_j t_j}. \tag{6}$$

(Recall that, throughout the paper, when considering this case we shall assume that $\sum_{j\in\mathcal{I}} \lambda_j > \mu$.)

In the case of only TCP connections ($\mathcal{I} = \emptyset$) operating in the linear region, we have

$$q = q_{\min} + \frac{\left(-\mu + \sqrt{\mu^2 + 4\alpha \sum_{j\in\mathcal{N}}\left(\frac{1}{R_j\sqrt{t_j}}\right)\sum_{j\in\mathcal{N}}\left(\frac{\sqrt{t_j}}{R_j}\right)}\right)^2}{4\alpha\left(\sum_{j\in\mathcal{N}}\frac{\sqrt{t_j}}{R_j}\right)^2}$$

and (7)

$$p_i = t_i \frac{\left(-\mu + \sqrt{\mu^2 + 4\alpha \sum_{j\in\mathcal{N}}\left(\frac{1}{R_j\sqrt{t_j}}\right)\sum_{j\in\mathcal{N}}\left(\frac{\sqrt{t_j}}{R_j}\right)}\right)^2}{4\alpha\left(\sum_{j\in\mathcal{N}}\frac{\sqrt{t_j}}{R_j}\right)^2}. \tag{8}$$

IV. Utility, pricing and equilibrium

We denote a strategy vector by $\mathbf{t}$ for all flows such that jth entry is t_j. By $(t_i, [\mathbf{t}]_{-i})$, we define a strategy where flow i uses t_i and all other flows $j \neq i$ use t_j from vector $[\mathbf{t}]_{-i}$.

We associate to flow i a utility U_i. The utility will be a function of the QoS parameters and the price payed by flow i, and is determined by the actions of all flows. More precisely, $U_i(t_i, [\mathbf{t}]_{-i})$ is given by

$$a_i\lambda_i(1 - p(t_i, [\mathbf{t}]_{-i})) - b_i p(t_i, [\mathbf{t}]_{-i}) - d(t_i)$$

where the first term stands for the utility for the goodput, the second term stands for the dis-utility for the loss rate and the last term corresponds to the price $d(t_i)$ to be paid by flow i to the network.

In particular, we find it natural to assume that a TCP flow i has $b_i = 0$ (as lost packets are retransmitted anyhow, and their impact is already taken into account in the throughput). Moreover, since λ_i for TCP already includes the loss term $p_i(t_i, [\mathbf{t}]_{-i})$, the utility function of TCP is assumed to be

$$U_i(t_i, [\mathbf{t}]_{-\mathbf{i}}) = a_i\lambda_i(1 - p(t_i, [\mathbf{t}]_{-i})) - d(t_i).$$

We assume that the strategies or actions available to session i are given by a compact set of the form:

$$t_i \in \left[t_{min}^i, t_{max}^i\right], \ i \in \mathcal{I} \cup \mathcal{N}.$$

Each flow of the network strives to find its best strategy so as to maximize its own objective function. Nevertheless its objective function depends upon its own choice but also upon the choices of the other flows. In this situation, the solution concept widely accepted is the concept of Nash equilibrium.

Definition 1: A Nash equilibrium of the game is a strategy profile $\mathbf{t} = (t_1, t_2, .., t_M)$ where $M = I + N$ from which no flow have any incentive to deviate. More precisely the strategy profile, $\mathbf{t}$ is a Nash equilibrium, if the following holds true for any i

$$t_i \in \arg \max_{\bar{t}_i \in [t_{min}^i, t_{max}^i]} U_i(\bar{t}_i, [\mathbf{t}]_{-\mathbf{i}}).$$

t_i is the best flow i can do if the other flows choose the strategies $[\mathbf{t}]_{-i}$.

Note that the network income is given by $\sum_{i \in \mathcal{I} \cup \mathcal{N}} d(t_i)$. Since the $p_i(t_i, [\mathbf{t}]_{-i})$'s are functions of t_i and $[\mathbf{t}]_{-\mathbf{i}}$, d can include pricing per volume of traffic successfully transmitted. In particular, we allow for d to depend on the uncontrolled arrival rates of real-time sessions (but since these are constants, we do not make them appear as an argument of the function d).

We shall sometimes find it more convenient to represent the control action of connection i as $T_i = 1/t_i$ instead of as t_i. Clearly, properties such as existence or uniqueness of equilibrium in terms of t_i directly imply the corresponding properties with respect to T_i.

V. Equilibrium for only Real-Time sessions or only TCP connections

We assume throughout that $t_{max}^i \leq 1/(q_{max} - q_{min})$ for all connections. The bound for t_{max}^i is given so that we have $t_{\max}^i(q_{\max} - q_{\min}) \leq 1$. From (2) we see that $p_i \leq 1$ with equality obtained only for the case $t_i = 1/(q_{max} - q_{min})$. [2]

In our analysis, we are interested mainly in the linear region. For only real-time sessions or only TCP connections, we state the assumptions and describe the conditions for linear region operations and we show the existence of a Nash equilibrium.

[2]Note that if the assumption does not hold then for some value $q' < q_{max}$ we would already have for some $i, p_i = 1$ so one could redefine q_{max} to be q'. An important feature in our model is that the queue length beyond which $p_j = 1$ should be the same for all j.

Theorem 1: A sufficient condition for the system to operate in linear region is that for all i:
1- For only real time connections :

$$\lambda > \mu \text{ and } t_{\min}^i > \frac{\lambda - \mu}{\lambda(q_{\max} - q_{\min})}. \qquad (9)$$

2- For only TCP connections :

$$t_{min}^i > \left(\frac{-\mu + \sqrt{\mu^2 + 4\alpha(\sum_{j \in \mathcal{N}} \frac{1}{R_j})^2}}{4\sqrt{\alpha \Delta q} \sum_{j \in \mathcal{N}} \frac{1}{R_j}} \right)^2 \qquad (10)$$

where $\lambda = \sum_{j \in \mathcal{I}} \lambda_j$ and $\Delta q := q_{max} - q_{min}$.

Proof: The condition (9) (resp. (10)) will ensure that the value of q obtained in the linear region (see (5) (resp.(7))) is not larger that q_{max}. Indeed, for real time connections, (9) implies that

$$\sum_{j \in \mathcal{I}} \lambda_j t_j > \frac{\lambda - \mu}{q_{max} - q_{min}}$$

which implies together with (6) that $q < q_{max}$.

Finally the fact that we are not below the lower extreme of the linear region (i.e. $p_i > 0$ for all i) is a direct consequence of $\lambda > \mu$.

The case of only TCP connections is proved in Appendix X-A. ∎

The following result establishes the existence of Nash equilibrium for only real time sessions or only TCP connections.

Theorem 2: Assume that the functions d are convex in $T_i := 1/t_i$. Then a Nash equilibrium exists.
Proof: See Appendix X-B.

A. Supermodular Games

In Theorem 3 (resp. Theorem 5) we present alternative conditions that provide sufficient conditions for a supermodular structure for real-time connections (resp. for only TCP connections). This implies in particular the existence of an equilibrium. Another implication of supermodularity is that a simple, so-called tatônnement or Round Robin scheme, for best responses converges to the equilibrium. To describe it, we introduce the following asynchronous dynamic greedy algorithm (GA).

Greedy Algorithm: Assume a given initial choice $\mathbf{t^0}$ for all flows. At some strictly increasing times τ_k, $k = 1, 2, 3, ...,$ flows update their actions; the actions t_i^k at time $\tau_k > 0$ are obtained as follows. A single flow i at time τ_{k+1} updates its t_i^{k+1} so as to optimize $U_i(., [\mathbf{t}^k]_{-i})$ where $[\mathbf{t}^k]_{-i}$ is the vector of actions of the other flows $j \neq i$. We assume that each flow updates its actions infinitely often. In particular, for the case of only real time sessions, we update t_i^{k+1} as follows:

$$t_i^{k+1} = \underset{t_i \in [t_{\min}^i, t_{\max}^i]}{\arg\max} a_i\lambda_i(1-p_i) - b_i p_i - d(t_i) \quad (11)$$

where p_i in (11) is given by (6).

For the TCP-only case, we update t_i^{k+1} as follows:

$$t_i^{k+1} = \underset{t_i \in [t_{\min}^i, t_{\max}^i]}{\arg\max} \frac{a_i}{R_i}\sqrt{\frac{\alpha}{p_i}}(1-p_i) - d(t_i) \quad (12)$$

where p_i in (12) is given by (8).

Remark 1: For the case of real-time sessions, we could obtain closed form solution for t_j^{k+1} with specific cost function $d(t_i)$ such as $\frac{d}{t_i}$ which will lead to update of t_i^{k+1} as follows,

$\delta_i^k = \frac{\sum_{j\neq i} \lambda_j t_j^k}{\sqrt{(a_i\lambda_i+b_i)(\sum_{j\in\mathcal{I}}\lambda_j-\mu)(\sum_{j\neq i}\lambda_j t_j^k)-\lambda_i\sqrt{d}}}$ where δ_i^k is such that $\frac{\partial U_i}{\partial t_i}\Big|_{t_i=\delta_i^k} = 0$ and U_i corresponds to utility function of real time session i. Then t_i^{k+1} is given by :

$$t_i^{k+1} = \begin{cases} t_{\min}^i & \text{if } \delta_i^k < 0, \\ t_{\max}^i & \text{if } \delta_i^k < t_{\min}^i, \delta_i^k \geq 0, \\ t_{\min}^i & \text{if } \delta_i^k > t_{\max}^i, \delta_i^k \geq 0, \\ \delta_i & \text{otherwise} \end{cases}$$

Theorem 3: For the case of only real-time connections we assume that $\forall j$, $\lambda_{\min} \leq \lambda_j \leq \lambda_{\max}$, and

$$(I-1)\lambda_{\min}t_{\min} \geq \lambda_{\max}t_{\max},$$

where $t_{\min} = \min_{i\in\mathcal{I}}\{t_{\min}^i\}$ and $t_{\max} = \max_{i\in\mathcal{I}}\{t_{\max}^i\}$. Then there is smallest equilibrium $\underline{t}$ and largest equilibrium $\underline{t}$, and the **GA** dynamic algorithm converges to $\underline{\mathbf{t}}$ (resp. $\overline{\mathbf{t}}$) provided it starts with $t_{\min}^j$ for all j (resp. $t_{\min}^j$ for all j)

Proof: Both statements will follow by showing that the game is super-modular, see [11], [12]. A sufficient condition is that

$$\frac{\partial^2 U_i}{\partial t_i \partial t_j} = -(a_i\lambda_i + b_i)\frac{\partial^2 p_i}{\partial t_i \partial t_j} \geq 0.$$

We have

$$\frac{\partial p_i}{\partial t_i} = \left(\sum_j \lambda_j - \mu\right)\left(\frac{1}{\sum_{j\in I}\lambda_j t_j} - \frac{t_i\lambda_i}{(\sum_{j\in I}\lambda_j t_j)^2}\right)$$

leading to

$$\frac{\partial^2 p_i}{\partial t_i \partial t_k} = \lambda_k\left(\sum_{j\in\mathcal{I}}\lambda_j - \mu\right)\frac{-\sum_{j\in\mathcal{I}}\lambda_j t_j + 2t_i\lambda_i}{(\sum_{j\in\mathcal{I}}\lambda_j t_j)^3}.$$

It is non-positive if and only if $\sum_{j\neq i}\lambda_j t_j \geq \lambda_i t_i$. A sufficient condition is that $(I-1)\lambda_{\min}t_{\min} \geq \lambda_{\max}t_{\max}$. Thus the game is super-modular. The result then follows from standard theory of super-modular games [11], [12]. ■

Theorem 4: For the case of only real-time connections, we assume that $\forall j, \lambda_{\min} \leq \lambda_j \leq \lambda_{\max}$, and $2t_{\min}^3\lambda_{\min}^2 > t_{\max}^3\lambda_{\max}^2$. Under supermodular condition, the Nash equilibrium is unique.

Proof See Appendix X-C

Theorem 5: For the case of only TCP connections, assume that $\forall j, t_{min} \leq t \leq t_{max}$ and

$$(3+p_i)\frac{\partial p_i}{\partial t_i}\frac{\partial p_i}{\partial t_j} \geq 2p_i(p_i+1)\frac{\partial^2 p_i}{\partial t_i \partial t_j}\ \forall i,j,\ i\neq j. \quad (13)$$

Then the game is super-modular.

Proof: See Appendix X-D. ■

Remark 2: It would also be interesting to consider a price per unit of received volume, i.e., of the form $d(t_i)\lambda_i(1-p_i)$. However, looking at the super-modularity of the utility function gives a condition depending on $d'(t_i)$, $d(t_i)$ and the t_j that does not seem tractable. On the other hand, we can consider a pricing per unit of **sent volume**, i.e., of the form $d(t_i)\lambda_i$ (since λ_i is fixed), Conditions of Theorems 2-3 then hold to provide a Nash equilibrium.

VI. Symmetric Users

In this section, we assume that all flows have the same utility function (for all i, $a_i = a$, $\lambda_i = \bar{\lambda}$ and $b_i = b$ for real-time sessions and $a_i = a$ and $R_i = R$ for TCP connections) and the same intervals for strategies ($t_{\min}^i = t_{\min}$ and $t_{\max}^i = t_{\max}$).

Algorithm for Symmetric Nash Equilibrium:

For symmetric Nash equilibrium, we are interested in finding a symmetric equilibrium strategy $\mathbf{t}^* = (t^*, t^*, .., t^*)$ such that for any flow i and any strategy t_i for that flow (real-time session or TCP connection),

$$U(\mathbf{t}^*) \geq U(t_i, [\mathbf{t}^*]_{-i}). \quad (14)$$

Next we show how to obtain an equilibrium strategy. We first note that due to symmetry, to see whether $\mathbf{t}^*$ is an equilibrium it suffices to check (14) for a single flow. We shall thus assume that there are $L+1$ flows all together, and that the first L flows use the strategy $\mathbf{t}^o = (t^o, ..., t^o)$ and flow $L+1$ use t_{L+1}. Define the set

$$\mathcal{Q}_{L+1}(\mathbf{t^o}) = \arg\max_{t_{L+1}\in[t_{\min},t_{\max}]} \Big(U(t_{L+1}, [\mathbf{t^o}]_{-(L+1)})\Big),$$

where $\mathbf{t^o}$ denotes (with some abuse of notation) the strategy where all flows use t^o, and where the maximization is taken with respect to t_{L+1}. Then $\mathbf{t}^*$ is a symmetric equilibrium if

$$t^* \in \mathcal{Q}_{L+1}(\mathbf{t}^*).$$

Theorem 6: Consider real time connections operating in linear region. The symmetric equilibrium t^* satisfies:

$$T^* \left. \frac{\partial \hat{d}(T)}{\partial T} \right|_{T=T^*} = \frac{a\lambda + b}{(I\bar{\lambda})^2} \tag{15}$$

where $T^* = 1/t^*$ and $\hat{d}(T) = d(\frac{1}{T})$.

Proof: Recall that $\lambda = I\bar{\lambda}$. Then for real time connections, we have

$$U = a\bar{\lambda} - (a\bar{\lambda} + b)\frac{(\lambda - \mu)}{\bar{\lambda} + T_i \sum_{j \neq i} \bar{\lambda}/T_j} - \hat{d}(T_i).$$

which gives when taking the derivative

$$\frac{\partial U}{\partial T_i} = (a\bar{\lambda} + b)\frac{(\lambda - \mu) \sum_{j \neq i} \bar{\lambda}/T_j}{(\bar{\lambda} + T_i \sum_{j \neq i} \bar{\lambda}/T_j)^2} - \frac{\partial \hat{d}(T_i)}{\partial T_i}$$

Equating $\frac{\partial U}{\partial T_i} = 0$ we obtain (15). ■

VII. Optimal pricing

The goal here is to determine the pricing that maximizes the network's benefit. Assume that we are in the situation of this last remark. The goal is then to find out a function

$$c(\mathbf{t}^*) = \arg\max_d \sum_{i=1}^{I} d(t_i^*),$$

where $\mathbf{t}^*$ is a Nash equilibrium which can be obtained when considering special classes function of d. For instance, consider the set of functions $d(t) = d/e^t$. We then obtain a system of equations that can be solved numerically (to get the t^* satisfying the Nash equilibrium). Then a numerical optimization over the parameter d can be obtained.

Nevertheless, an assumption of this optimization problem is that the network knows the number of flows and the parameters a_i, b_i and $R_i \forall i$.

A more likely situation is when the network only knows the distribution of the number of players I (now a random variable) and the distribution of parameters a_i, b_i and R_i (assumed independent and independent between flows for convenience). A numerical investigation of optimal parameters can be realized as well.

VIII. Numerical Examples

In the following simulations, we obtain a unique Nash equilibrium for only real-time sessions or only TCP connections without satisfying the conditions in Theorem 4. Moreover, the GA algorithm converges without satisfying the conditions of supermodularity. All the conditions of supermodular games (Theorem 3 and Theorem 5) and uniqueness of Nash equilibrium (Theorem 2 and Theorem 3) are only sufficient but not necessary as shown in the numerical results.

The pricing function that we use for player i throughout this section is $d/\exp(t_i)$. We shall investigate how the choice of the constant d will affect the revenue of the network.[3]

A. Symmetric Real-Time flows

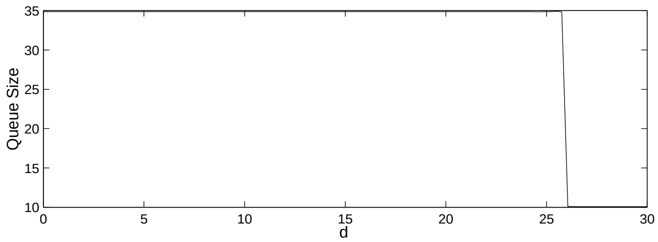
In the following numerical evaluations, we show the variation of different metrics as function of d. Figs. 2, 3 and 4 correspond to a unique symmetric Nash equilibrium case in which all the real time flows have $\lambda_i =$ 2Mbps with $t_{\min} = 0.01, t_{\max} = 100, \mathcal{I} = 20, q_{\min} = 10, q_{\max} = 40, \mu = 30$Mbps. Here we set the values of parameters to ensure that the system operates in linear region such as $t_{\min} > \frac{1}{\Delta q}(1 - \frac{\mu}{\sum_{j \in \mathcal{I}} \lambda_j}) = 0.0083$. The bound on $t_{\max}$ is needed only to limit the value of loss probability to 1. The value of d which maximizes the network revenue occurs at $d = 25.75$. All the flows attain a loss rate of 0.25. Note that for real time flows symmetric case, $p_i^* = (\sum_{j \in \mathcal{I}} \lambda_j - \mu)/\sum_{j \in \mathcal{I}} \lambda_j$ at the Nash equilibrium is a constant. The average queue size, given by $q_{\min} + p_i^*/t_i^*$, is shown in Fig. 2. We observe the value of t^* at which maximum network income is achieved is close to $t_{\min}$ while the system operates in the linear region of RED throughout.

We plot in Fig. 4 sample paths of a connection that uses the Algorithm for symmetric users (Sec. VI) (the evolution for all connections is the same). The figure shows convergence to the same Nash equilibrium when t^0 started from $t_{\min}$ or $t_{\max}$. We plot it for $d = 41.525$. In Figure 4(a), the value of t^* is 3.6163, and in Figure 4(b), it is 3.6162.

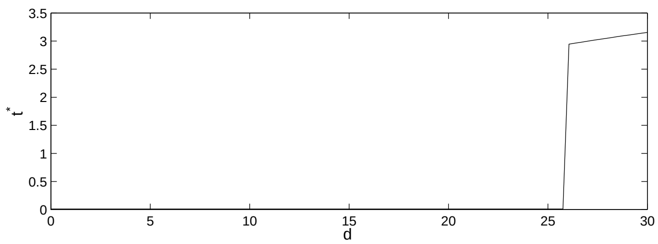
B. Non-symmetric real-time flows

In the next experiment, instead of having symmetric case, the rates, λ_i are drawn uniformly from $[1, 10]$ Mbps

[3]We note that it is desirable to have a "nontrivial" parameterized pricing function that leads to an optimal revenue for some parameter. We also tested other pricing functions that did turned out to be "trivial" in the sense that the benefit was always monotone in the parameter; an example of such a function is $\exp(-\beta t_i)$ and the network optimizes with respect to β.

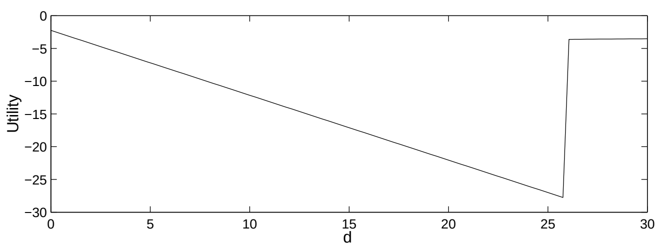


(a) queue size vs. d

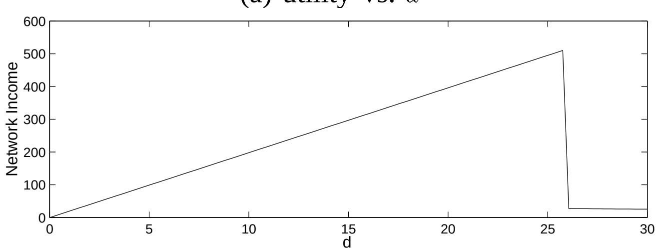


(b) t^* vs. d

Fig. 2. Symmetric Real Time flows: (a) queue size and (b) t^* vs. d



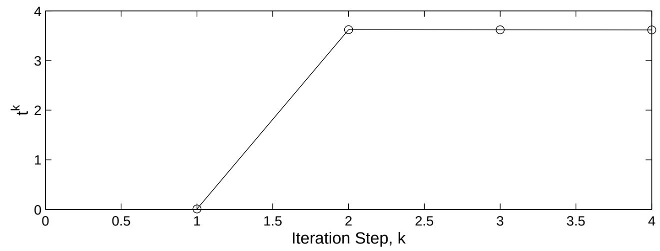
(a) utility vs. d



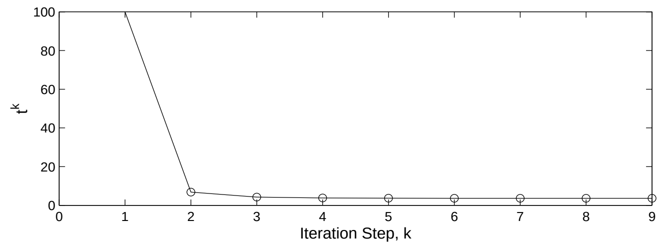
(b) network income vs. d

Fig. 3. Symmetric Real Time flows: (a) utility and (b) network income vs. d

with $t_{\min} = 1, t_{\max} = 100, q_{\max} = 40, q_{\min} = 10, \mathcal{I} = 20, \mu = 30$Mbps. Figures 5, 6 and 7 show how different metrics vary with d at unique Nash equilibrium. To ensure that the flows operate in linear region, we need $t_{\min} > \frac{1}{\Delta q} \geq \frac{1}{\Delta q}(1 - \frac{\mu}{\sum_j \lambda_j})$. We observe that $d = 27.27$ maximizes the network revenue. Figure 5(b) shows that values of t^* for flows having higher rates increase slower than that of flows having lower rates, i.e., higher rate flows experience less loss rates. Figure 6(a) shows that flows having different rates gains similarly in their utility functions. We plot the average loss rate in Figure 7. We confirm in these experiments about uniqueness of Nash equilibrium, although the sample path of different



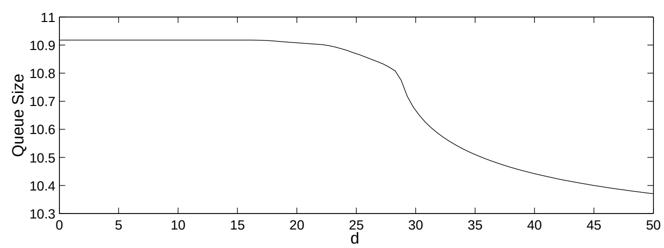
(a) $t^0 = t_{\min}$



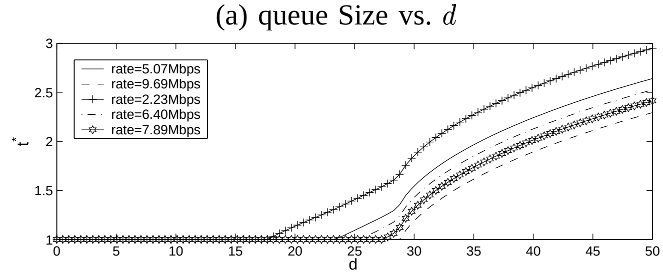
(b) $t^0 = t_{\max}$

Fig. 4. Symmetric Real Time flows: Convergence to Nash equilibrium

connections will depend on the connection rates.



(a) queue Size vs. d

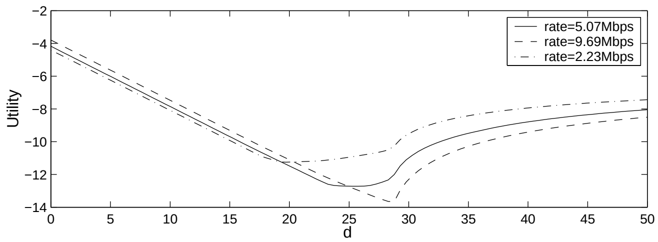


(b) t^* vs. d

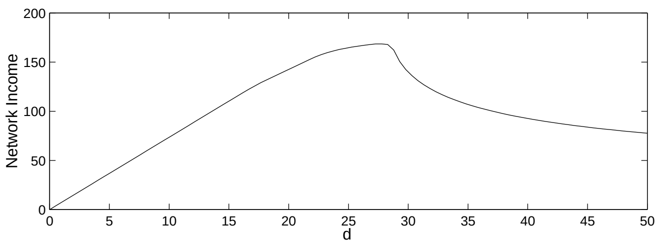
Fig. 5. Non-symmetric Real Time flows: (a) queue size and (b) t^* vs. d

C. Symmetric TCP Connections

For symmetric TCP connections we have considered $R_i = R = 20$ms for all connections with $t_{min} = 0.1$, $t_{max} = 100, \mu = 30$Mbps$, N = 20$. Figures 8, 9, and 10 show the corresponding figures. The maximum value of network revenue is found at $d = 0.4040$. In this



(a) utility vs. d



(b) network income vs. d

Fig. 6. Non-symmetric Real Time flows: (a) utility and (b) network income vs. d

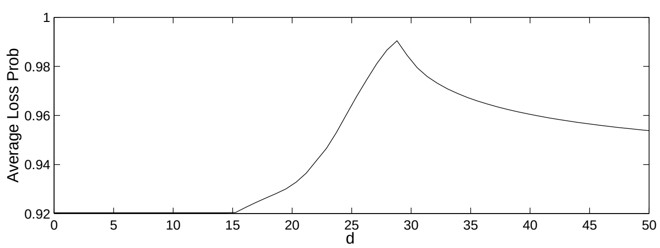


Fig. 7. Non-Symmetric Real Time flows: Average Loss Prob vs. d

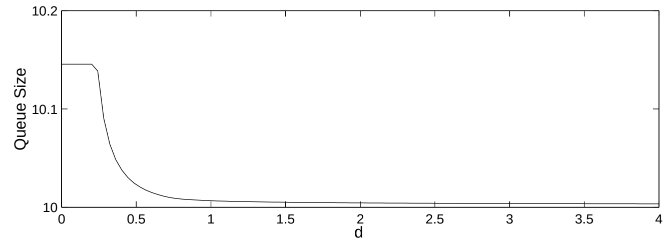
symmetric case, the loss probability is given by

$$p^* = \frac{R^2}{3N^2}\left\{\mu^2 + \frac{3N^2}{R^2} - \mu\sqrt{\mu^2 + \frac{6N^2}{R^2}}\right\} = 0.0017$$

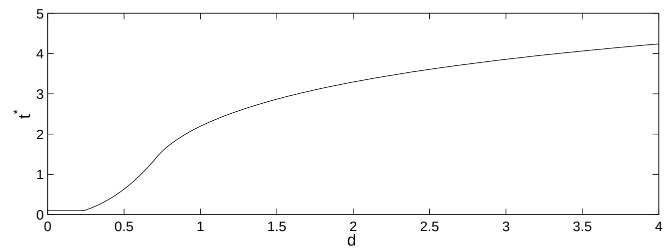
To ensure that the symmetric TCP flows operate in the linear region, we satisfy the condition on $t_{\min} > \left(\frac{-\mu + \sqrt{\mu^2 + 4(\sum_{j\in\mathcal{N}}\frac{1}{R_j})^2}}{4\sqrt{\alpha\Delta q}\sum_{j\in N}\frac{1}{R_j}}\right)^2 = 4.6271 \times 10^{-5}$.
We plot sample paths of a connection which show convergence to Nash equilibrium when t^0 started from $t_{\min}$ or $t_{\max}$. We plot it for $d = 3.821$. In Figure 10(a), the value of t^* is 3.4480, and in Figure 10(b), it is 3.4481.

D. Non-symmetric TCP connections

We present a non-symmetric case in Figures 11,12 and 13 in which R_is are drawn uniformly from $[1, 20]$ms with $t_{\min} = 1, t_{\max} = 100, \mu = 30$Mbps, $N = 20$. The value of d at which network revenue is highest is 0.9321. We

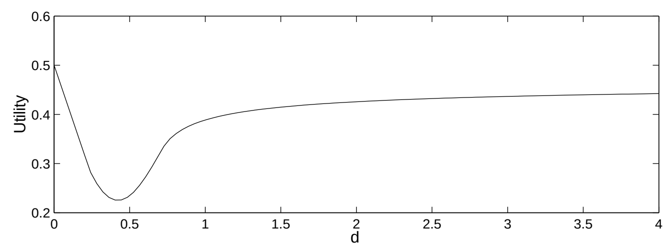


(a) TCP: queue Size vs. d

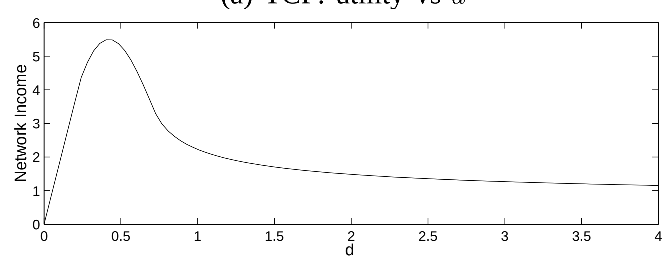


(b) TCP: t^* vs. d

Fig. 8. Symmetric TCP flows: (a) queue size and (b) t^* vs. d



(a) TCP: utility vs d



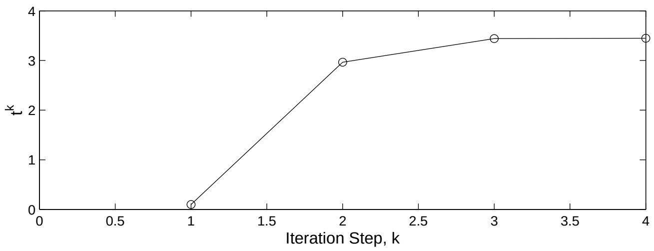
(b) TCP: network income vs. d

Fig. 9. Symmetric TCP flows: (a) utility and (b) network income vs. d

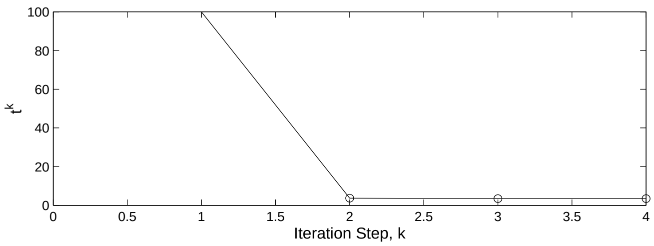
ensure that the non-symmetric connections operate in linear region by setting $t_{\min} > \left(\frac{-\mu + \sqrt{\mu^2 + 4(\sum_{j\in\mathcal{N}}\frac{1}{R_j})^2}}{4\sqrt{\alpha\Delta q}\sum_{j\in\mathcal{N}}\frac{1}{R_j}}\right)^2 = 0.5476$.

E. Real-time connections and TCP flows

In this experiment, we combine both real-time and TCP connections. We have $I = 15$, $N = 15$, $\mu = 13$Mbps, RTT=10ms, $t^{real}_{\min} = 5, t^{real}_{\max} = 11, t^{TCP}_{\min} = 5, t^{TCP}_{\max} = 11, \lambda = 1$Mbps, $q_{\min} = 10, q_{\max} = 40$. The highest network revenue is achieved at $d = 40.40, t^{real} = 7.69, t^{TCP} = 5$. In the simulations, we

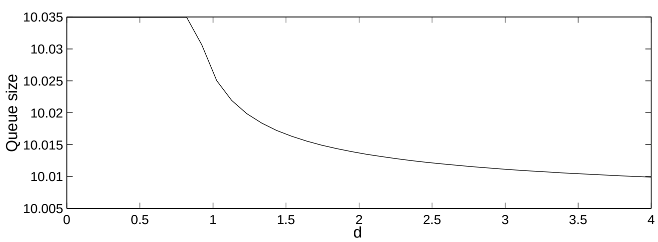


(a) $t^0 = t_{\min}$

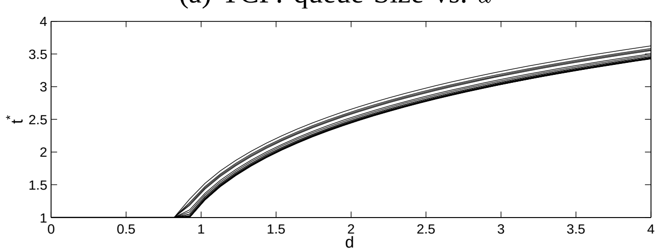


(b) $t^0 = t_{\max}$

Fig. 10. Symmetric TCP fbws: Convergence to Nash equilibrium

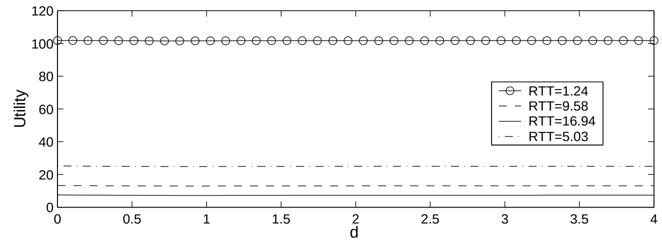


(a) TCP: queue Size vs. d

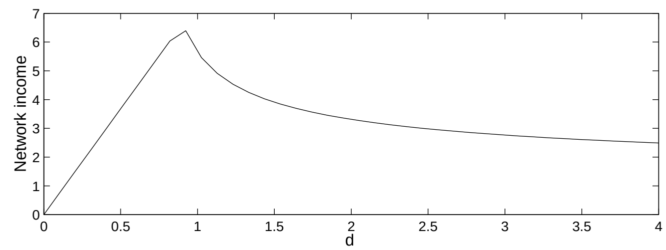


(b) TCP: t^* vs. d

Fig. 11. Non-Symmetric TCP: (a) queue size and (b) t^* vs d

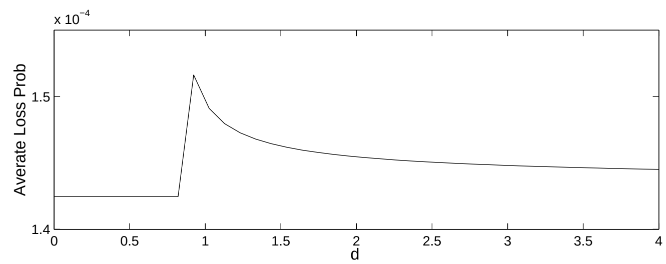


(a) TCP: utility vs d

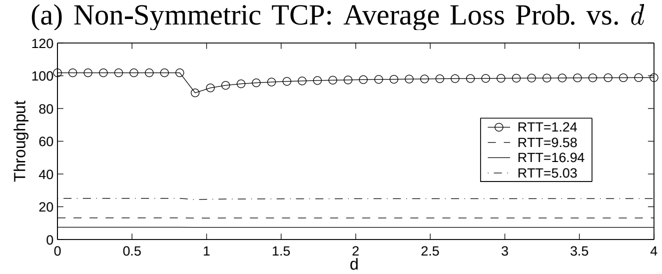


(b) TCP: network income vs. d

Fig. 12. Non-Symmetric TCP: (a) utility and (b) network pricing vs. d



(a) Non-Symmetric TCP: Average Loss Prob. vs. d



(b) Non-Symmetric TCP: Throughput vs. d

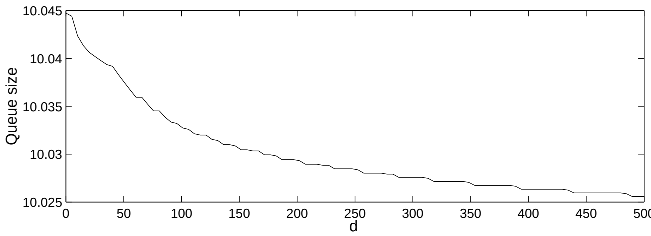
Fig. 13. Non-Symmetric TCP: Average Loss prob and throughputs vs. d

observe the values of $q < q_{\max}$ and since there is atleast one TCP flow i with throughput, $\lambda_i > 0$, it implies that the flow has loss probability, $p_i > 0$ and average queue length, $q > q_{\min}$. We conclude that system operates in linear region. Our objective in this set of experiments is to show that there exists a Nash equilibrium for both real-time and TCP connections.

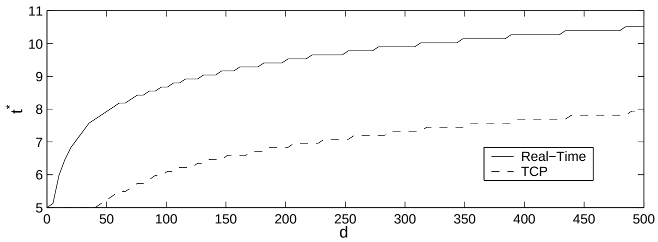
IX. Conclusions and Future Work

We have studied in this paper a fluid model of the RED buffer management algorithm with different drop probabilities applied to both UDP and TCP traffic. We first computed the performance measures for fixed drop policies. We then investigated how the drop policies are determined. We modeled the decision process as a non-cooperative game and obtained its equilibria. We showed the existence of the equilibria under various conditions, and provided ways for computing them (establishing also convergence properties of best-response dynamics). The equilibrium depends on the pricing strategy of the network provider. We finally addressed the problem of optimizing the revenue of the network provider.

Concerning the future work, we are working on deriving sufficient and necessary conditions for operating at the linear region when there are both real time and

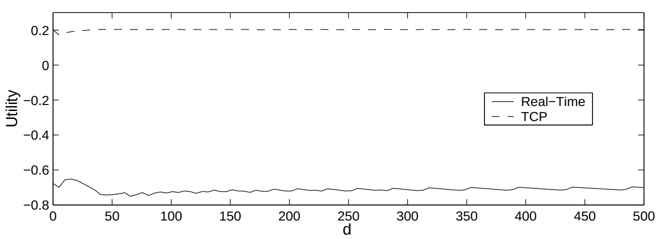


(a) queue Size vs. d

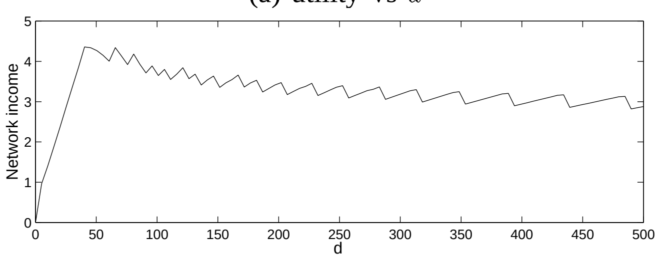


(b) t^* vs. d

Fig. 14. Real-time and TCP: (a) queue size and (b) t^* vs d



(a) utility vs d



(b) network income vs. d

Fig. 15. Real-time and TCP: (a) utility and network income vs. d

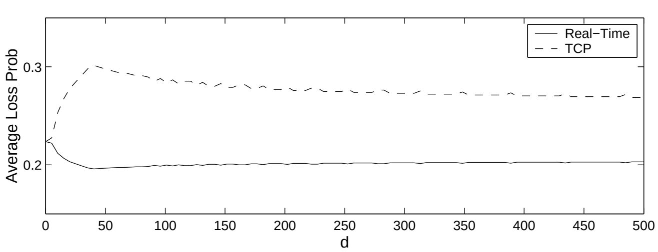


Fig. 16. Real-time and TCP: Average Loss prob vs. d

TCP connections; these seem to be more involved than the conditions we have obtained already. We will further study the impact of buffer management schemes on the performance and on the revenues of the network; in particular, other versions of RED will be considered (such as the gentle-RED variant). We will also examine how well the fluid model is suitable for the packet-level model that it approximates.

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X. APPENDIX

A. Proof of part 2 of Theorem 1

For only TCP connections, we have,

$$\sqrt{q - q_{\min}} \leq \sqrt{q_{\max} - q_{\min}} = \sqrt{\Delta q} \tag{16}$$

From equation (7), we get the following sufficient and necessary condition for $q \leq q_{max}$:

$$\frac{-\mu + \sqrt{\mu^2 + 4\alpha(\sum_j \frac{\sqrt{t_j}}{R_j})(\sum_j \frac{1}{R_j\sqrt{t_j}})}}{2\sqrt{\alpha}(\sum_j \frac{\sqrt{t_j}}{R_j})} \leq \sqrt{\Delta q}$$

or equivalently,

$$\sqrt{\mu^2 + 4\alpha(\sum_j \frac{\sqrt{t_j}}{R_j})(\sum_j \frac{1}{R_j\sqrt{t_j}})} \leq \mu + 2\sqrt{\alpha}(\sum_j \frac{\sqrt{t_j}}{R_j})\sqrt{\Delta q}$$

which is equivalent to

$$\mu^2 + 4\alpha(\sum_j \frac{\sqrt{t_j}}{R_j})(\sum_j \frac{1}{R_j\sqrt{t_j}}) \leq \mu^2$$

$$+4\sqrt{\alpha}\mu(\sum_j \frac{\sqrt{t_j}}{R_j})\sqrt{\Delta q} + 4\alpha\Delta q(\sum_j \frac{\sqrt{t_j}}{R_j})^2$$

or equivalently,

$$\alpha(\sum_j \frac{1}{R_j\sqrt{t_j}}) \leq \mu\sqrt{\alpha\Delta q} + \alpha\Delta q(\sum_j \frac{\sqrt{t_j}}{R_j}).$$

A sufficient condition for the latter is

$$\sum_{j\in\mathcal{N}} \frac{1}{R_j\sqrt{t_{\min}}} \leq \mu\sqrt{\alpha\Delta q} + \alpha\Delta q \sum_{j\in\mathcal{N}} \frac{\sqrt{t_{\min}}}{R_j} \qquad (17)$$

Solving the quadratic equation (17) for $t_{\min}$, we see that this is implied by (10).

Finally the fact that we are not below the lower extreme of the linear region (i.e. $p_i > 0$ for all i) is a direct consequence of the fact that zero loss probability would imply infinite throughput (see eq (1)), which is impossible since the link capacity μ is finite.

B. Proof of Theorem 2

We first show that the utility function is concave in the case of only real time sessions. Replacing t_i by $1/T_i$ in Equation (6), we obtain

$$p_i = \frac{\sum_{j\in\mathcal{I}} \lambda_j - \mu}{\lambda_i + T_i\sum_{j\neq i}\lambda_j/T_j},$$

which is convex in T_i. Hence U_i are concave in T_i and continuous in T_j. The existence then follows from [10]. For TCP connection, we have

$$\frac{\partial^2 U_i}{\partial T_i^2} = a_i\Big[\frac{\partial^2\lambda_i}{\partial T_i^2}(1-p_i) - 2\frac{\partial\lambda_i}{\partial T_i}\frac{\partial p_i}{\partial T_i} - \lambda_i\frac{\partial^2 p_i}{\partial T_i^2}\Big] - \frac{\partial^2\hat{d}(T_i)}{\partial T_i^2} \qquad (18)$$

where $\hat{d}(T_i) = d(1/t_i)$. On the other hand, (1) implies

$$\frac{\partial p_i}{\partial T_i} = -\frac{2\alpha}{R_i^2}\frac{\frac{\partial\lambda_i}{\partial T_i}}{\lambda_i^3}, \quad \text{and}$$

$$\frac{\partial^2 p_i}{\partial T_i^2} = -\frac{2\alpha}{R_i^2\lambda_i^4}\Big[\frac{\partial^2\lambda_i}{\partial T_i^2}\lambda_i - 3(\frac{\partial\lambda_i}{\partial T_i})^2\Big]$$

Then (18) becomes

$$\frac{\partial^2 U_i}{\partial T_i^2} = a_i\Big[\frac{\partial^2\lambda_i}{\partial T_i^2}(1 + \frac{\alpha}{R_i^2\lambda_i^2}) - (\frac{\partial\lambda_i}{\partial T_i})^2\frac{2\alpha}{R_i^2\lambda^3}\Big] - \frac{\partial^2\hat{d}(T_i)}{\partial T_i^2} \qquad (19)$$

Since the function $\hat{d}$ is convex in T_i, then form (19), it suffices to show that the second derivative of λ_i with respect to T_i is non-positive. We have

$$\begin{aligned}
\lambda_i &= \frac{1}{R_i}\sqrt{\frac{\alpha}{p_i}} \\
&= \frac{\frac{2\alpha}{R_i}\left(\sum_{j\in\mathcal{N}}\frac{\sqrt{t_j}}{R_j}\right)}{\sqrt{t_i}\left(-\mu + \sqrt{\mu^2 + 4\alpha\sum_{j\in\mathcal{N}}(\frac{1}{R_j\sqrt{t_j}})\sum_{j\in\mathcal{N}}(\frac{\sqrt{t_j}}{R_j})}\right)} \\
&= \frac{\frac{2\alpha}{R_i}\left(\frac{\sqrt{t_i}}{R_i} + C_1\right)}{\sqrt{t_i}\left(-\mu + \sqrt{\mu^2 + 4\alpha(\frac{1}{R_i\sqrt{t_i}} + C_2)(\frac{\sqrt{t_i}}{R_i} + C_1)}\right)} \\
&= \frac{\frac{2\alpha}{R_i}\sqrt{T_i}\left(\frac{1}{\sqrt{T_i}R_i} + C_1\right)}{\left(-\mu + \sqrt{\mu^2 + 4\alpha(\frac{\sqrt{T_i}}{R_i} + C_2)(\frac{1}{\sqrt{T_i}R_i} + C_1)}\right)} \\
&= \frac{\frac{\sqrt{T_i}}{2R_i}\left(\mu + \sqrt{\mu^2 + 4\alpha(\frac{\sqrt{T_i}}{R_i} + C_2)(\frac{1}{\sqrt{T_i}R_i} + C_1)}\right)}{(\frac{\sqrt{T_i}}{R_i} + C_2)} \\
&= \frac{1}{2R_i}\Big[\frac{\sqrt{T_i}\mu}{(\frac{\sqrt{T_i}}{R_i} + C_2)} + \\
&\quad \frac{\sqrt{T_i}\sqrt{\mu^2 + 4\alpha(\frac{\sqrt{T_i}}{R_i} + C_2)(\frac{1}{\sqrt{T_i}R_i} + C_1)}}{(\frac{\sqrt{T_i}}{R_i} + C_2)}\Big] \\
&= \frac{1}{2R_i}\Big[F_1(T_i) + F_2(T_i)\Big]
\end{aligned}$$

where $C_1 = \sum_{j\neq i}\frac{\sqrt{t_j}}{R_i}$ and $C_2 = \sum_{j\neq i}\frac{1}{\sqrt{t_j}R_i}$. Now, we must prove that the second derivative of the functions F_1 and F_2 are non-positive for all $C_1 \geq 0$ and $C_2 \geq 0$. We begin by taking the second derivative of F_1. After some simplification, we obtain

$$\frac{\partial^2 F_1(T_i)}{\partial T_i^2} = -1/4\frac{\mu R_i^2 C_2(3\sqrt{T_i} + C_2R_i)}{(T_i^{3/2}(\sqrt{T_i} + C_2R_i)^3)}$$

which is positive. For the second function F_2, since the function F_2 is positive, it suffices to show that the second derivative of function $[F^2(T_i)]^2$ is non-positive, we have

$$\frac{\partial^2[F_2(T_i)]^2}{\partial T_i^2} = -\frac{R_i}{2T_i^{3/2}(\sqrt{T_i}+C_2R_i)^4}(6T_i\alpha C_2 + 8\sqrt{T_i}\alpha C_2^2 R_i + 2\alpha T_i^2 C_1 \ + 8T_i^{3/2}\alpha C_2 R_i C_1 + 6\alpha T_i R_i^2 C_1 C_2^2 + 2\alpha R_i^2 C_2^3 + 3T_i\mu^2 R_i^2 C_2)$$

which is non-positive.

C. Proof of Theorem 4

Under supermodular condition, to show the uniqueness of Nash equilibrium, it suffices to show that [4],

$$-\frac{\partial^2 U_i}{(\partial T_i)^2} \geq \sum_{j\neq i}\frac{\partial^2 U_i}{\partial T_i \partial T_j}. \tag{20}$$

or equivalently,

$$\frac{\partial^2 p_i}{(\partial T_i)^2} + \sum_{j\neq i}\frac{\partial^2 p_i}{\partial T_i \partial T_j} \geq 0. \tag{21}$$

For the case of only real time sessions, $p_i = \frac{\sum \lambda_j - \mu}{\lambda_i + T_i \sum_{j\neq i}\lambda_j \frac{1}{T_j}}$. We have ,

$$\frac{\partial p_i}{\partial T_i} = -\frac{(\lambda-\mu)\sum_{k\neq i}\frac{\lambda_k}{T_k}}{\left(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^2}$$

$$\frac{\partial^2 p_i}{\partial T_i^2} = \frac{2(\lambda-\mu)\left(\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^2}{(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k})^3}$$

$$\frac{\partial^2 p_i}{\partial T_i \partial T_j} = (\lambda-\mu)\frac{\lambda_j}{T_j^2}\frac{\lambda_i - T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}}{\left(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^3}$$

Therefore, in order to get the uniqueness, we need that

$$\frac{\partial^2 p_i}{\partial T_i^2} + \sum_{j\neq i}\frac{\partial^2 p_i}{\partial T_i \partial T_j} = \frac{2(\lambda-\mu)\left(\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^2}{\left(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^3} + (\lambda-\mu)\frac{\lambda_i - T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}}{\left(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^3}\sum_{j\neq i}\frac{\lambda_j}{T_j^2}$$

$$= \frac{(\lambda-\mu)}{\left(\lambda_i + T_i\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^3}\left[\lambda_i\sum_{j\neq i}\frac{\lambda_j}{T_j^2} - T_i\left(\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)\left(\sum_{k\neq i}\frac{\lambda_k}{T_k^2}\right) + 2\left(\sum_{k\neq i}\frac{\lambda_k}{T_k}\right)^2\right] \geq 0$$

This leads to the sufficient condition:

$$\frac{\lambda_{min}^2}{T_{min}^2} - T_{max}\frac{\lambda_{max}^2(I-1)^2}{T_{min}^3} + 2\frac{\lambda_{min}^2}{T_{max}^2} \geq 0$$
$$2\frac{\lambda_{min}^2}{T_{max}^2} > T_{max}\frac{\lambda_{max}^2}{T_{min}^3}$$
$$2T_{min}^3\lambda_{min}^2 > T_{max}^3\lambda_{max}^2$$

D. Proof of Theorem 5

For supermodularity on TCP connections, we consider the sufficient condition that $\frac{\partial^2 U_i}{\partial t_i \partial t_j} \geq 0$. It follows that

$$U_i = \frac{a_i}{R_i}\sqrt{\frac{\alpha}{p_i}}(1-p_i) - d(t_i)$$
$$= \frac{a_i}{R_i}\sqrt{\alpha}(p_i^{-1/2} - p_i^{1/2}) - d(t_i).$$

Then, for $j \neq i$,

$$\frac{\partial U_i}{\partial t_j} = \frac{a_i\sqrt{\alpha}}{R_i}\left(\frac{-p_i^{-3/2}}{2}\frac{\partial p_i}{\partial t_j} - \frac{p_i^{-1/2}}{2}\frac{\partial p_i}{\partial t_j}\right)$$
$$\frac{\partial^2 U_i}{\partial t_i \partial t_j} = \frac{a_i\sqrt{\alpha}}{R_i}\left[\left(\frac{3p_i^{-5/2}}{4} + \frac{p_i^{-3/2}}{4}\right)\frac{\partial p_i}{\partial t_i}\frac{\partial p_i}{\partial t_j} - \left(\frac{p_i^{-3/2}}{2} + \frac{p_i^{-1/2}}{2}\right)\frac{\partial^2 p_i}{\partial t_i \partial t_j}\right].$$

Thus a sufficient condition for supermodularity ($\frac{\partial^2 U_i}{\partial t_i \partial t_j} \geq 0, \ \forall i,j,j\neq i$)is

$$(3+p_i)\frac{\partial p_i}{\partial t_i}\frac{\partial p_i}{\partial t_j} \geq 2p_i(p_i+1)\frac{\partial^2 p_i}{\partial t_i \partial t_j}, \ \forall i,j,j\neq i$$

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