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A particle system in interaction with a rapidly varying environment: Mean field limits and applications

Charles Bordenave*, David McDonald† and Alexandre Proutiere‡

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Abstract

We study an interacting particle system whose dynamics depends on an interacting random environment. As the number of particles grows large, the transition rate of the particles slows down (perhaps because they share a common resource of fixed capacity). The transition rate of a particle is determined by its state, by the empirical distribution of all the particles and by a rapidly varying environment. The transitions of the environment are determined by the empirical distribution of the particles. We prove the propagation of chaos on the path space of the particles and establish that the limiting trajectory of the empirical measure of the states of the particles satisfies a deterministic differential equation. This deterministic differential equation involves the time averages of the environment process.

We apply our results to analyze the performance of communication networks where users access some resources using random distributed multi-access algorithms. For these networks, we show that the environment process corresponds to a process describing the number of clients in a certain loss network, which allows us provide simple and explicit expressions of the network performance.

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1 Introduction and motivation

In this paper we are interested in the mean field limit of a fairly general stochastic particle system whose dynamics depends on an environment. A first key feature of the model is that the particle system is interacting: the evolution of each particle depends the empirical distribution of all the particles and also depends on an environment variable. Secondly, the environment is interacting with the particle system: its dynamics depends on the evolution of the particles and is given by a Markov transition kernel which depends on the actual state of the particle system. A last key feature is that the environment is rapidly varying: it evolves at rate 1 whereas the particles evolve at rate $1/N$, where N is the total number of particles.

We analyze this particle system when N, the number of particles, goes to infinity. The limiting system is known as the mean field limit of the particle system. We use a method developed by Sznitman [25] to prove a path space convergence of the trajectory of the empirical measures of the states of the particles. For exchangeable systems, this convergence turns out to be equivalent to the particle system being chaotic.

*University of California at Berkeley, United States.
†University of Ottawa, Canada.
‡KTH, Sweden.

The initial motivation for this model comes from the analysis of the way users communicate in wired and wireless Local Area Networks (LANs). LANs are computer networks covering a local area such as a home, an office, or a group of buildings. In such networks, users share a common resource, the channel, in a distributed manner using random multiple access algorithms, such as Aloha [3] and the exponential back-off algorithm used in all LANs today [1, 2]. A particle is a user of a communication channel shared between N users so each user accesses the channel roughly one in N time slots. At each time slot, a user tries to access the common channel with a certain probability if the channel is available; i.e. no user is currently transmitting a packet. When the channel is available two users may attempt to grab the channel at the same time. This will produce a collision detected by both users. Both users will then back-off by reducing their access probabilities. Therefore probability of accessing the channel of each user evolves according to its past collisions and successful attempts and will necessarily settle around $1/N$; i.e. they slow down as N increases.

It may happen that some users are hidden from others: two users hidden from each other can successfully access the channel simultaneously. This phenomenon is frequent in wireless LANs where users interact through interference depending on their relative positions. To model this we assume each user has a class determined by the region in the network where it is located. The state of a user is its class and its probability to access the channel. Users interact depending on their class: users in the same region or in two adjacent regions are not able to successfully access the channel at the same time, whereas users located in two non-adjacent regions may access the channel simultaneously. An environment variable indicates whether the channel is available or not for users in each class. This variable depends on whether or not an interacting user is currently transmitting successfully. The environment variable varies from time slot to time slot; i.e. it is quickly varying.

Mean field models have been used in statistical physics for some time; see [11] for example. An extensive literature exists on the mean fields analysis of stochastic genetic models. A particle is an individual and its state represents its genetic types and its position. The most important related model is the Fleming-Viot process; see for [12] for example. This new model could also help to analyze genetics models. The environment variable may represent available resources at each location. The environment evolves according to the empirical distribution of the individuals and individuals interact with each other and their environment.

Another potential field of application is microscopic models in economic theory and stochastic market evolution, also known as "econophysics", see for example the work by Karatzas [19] or Cordier [10]. In a simple market economy or in a financial market, a particle is an economic agent and its states represents its goods and its savings. The environment is the prices of the various available goods. Agents may exchange, borrow or lend money. Both prices and the purchase decisions of agents are interacting. In some markets, like financial markets, the prices are fluctuating roughly N times faster than decisions of each individual agent; i.e. this framework is consistent with our model.

The remainder of the paper is organized as follows. In Section 2.1, we define rigourously the stochastic particle system and in Section 2.2 we state our main results. The proofs are in Section 3. In Section 4, we discuss the assumptions of the models and define a convenient sufficient condition for checking most of them. Lastly, in Section 5, we apply our results to the analysis of random multiple access algorithms.

Notations Let $\mathcal{Y}$ be a separable, complete metric space, $\mathcal{P}(\mathcal{Y})$ denotes the space of probability measures on $\mathcal{Y}$. $\mathcal{L}(X)$ is the law of the $\mathcal{Y}$-valued random variable X. $D(\mathbb{R}^+, \mathcal{Y})$ the space of right-continuous functions with left-handed limits, with the Skorohod topology associated with the metric d^0_∞, see [7] p 168. With this metric, $D(\mathbb{R}^+, \mathcal{Y})$ is complete and separable. We extend a discrete time trajectory $(X(k)), k \in \mathbb{N}$, in $D(\mathbb{N}, \mathcal{Y})$ in a continuous time trajectory in $D(\mathbb{R}^+, \mathcal{Y})$ by setting for $t \in \mathbb{R}^+$, $X(t) = X([t])$, where $[\cdot]$ denotes the integer part. $(\mathcal{F}_t), t \in \mathbb{R}^+$ or $\mathbb{N}$, will denote the natural filtration with respect to the processes considered. $\|\cdot\|$ denotes the norm in total variation of measures. Finally, for any measure $Q \in \mathcal{P}(\mathcal{Y})$ and any measurable function f on $\mathcal{Y}$, $\langle f, Q\rangle = Q(f) = \int f dQ$ denotes the usual duality brackets.

We recall that a sequence of random variables $(X_i^N)_{i\in\{1,\dots,N\}} \in \mathcal{Y}^N$ is exchangeable if $\mathcal{L}((X_i^N)_{i\in\{1,\dots,N\}}) = \mathcal{L}((X_{\sigma(i)}^N)_{i\in\{1,\dots,N\}})$ where σ is any permutation of $\{1,\dots,N\}$. Moreover the sequence is Q-chaotic if for all subsets $I \subset \mathbb{N}$ of finite cardinal $|I|$,

$$\lim_{N\to\infty} \mathcal{L}\left((X_i^N)_{i\in I}\right) = Q^{\otimes |I|} \quad \text{weakly in } \mathcal{P}(\mathcal{Y}^{|I|}). \tag{1}$$

2 An interacting particle system in a varying environment

In this section, we first provide a precise description of the interacting particle system under consideration. We then state the main results, giving the system behavior in the mean field limit when the number of particles grows to infinity. The proofs of these results are postponed to subsequent sections.

2.1 Model description

The particles We consider N particles evolving in a countable state space $\mathcal{X}$ at discrete time slots $k \in \mathbb{N}$. For simplicity we assume the particles are exchangeable. At time k, the state of the i-th particle is $X_i^N(k) \in \mathcal{X}$. The state of the system at time k is described by the empirical measure $\nu^N(k) \in \mathcal{P}(\mathcal{X})$ while the entire history of the process is described by the empirical measure ν^N on path space $\mathcal{P}(D(\mathbb{N}, \mathcal{X}))$:

$$\nu^N(k) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(k)} \quad \text{and} \quad \nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}.$$

The interacting environment In the system considered, the evolution of the particles depends not only on the state of the particle system but also on a background Markovian process $Z^N \in D(\mathbb{N}, \mathcal{Z})$, where $\mathcal{Z}$ is an at most countable state space. This process evolves as follows:

$$\mathrm{P}(Z^N(k+1) = z|\mathcal{F}_k) = K^N_{\nu^N(k)}(Z^N(k), z),$$

where K^N_μ is a transition kernel on $\mathcal{Z}$ depending on a probability measure μ on $\mathcal{P}(\mathcal{X})$, and where $\mathcal{F}_k = \sigma((\nu^N(0), Z^N(0)), \cdots, (\nu^N(k), Z^N(k)))$. The latter filtration depends on N, but as pointed out above, without possible confusion, $\mathcal{F}_k$ will always denote the underlying natural filtration of the processes. In words, Z^N is a Markov chain whose transition kernel evolves with the empirical measure of the state of the particle system.

Evolution of the particles We represent the possible transitions for a particle by a countable set $\mathcal{S}$ of mappings from $\mathcal{X}$ to $\mathcal{X}$. A s-transition for a particle in state x leads this particle to the state $s(x)$. In each time slot the state of a particle has a transition with probability $1/N$ independently of everything else. If a transition occurs, this transition is a s-transition with probability $F_s^N(x, \nu, z)$, where x, ν, and z are respectively the state of the particle, the empirical measure, and the state of the background process before the transition. Hence, in this state, a s-transition occurs with probability:

$$\frac{1}{N}F_s^N(x,\nu,z). \tag{2}$$

with $\sum_{s\in\mathcal{S}} F_s^N(x,\alpha,z) = 1$ for all $(x,\alpha,z)\in\mathcal{X}\times\mathcal{P}(\mathcal{X})\times\mathcal{Z}$. Note that, due to (2), the process Z^N evolves quickly while the empirical measure $\nu^N(k)$ evolves slowly. Also note that the s-transitions of the various particles may be correlated. Finally the process Z^N may depend on the transitions of the particles. The particle system is thus in interaction with its environment.

We make the following additional assumptions on the system evolution.

Assumptions

A1. Uniform convergence of F_s^N to F_s:
$\lim_{N\to\infty}\sup_{(x,\alpha,z)\in\mathcal{X}\times\mathcal{P}(\mathcal{X})\times\mathcal{Z}}\sum_{s\in\mathcal{S}}|F_s^N(x,\alpha,z)-F_s(x,\alpha,z)|=0$.

A2. The functions F_s is uniformly Lipschitz:
$\sup_{(x,z)\in\mathcal{X}\times\mathcal{Z}}\sum_{s\in\mathcal{S}}|F_s(x,\alpha,z)-F_s(x,\beta,z)|\le C\|\alpha-\beta\|$.

A3. Uniform convergence in total variation of K_α^N to K_α:
$\lim_{N\to\infty}\sup_{(\alpha,z)\in\mathcal{P}(\mathcal{X})\times\mathcal{Z}}\|K_\alpha^N(z,\cdot)-K_\alpha(z,\cdot)\|=0$.

A4. The mapping $\alpha\mapsto K_\alpha$ is uniformly Lipschitz:
$\sup_{z\in\mathcal{Z}}\|K_\alpha(z,\cdot)-K_\beta(z,\cdot)\|\le C\|\alpha-\beta\|$.

A5. The Markov chains with kernels K_α are ergodic uniformly in α, i.e., they are ergodic and, $\lim_{n\to\infty}\sup_{\alpha\in\mathcal{P}(\mathcal{X})}\|P_\alpha^n(z_0,\cdot)-\pi_\alpha\|=0,$ for some fixed $z_0\in\mathcal{Z}$, where π_α is the stationary probability of K_α and $P_\alpha^n(z,\cdot)=(K_\alpha)^n(z,\cdot)$.

A6. We define $\tau^N(t)=\inf\{k\ge 0: Z^N(t+k)=z_0\}$, where z_0 has been defined in A5. For all $Z^N(0)$ and $\nu^N(0)$, uniformly in t and N, there exists $C=C(Z^N(0),\nu^N(0))$ such that $\mathrm{E}[\tau^N(t)]\le C$.

A7. For all α, β in $\mathcal{P}(\mathcal{X})$: $\|\pi_\alpha-\pi_\beta\|\le C\sup_{z\in\mathcal{Z}}\|K_\alpha(z,\cdot)-K_\beta(z,\cdot)\|$.

A8. For all For all α, β in $\mathcal{P}(\mathcal{X})$: $\|\pi_\alpha-\pi_\beta\|\le C\sup_{z\in\mathcal{Z}}\|K_\alpha(z,\cdot)-K_\beta(z,\cdot)\|$.

We discuss in Section 4 how the above assumptions may be checked.

2.2 Main Results

The main result of this paper is to provide a mean field analysis of the system described above, i.e, to characterize the evolution of the system when the number of particles grows. According to (2), as $N\to\infty$, the chains $X_i^N(t)$ slow down hence to derive a limiting behavior we define:

$$q_i^N(t)=X_i^N([Nt]) \quad \text{and} \quad \mu^N=\frac{1}{N}\sum_{i=1}^N\delta_{q_i^N}\in\mathcal{P}(D(\mathbb{R}_+,\mathcal{X})).$$

2.2.1 Transient regimes

The following theorem provides the limiting behavior of the system in transient regimes.

Theorem 1 *Assume that the Assumptions A1-A7 hold, that $\mathcal{L}(q_1^N(\cdot))$ is tight in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$, and that the initial values $q_i^N(0)$, $i = 1, \ldots, N$, are exchangeable and such that their empirical measure μ_0^N converges in distribution to a deterministic limit $Q_0 \in \mathcal{P}(\mathcal{X})$. There exists a probability measure Q on $D(\mathbb{R}^+, \mathcal{X})$ such that the processes $(q_i^N(.), i \in \{1, \ldots, N\})$ are Q-chaotic.*

In [25], Sznitman proved that if $q_i^N(0)$, $i = 1, \ldots, N$, are exchangeable, their empirical measure μ_0^N converges in distribution to a deterministic limit $Q_0 \in \mathcal{P}(\mathcal{X})$ if and only if $q_i^N(0)$, $i = 1, \ldots, N$, are Q_0-chaotic. Then, the above theorem states that if the particles are initially asymptotically independent, then they remain asymptotically independent. This phenomenon is also known as the propagation of chaos.

In most applications, the tightness of $\mathcal{L}(q_1^N(\cdot))$ in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$ should not be a major issue. Indeed, note that the inter-arrival times between two transitions of $q_1^N(.)$ are independent Binomial $(N, 1/N)$ variables (which converges to exponential (1) variables). Hence, if for example the state space $\mathcal{X}$ or the set of transitions $\mathcal{S}$ is finite, we may apply the tightness criterion Theorem 7.2 in Ethier-Kurtz [14] p.128.

The independence allows us to derive an explicit expression for the system state evolution. As explained earlier, intuitively, when N is large, the evolution of the background process is very fast compared to that of the particle system. The particles then see a time average of the background process. The following theorem formalizes this observation. For $\alpha \in \mathcal{P}(\mathcal{X})$, let π_α denote the stationary distribution of the Markov chain with transition kernel K_α. We define the average transition rates for a particle in state x by

$$\overline{F}_s(x, \alpha) = \sum_{z \in \mathcal{Z}} F_s(x, \alpha, z) \pi_\alpha(z). \tag{3}$$

Define $Q^n(t) = Q(t)(\{x_n\})$ where $\mathcal{X} = \{x_n, n \in \mathbb{N}\}$. $Q^n(t)$ is the limiting (when $N \to \infty$) proportion of particles in state x_n at time t.

Theorem 2 *Under the assumptions of Theorem 1, the limiting proportions $Q^n(t)$ of the particles in the various states satisfy: $Q^n(0) = Q_0(\{x_n\})$ and for all time $t > 0$, for all $n \in \mathbb{N}$,*

$$\frac{dQ^n}{dt} = \sum_{s \in \mathcal{S}} \sum_{m : s(x_m) = x_n} Q^m(t) \overline{F}_s(x_m, Q(t)) - Q^n(t) \overline{F}_s(x_n, Q(t)), \tag{4}$$

The differential equations (4) have the following interpretation: if $s(x_m) = x_n$ then $Q^m(t)\overline{F}_s(x_m, Q(t))$ is a mean flow of particles from state x_m to x_n.
Hence, $\sum_{s \in \mathcal{S}} \sum_{m : s(x_m) = x_n} Q^m(t)\overline{F}_s(x_m, Q(t))$, is the total mean incoming flow of particle to x_n and $\sum_{s \in \mathcal{S}} Q^n(t)\overline{F}_s(x_n, Q(t))$ is the mean outgoing flow from x_n.

2.3 Stationary regime

We now characterize the stationary behavior of the system in the mean field limit. To do so, we make two additional assumptions:

A8. For all N, the Markov chain $((X_i^N(k), Z^N(k))_{k \in \mathbb{N}}$ is positive recurrent. The set of the stationary distributions of the empirical measures $\mathcal{L}_{\rm st}(\mu^N)$ is tight.

A9. The dynamical system (4) is globally stable: there exists a measure $Q^{st} = (Q^n_{st}) \in \mathcal{P}(\mathcal{X})$ satisfying for all n:

$$\sum_{s\in\mathcal{S}} \sum_{m:s(x_m)=x_n} Q^m_{st}\overline{F}_s(x_m, Q_{st}) = Q^n_{st} \sum_{s\in\mathcal{S}} \overline{F}_s(x_n, Q_{st}), \tag{5}$$

and such that for all $Q \in \mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$ satisfying (4), for all n, $\lim_{t\to+\infty} Q^n(t) = Q^n_{st}$.

Then the asymptotic independence of the particles also holds in the stationary regime:

Theorem 3 *Under Assumptions A1-A9, for all subsets $I \subset \mathbb{N}$ of finite cardinal $|I|$,*

$$\lim_{N\to\infty} \mathcal{L}_{\mathrm{st}} \left((q_i^N(.))_{i\in I}\right) = Q_{\mathrm{st}}^{\otimes |I|} \quad \textit{weakly in } \mathcal{P}(D(\mathbb{R}^+, \mathcal{X})^{|I|}).$$

3 Proof of Theorems 1, 2 and 3

In the following, we extensively use the notation:

$$A_i^{N,s}(k) = \{s\text{-transition occurs for the particle } i \text{ between } k \text{ and } k+1\}. \tag{6}$$

By definition, we have:

$$\mathrm{P}(A_i^{N,s}(k)|\mathcal{F}_k) = \frac{1}{N} F_s^N \left(q_i^N(\frac{k}{N}), \mu^N(\frac{k}{N}), Z(k) \right).$$

We also define:

$$A_i^N(k) = \{\text{a transition occurs for particle } i \text{ between times } k \text{ and } k+1\}. \tag{7}$$

We have: $A_i^N(k) = \cup_{s\in\mathcal{S}} A_i^{N,s}(k)$.

3.1 Proof of Theorems 1 and 2

By Proposition 2.2. in Sznitman [25], Theorem 1 is equivalent to

$$\lim_{N\to\infty} \mathcal{L}(\mu^N) = \delta_Q \quad \text{weakly in } \mathcal{P}(\mathcal{P}(D(\mathbb{R}_+, \mathcal{X}))). \tag{8}$$

To establish (8), we first prove the tightness of the sequence $\mathcal{L}(\mu^N)$. We then show that any accumulation point of the previous sequence is the unique solution of a martingale problem. We finally prove the desired convergence.

3.1.1 Step 1 : Tightness

First we check that the sequence $\mathcal{L}(\mu^N)$ is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B})))$. Thanks again to Sznitman [25] Proposition 2.2, this a consequence of the tightness of $\mathcal{L}(q_1^N(.))$ in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$.

3.1.2 Step 2 : Convergence to the solution of a martingale problem

We will follow the Step 2 in Graham [18]. We show that any accumulation point of $\mathcal{L}(\mu^N)$ satisfies a certain martingale problem. For $f \in L^\infty(\mathcal{X})$, the bounded and forcibly measurable functions of $\mathcal{X} \to \mathbb{R}$. For each $s \in \mathcal{S}$, we define

$$f^s(x) = f(s(x)) - f(x).$$

Now, for $f \in L^\infty(\mathcal{X})$ and $T \geq 0$,

$$\begin{aligned} f(q_i^N(T)) \quad - \quad f(q_i^N(0)) &= \sum_{k=0}^{[NT]-1} (f(q_i^N(\frac{(k+1)}{N}) - f(q_i^N(\frac{k}{N})) \\ &= \sum_{s\in\mathcal{S}} \sum_{k=0}^{[NT]-1} f^s(q_i^N(\frac{k}{N})) \left(\mathbf{1}(A_i^{N,s}(k)) - \mathrm{P}(A_i^{N,s}(k)|\mathcal{F}_k) \right) \\ &\quad + \sum_{s\in\mathcal{S}} \sum_{k=0}^{[NT]-1} f^s(q_i^N(\frac{k}{N}))\mathrm{P}(A_i^{N,s}(k)|\mathcal{F}_k). \end{aligned} \tag{9}$$

Then we define $M_i^{f,N}(t) = \sum_{s\in\mathcal{S}} M_i^{f,N,s}(t)$ with

$$M_i^{f,N,s}(t) = \sum_{k=0}^{[Nt]-1} f^s(q_i^N(\frac{k}{N})) \left(\mathbf{1}(A_i^{N,s}(k)) - \mathrm{P}(A_i^{N,s}(k)|\mathcal{F}_k)\right) \tag{10}$$

and

$$\mathcal{G}_i^{N,s} f(k) = f^s(q_i^N(\frac{k}{N}))F_s^N\left(q_i^N(\frac{k}{N}), \mu^N(\frac{k}{N}), Z(k)\right).$$

So that, we may rewrite Equation (9) as

$$f(q_i^N(T)) - f(q_i^N(0)) \quad = \quad M_i^{f,N}(T) + \frac{1}{N}\sum_{k=0}^{[NT]-1} \sum_{s\in\mathcal{S}} \mathcal{G}_i^{N,s} f(k) \tag{11}$$

The proofs of the two following lemmas are given at the end of this section.

Lemma 1 *$M_i^{f,N}(t)$ defined at (10) is a square-integrable martingale. There exists a $C > 0$ such that the Doob-Meyer brackets $\langle M_i^{f,N}, M_j^{f,N}\rangle_t \leq Ct\|f\|_\infty^2/N$ uniformly in i, j with $i \neq j$.*

The next lemma is main technical contribution of the proof.

Lemma 2 *The martingale $M_i^{f,N}(t)$ defined at (10) satisfies*

$$M_i^{f,N}(T) = f(q_i^N(T)) - f(q_i^N(0)) - \int_0^T \mathcal{G}f(q_i^N(t), \mu^N(t))dt + \varepsilon_i^{f,N}(T), \tag{12}$$

where

$$\mathcal{G}f(x, \alpha) = \sum_{s\in\mathcal{S}} f^s(x)\overline{F}_s(x, \alpha), \tag{13}$$

and where, for all T, $\mathrm{E}|\varepsilon_i^{f,N}(T)| \leq \epsilon_N T\|f\|_\infty$ and $\lim_{N\to\infty} \epsilon_N = 0$.

Remind that $\overline{F}_s(x,\alpha)$ was defined by Equation (3).

Now assume that Lemmas 1 and 2 hold, and let $Q \in \mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$ be an arbitrary distribution in the support of an accumulation point Π^∞ of the sequence $\mathcal{L}(\mu^N)$.

Lemma 3 *Q satisfies a non-linear martingale problem starting at Q_0. Specifically, for all $f \in L^\infty(\mathcal{X})$,*

$$M^f(T) = f(X(T)) - f(X(0)) - \int_0^T \mathcal{G}f(X(t), Q(t))dt \tag{14}$$

is a Q-martingale, where $X = (X(t))_{t\geq 0}$ denotes a canonical trajectory in $D(\mathbb{R}^+, \mathcal{X})$, and $Q(0) = Q_0$, Π^∞-a.s..

Proof. The proof is similar to Step 2 of Theorem 3.4 of Graham [18] or of Theorem 4.5 of Graham and Méléard [17]. However, here our assumptions are weaker so we detail the proof.

From Lemma 7.1 in Ethier and Kurtz [14], the projection map $X \mapsto X(t)$ is Q-a.s. continuous for all t except perhaps in at most a countable subset D_Q of $\mathbb{R}_+$. Further it is shown easily that $D = \{t \in \mathbb{R}_+ : \Pi^\infty(\{Q : t \in D_Q\}) > 0\}$ is at most countable (see the argument in the proof of Theorem 4.5 of Graham and Méléard [17]).

Take $0 \leq t_1 < t_2 < \cdots t_k \leq t < T$ outside D and $g \in L^\infty(\mathcal{X}^k)$. Take $f \in L^\infty(\mathcal{X})$. The map $G : \mathcal{P}(D(\mathbb{R}^+, \mathcal{X})) \to \mathbb{R}$ defined by

$$R \mapsto \langle \left(f(X(T)) - f(X(t)) - \int_t^T \mathcal{G}f(R(u), X(u))du \right) g(X(t_1), \ldots, X(t_k)), R\rangle$$

is Π^∞-a.s. continuous. We will prove that

$$\Pi^\infty\text{-a.s}, \quad G(Q) = 0. \tag{15}$$

Now assume (15) holds for arbitrary $0 \leq t_1 < t_2 < \cdots t_k \leq t < T$ outside a countable set D and $g \in C_b(\mathcal{X}^k)$. It implies that for all $A \subset \mathcal{F}_t$, $\langle M^f(T)\mathbf{1}_A, Q\rangle = \langle M^f(t)\mathbf{1}_A, Q\rangle$. Therefore, by definition, $M^f(t)$ is a Q-martingale and Q satisfies the non-linear martingale problem (14).

It remains to prove (15). Let Π^N be the law of $\mu^N = 1/N \sum_{i=1}^N \delta_{q_i^N}$, we write :

$$\begin{aligned}
\langle |G|, \Pi^N\rangle &= \mathrm{E}|G(\frac{1}{N}\sum_{i=1}^N \delta_{q_i^N})| \\
&= \mathrm{E}\,\Big| \frac{1}{N}\sum_{i=1}^N \left(f(q_i^N(T)) - f(q_i^N(t)) - \int_t^T \mathcal{G}f(\mu^N(u), q_i^N(u))du \right) \\
&\quad \times g(q_i^N(t_1), \ldots, q_i^N(t_k)) \Big| \\
&= \mathrm{E}\,\Big| \frac{1}{N}\sum_{i=1}^N \left(M_i^{f,N}(T) - M_i^{f,N}(t) - (\varepsilon_i^{f,N}(T) - \varepsilon_i^{f,N}(t)) \right) \\
&\quad \times g(q_i^N(t_1), \ldots, q_i^N(t_k)) \Big| \\
&\leq \mathrm{E}\,\Big| \frac{1}{N}\sum_{i=1}^N (M_i^{f,N}(T) - M_i^{f,N}(t))g_i^N \Big| \\
&\quad + \mathrm{E}\,\Big| \frac{1}{N}\sum_{i=1}^N (\varepsilon_i^{f,N}(T) - \varepsilon_i^{f,N}(t))g_i^N \Big| \\
&\leq \mathrm{I} + \mathrm{II},
\end{aligned} \tag{16}$$

where $g_i^N = g(q_i^N(t_1), \ldots, q_i^N(t_k))$.

Using exchangeability and the Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned} \mathrm{I}^2 &\leq \frac{\|g\|_\infty^2}{N} \mathrm{E}\left(M_1^{f,N}(T) - M_1^{f,N}(t)\right)^2 \\ &\quad + \frac{N-1}{N} \mathrm{E}\left((M_1^{f,N}(T) - M_1^{f,N}(t)) g_1^N (M_2^{f,N}(T) - M_2^{f,N}(t)) g_2^N \right). \end{aligned}$$

Lemma 1 implies that I tends to 0. Similarly, we have:

$$\mathrm{II} \leq \|g\|_\infty \mathrm{E}|\varepsilon_1^{f,N}(T) - \varepsilon_1^{f,N}(t)|$$

Lemma 2 implies that II tends to 0. Hence, from (16) and Fatou's Lemma, $\langle |G|, \Pi^\infty \rangle \leq \lim_N \langle |G|, \Pi^N \rangle = 0$ and thus Π^∞-a.s, $G(Q) = 0$, (15) is proved.

To conclude the proof of Lemma 3, note that the continuity of $X \to X(0)$ implies $Q(0) = Q_0$, Π^∞-a.s.. □

3.1.3 Step 3 : Uniqueness of the solution of martingale problem

We now show the solution to (14) is unique. Here, we will use Proposition 2.3 in Graham [18] (which is an extension of Lemma 2.3 in Shiga and Tanaka [24]) to show uniqueness. We remark that $\mathcal{G}f(x,\alpha) = \int_{\mathcal{X}} (f(y) - f(x)) J_{x,\alpha}(dy)$ where

$$J_{x,\alpha} = \sum_{s \in \mathcal{S}} \overline{F}_s(x,\alpha) \delta_{s(x)}.$$

Next, $\|J_{x,\alpha}\| = \sum_{s\in\mathcal{S}} \overline{F}_s(x,\alpha) = 1$ and $\|J_{x,\alpha} - J_{x,\beta}\| = \sup \int_{\mathcal{X}} \varphi(y) J_{x,\alpha}(dy) - \int_{\mathcal{X}} \varphi(y) J_{x,\beta}(dy)$ where the supremum is over the functions $\varphi \in L^\infty(\mathcal{X})$ with $\|\varphi\|_\infty \leq 1$.

$$\begin{aligned} |J_{x,\alpha}(\varphi) &- J_{x,\beta}(\varphi)| = \Big| \sum_{s\in\mathcal{S}} \varphi(s(x)) \left(\overline{F}_s(x,\alpha) - \overline{F}_s(x,\beta) \right) \Big| \\ &= \Big| \sum_{s\in\mathcal{S}} \varphi(s(x)) \left(\int_{\mathcal{Z}} F_s(x,\alpha,z)\pi_\alpha(dz) - \int_{\mathcal{Z}} F_s(x,\beta,z)\pi_\beta(dz) \right) \Big| \\ &\leq \Big| \sum_{s\in\mathcal{S}} \varphi(s(x)) \left(\int_{\mathcal{Z}} F_s(x,\alpha,z)\pi_\alpha(dz) - \int_{\mathcal{Z}} F_s(x,\alpha,z)\pi_\beta(dz) \right) \Big| \\ &\quad + \Big| \sum_{s\in\mathcal{S}} \varphi(s(x)) \left(\int_{\mathcal{Z}} F_s(x,\alpha,z)\pi_\beta(dz) - \int_{\mathcal{Z}} F_s(x,\beta,z)\pi_\beta(dz) \right) \Big| \\ &\leq \mathrm{I} + \mathrm{II}. \end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned} \mathrm{I} &= \Big| \int_{\mathcal{Z}} \sum_{s\in\mathcal{S}} \varphi(s(x)) F_s(x,\alpha,z)\pi_\alpha(dz) - \int_{\mathcal{Z}} \sum_{s\in\mathcal{S}} \varphi(s(x)) F_s(x,\alpha,z)\pi_\beta(dz) \Big| \\ &\leq \|\sum_{s\in\mathcal{S}} \varphi(s(x)) F_s(x,\alpha,\cdot)\|_\infty \|\pi_\alpha - \pi_\beta\|. \end{aligned}$$

Since $|\varphi(s(x))| \leq 1$, $F_s(x,\alpha,z) \geq 0$ and $\sum_{s\in\mathcal{S}} F_s(x,\alpha,z) = 1$,

$$\|\sum_{s\in\mathcal{S}} \varphi(s(x)) F_s(x,\alpha,\cdot)\|_\infty \leq 1.$$

Thus applying Assumptions A4-A7, we deduce:

$$\mathrm{I} \leq \|\pi_\alpha - \pi_\beta\| \leq C\|\alpha - \beta\|.$$

Using Assumption A2,

$$\begin{aligned} \mathrm{II} &\leq \int_{\mathcal{Z}} \sum_{s\in\mathcal{S}} |F_s(x,\alpha,z) - F_s(x,\beta,z)| \pi_\beta(dz) \\ &\leq C\|\alpha - \beta\|. \end{aligned}$$

So finally, we have checked that:

$$\|J_{x,\alpha} - J_{x,\beta}\| \leq C\|\alpha - \beta\|.$$

We then use Proposition 2.3 in Graham [18] to establish the solution to the martingale problem (14) is unique.

3.1.4 Step 4 : Weak convergence and Evolution equation

In the three first steps we have proved that $\mathcal{L}(\mu^N)$ converges weakly to δ_Q, where Q is the unique solution of the martingale problem (14) starting at Q_0.

We can now identify the evolution equation satisfied by Q. Since Q satisfies the martingale problem then $(Q(t))_{t\geq 0}$ solves the non-linear Kolmogorov equation derived by taking the expectations in (14):

$$\langle f, Q(T)\rangle - \langle f, Q(0)\rangle = \int_0^T \langle \mathcal{G}f(\cdot, Q(t)), Q(t)\rangle dt. \tag{17}$$

Applying (17) to $f = \mathbf{1}_{x_n}$ for all n, we get the set of differential equations (4).

3.1.5 Proof of Lemma 1

First, $M_i^{f,N}(t)$ is a square-integrable martingale by the Dynkin formula. Recall that $A_i^{N,s}(k)$ is defined in Equation (6) and that $A_i^N(k) = \cup_{s\in\mathcal{S}} A_i^{N,s}(k)$. Since at time k, at most one transition occurs:

$$\mathbf{1}_{A_i^N(k)} = \sum_{s\in\mathcal{S}} \mathbf{1}_{A_i^{N,s}(k)}. \tag{18}$$

Hence, note in particular for $i \neq j$,

$$\mathrm{P}(A_i^N(k) A_j^N(k)) = \frac{1}{N^2}. \tag{19}$$

In the sequel, $\mathrm{E}_{\mathcal{F}(k)}[.]$ will denote $\mathrm{E}[.|\mathcal{F}_k]$.
With this notation, $\mathrm{E}_{\mathcal{F}(k)}[\mathbf{1}_{A_i^N(k)}] = 1/N$, and we can rewrite Equation (10) as:

$$M_i^{f,N}(t) = \sum_{k=0}^{[Nt]-1} \sum_{s\in\mathcal{S}} f^s(q_i^N(\frac{k}{N})) \left(\mathbf{1}_{A_i^{N,s}(k)} - \mathrm{E}_{\mathcal{F}(k)} \mathbf{1}_{A_i^{N,s}(k)} \right).$$

To prove Lemma 1, we need to compute $\mathrm{E}[M_1^{f,N}(t)M_2^{f,N}(t)]$.
Since $(M_i^{f,N}(t))_{t\in\mathbb{R}^+}$ is a martingale this product is equal to:

$$\begin{aligned}\mathrm{E}[M_1^{f,N}(t)M_2^{f,N}(t)] &= \sum_{k=0}^{[Nt]-1}\sum_{s,s'\in\mathcal{S}} \mathrm{E} f^s(q_1^N(\frac{k}{N}))\left(\mathbf{1}_{A_1^{N,s}(k)} - \mathrm{E}_{\mathcal{F}(k)}\mathbf{1}_{A_1^{N,s}(k)}\right) \\ &\quad \times f^{s'}(q_2^N(\frac{k}{N}))\left(\mathbf{1}_{A_2^{N,s'}(k)} - \mathrm{E}_{\mathcal{F}(k)}\mathbf{1}_{A_2^{N,s'}(k)}\right).\end{aligned}$$

Now, let

$$\begin{aligned}I_k^N &= \Big| \sum_{s,s'\in\mathcal{S}} \mathrm{E}\Big[f^s(q_1^N(\frac{k}{N}))(\mathbf{1}_{A_1^{N,s}(k)} - \mathrm{E}_{\mathcal{F}(k)}\mathbf{1}_{A_1^{N,s}(k)}) \\ &\quad \times f^{s'}(q_2^N(\frac{k}{N}))(\mathbf{1}_{A_2^{N,s'}(k)} - \mathrm{E}_{\mathcal{F}(k)}\mathbf{1}_{A_2^{N,s'}(k)})\Big]\Big|.\end{aligned}$$

We have:

$$\begin{aligned}I_k^N &\leq 4\|f\|_\infty^2\Big(\sum_{s,s'\in\mathcal{S}}\mathrm{E}\mathbf{1}_{A_1^{N,s}(k)}\mathbf{1}_{A_2^{N,s'}(k)} + 2\sum_{s,s'\in\mathcal{S}}\mathrm{E}\mathbf{1}_{A_1^{N,s}(k)}\mathrm{E}_{\mathcal{F}_k}\mathbf{1}_{A_2^{N,s'}(k)} \\ &\quad + \sum_{s,s'\in\mathcal{S}}\mathrm{E}[\mathrm{E}_{\mathcal{F}_k}\mathbf{1}_{A_1^{N,s}(k)}\mathrm{E}_{\mathcal{F}_k}\mathbf{1}_{A_2^{N,s'}(k)}]\Big) \\ &= 16\|f\|_\infty^2\mathrm{E}\mathbf{1}_{A_1^N(k)}\mathbf{1}_{A_1^N(k)},\end{aligned}$$

where we have used (18) and the independence of $A_i^N(k)$ with respect to $\mathcal{F}_k$. Then, using (19), we get $I_k^N \leq C\|f\|_\infty^2/N^2$ and the lemma follows. □

3.1.6 Proof of Lemma 2

Let

$$\kappa_N = \sup_{(\alpha,z)} \|K_\alpha^N(z,\cdot) - K_\alpha(z,\cdot)\|. \tag{20}$$

By Assumption A3, κ_N tends to 0 as $N\to\infty$.
In the sequel n_N will denote a sequence of integers satisfying:

$$\lim_{N\to\infty} n_N = \infty\ , \qquad \lim_{N\to\infty} n_N\kappa_N = 0 \quad \text{and} \quad \lim_{N\to\infty}\frac{n_N^2}{N} = 0.$$

We start by proving additional intermediary lemmas.

Lemma 4

$$\mathrm{E}\left[\sup_{0\leq t\leq \frac{n}{N}} \|\mu^N(t) - \mu^N(0)\|\right] \leq \frac{n}{N}. \tag{21}$$

Proof. The total variation distance between $\mu^N(t)$ and $\mu^N(0)$ is upper bounded by the total number of transitions between time 0 and tN, hence,

$$\sup_{0\leq t\leq\frac{n}{N}} \|\mu^N(t) - \mu^N(0)\| \leq \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{n-1}\mathbf{1}_{A_i^N(k)}.$$

Since $\mathrm{P}(A_i^N(k)) = 1/N$, we get (21). □

Lemma 5 *Let φ be a bounded measurable function from $\mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \mathcal{Z}$ to $\mathbb{R}$ such that there exists C such that for all x, z, α, β:*

$$|\varphi(x, \alpha, z) - \varphi(x, \beta, z)| \leq C\|\alpha - \beta\|.$$

Further, let $\varphi^N(x, \alpha, z)$ be a sequence of measurable functions from $\mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \mathcal{Z}$ to $\mathbb{R}$ such that for all x, α, z, $|\varphi^N(x, \alpha, z) - \varphi(x, \alpha, z)| \leq \delta_N$ with $\lim_N \delta_N = 0$, then uniformly in $(\mu^N(0), q_i^N(0))$,

$$\lim_{N\to\infty} \mathrm{E} \Big| \frac{1}{n_N} \sum_{k=0}^{n_N-1} \left(\varphi^N(q_i^N(\frac{k}{N}), \mu^N(\frac{k}{N}), Z(k)) - \varphi(q_i^N(0), \mu^N(0), Z(k)) \right) \Big| = 0.$$

Proof. If no transition occurs for the particle i between time 0 and time n, then

$$\sum_{k=0}^{n-1} \varphi(q_i^N(k/N), \mu^N(k/N), Z(k)) = \sum_{k=0}^{n-1} \varphi(q_i^N(0), \mu^N(k/N), Z(k)),$$

otherwise the difference is bounded by $2n\|\varphi\|_\infty$. Note also that

$$\begin{aligned} |\frac{1}{n}\sum_{k=0}^{n-1} \varphi(q_i^N(0), \mu^N(k/N), Z(k)) \quad &- \quad \frac{1}{n}\sum_{k=0}^{n-1} \varphi^N(q_i(0), \mu^N(0), Z(k))| \\ &\leq C \sup_{0\leq t\leq n/N} \|\mu^N(t) - \mu^N(0)\|. \end{aligned}$$

Thus we obtain that for all n,

$$\begin{aligned} &\Big| \frac{1}{n}\sum_{k=0}^{n-1} \varphi^N(q_i^N(\frac{k}{N}), \mu^N(\frac{k}{N}), Z(k)) - \frac{1}{n}\sum_{k=0}^{n-1} \varphi(q_i^N(0), \mu^N(0), Z(k)) \Big| \\ \leq \delta_N + &\Big| \frac{1}{n}\sum_{k=0}^{n-1} \varphi(q_i^N(\frac{k}{N}), \mu^N(\frac{k}{N}), Z(k)) - \frac{1}{n}\sum_{k=0}^{n-1} \varphi(q_i(0), \mu^N(0), Z(k)) \Big| \\ &\leq \delta_N + 2\|\varphi\|_\infty (1 - \prod_{k=0}^{n-1}(1 - \mathbf{1}_{A_i^N(k)})) + C \sup_{0\leq t\leq \frac{n}{N}} \|\mu^N(t) - \mu^N(0)\| \\ &\leq \delta_N + 2\|\varphi\|_\infty \sum_{k=0}^{n-1} \mathbf{1}_{A_i^N(k)} + C \sup_{0\leq t\leq \frac{n}{N}} \|\mu^N(t) - \mu^N(0)\|. \end{aligned}$$

It is easy to compute this last expression. First,

$$\mathrm{E}|\sum_{k=0}^{n-1} \mathbf{1}_{A_i^N(k)}| = \frac{n}{N}.$$

Secondly, by Lemma 4, the total variation distance between $\mu^N(t)$ and $\mu^N(0)$ is bounded: $\mathrm{E}[\sup_{0\leq t\leq n/N} \|\mu^N(t) - \mu^N(0)\|] \leq n/N$. Since n_N/N goes to 0 as N goes large, the lemma follows. □

The next Lemma is the cornerstone of the proof Lemma 2: it states that the process Z^N averages at rate n_N^{-1}, faster than N^{-1}.

Lemma 6 *Assume that $Z^N(0) = z_0$. Then the following uniform convergence in $(\mu^N(0), \psi) \in \mathcal{P}(\mathcal{X}) \times L^\infty(\mathcal{Z})$ holds:*

$$\lim_{N\to\infty} \sup_{\mu^N(0), \|\psi\|_\infty \le 1} \mathrm{E} \left| \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(Z^N(k)) - \pi_{\mu^N(0)}(\psi) \right| = 0.$$

Proof. The first step consists in coupling Z^N with a Markov chain $\tilde{Z}$ with kernel $K_{\mu^N(0)}$ on the interval $[0, n_N]$.

So consider a Markov chain $\tilde{Z}$ having kernel transition $K_{\mu^N(0)}$ and starting at $\tilde{Z}(0) = Z^N(0) = z_0$. The chain $\tilde{Z}$ is coupled with Z^N by forcing $(\tilde{Z}, Z^N)$ to make identical transitions when they are in a common state $\tilde{Z}(k) = Z^N(k) = z_1$ with probability

$$\begin{aligned} \sum_{z\in\mathcal{Z}} & \min\left(K^N_{\mu^N(\frac{k}{N})}(z_1, z), K_{\mu^N(0)}(z_1, z) \right) \\ &= 1 - \frac{1}{2} \sum_{z\in\mathcal{Z}} \left| K^N_{\mu^N(\frac{k}{N})}(z_1, z) - K_{\mu^N(0)}(z_1, z) \right| \\ &\ge 1 - \frac{\kappa_N}{2} - \frac{C}{2} \|\mu^N(\frac{k}{N}) - \mu^N(0)\|, \end{aligned}$$

(where in the last inequality we have used Assumption A4 and κ_N was defined in Equation (20)). Consequently, if $Z^N[0, n] = (Z^N(1), \cdots, Z^N(n))$, we have,

$$\begin{aligned} \mathrm{P}(Z^N[0, n] \neq \tilde{Z}[0, n]) &\le \sum_{k=1}^{n} \mathrm{P}(Z^N(k) \neq \tilde{Z}(k); Z^N(k-1) = \tilde{Z}(k-1)) \\ &\le \sum_{k=1}^{n} \frac{\kappa_N}{2} + \mathrm{E}\frac{C}{2} \|\mu^N(\frac{k}{N}) - \mu^N(0)\| \\ &\le \frac{n}{2} \left(\kappa_N + C\mathrm{E} \sup_{0\le t\le \frac{n}{N}} \|\mu^N(t) - \mu^N(0)\| \right). \end{aligned}$$

We deduce that, by Lemma 4,

$$\begin{aligned} \mathrm{E} \left| \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(Z^N(k)) \right. &- \left. \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(\tilde{Z}(k)) \right| \\ &\le 2\|\psi\|_\infty \mathrm{P}(Z^N[0, n_N] \neq \tilde{Z}[0, n_N]) \\ &\le n_N \|\psi\|_\infty \left(\kappa_N + C\mathrm{E} \sup_{0\le t\le \frac{n_N}{N}} \|\mu^N(t) - \mu^N(0)\| \right) \\ &\le \|\psi\|_\infty \left(n_N \kappa_N + C\frac{n_N^2}{N} \right) \end{aligned}$$

For our particular choice of n_N, we obtain, uniformly in $\mu^N(0)$ and $Z^N(0)$,

$$\lim_{N\to\infty} \mathrm{E} \left| \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(Z^N(k)) - \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(\tilde{Z}(k)) \right| = 0. \tag{22}$$

This last equation concludes the first step of the proof. The second step consists in proving that uniformly in $\mu^N(0)$ and $\psi \in L^\infty$, $\|\psi\|_\infty \le 1$:

$$\lim_{N\to\infty} \mathrm{E} \left| \frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi(\tilde{Z}(k)) - \pi_{\mu^N(0)}(\psi) \right| = 0. \tag{23}$$

Indeed Equations (22) and (23) implies the statement of the lemma. Now (23) follows from the ergodicity of the Markov chains with kernel K_α, i.e. Assumption A5. Indeed,

$$\begin{aligned} \mathrm{E}|\frac{1}{n}\sum_{k=0}^{n-1}\psi(\tilde{Z}(k)) - \pi_{\mu^N(0)}(\psi)| &\leq \frac{1}{n}\sum_{k=0}^{n-1}\mathrm{E}|\psi(\tilde{Z}(k)) - \pi_{\mu^N(0)}(\psi)| \\ &\leq \|\psi\|_\infty \frac{1}{n}\sum_{k=0}^{n-1}\|P^k_{\mu^N(0)}(z_0,\cdot) - \pi_{\mu^N(0)}\|. \end{aligned}$$

By Assumption A5, $\sup_{\alpha\in\mathcal{P}(\mathcal{X})}\|P^k_\alpha(z_0,\cdot)-\pi_\alpha\|$ converges to 0 as k tends to infinity. We thus deduce (23). □

Now, having proved these preliminary lemmas, we are ready to prove Lemma 2.

Proof of Lemma 2. By definition,

$$\varepsilon_i^{f,N}(T) = \frac{1}{N}\sum_{k=0}^{[NT]-1}\sum_{s\in\mathcal{S}}\mathcal{G}_i^{N,s}f(k) - \int_0^T \mathcal{G}f(q_i^N(t),\mu^N(t))dt. \tag{24}$$

Since T is fixed, we may suppose without loss of generality that n_N is chosen so that $[NT]/n_N \in \mathbb{N}$. We divide the interval $[0,[NT]-1]$ into $[NT]/n_N$ equal parts, we first prove the convergence of a Riemann approximation of the integral term in the expression of $\varepsilon_i^{N,f}(T)$. Specifically we prove that there exists C such that:

$$\mathrm{E}\left|\int_0^T \mathcal{G}f(q_i^N(t),\mu^N(t))dt - \frac{n_N}{N}\sum_{u=0}^{\frac{[NT]}{n_N}-1}\mathcal{G}f(q_i^N(\frac{un_N}{N}),\mu^N(\frac{un_N}{N}))\right| \leq CT\|f\|_\infty\frac{n_N}{N}. \tag{25}$$

Indeed, we have:

$$\begin{aligned} \int_0^T \mathcal{G}f(q_i^N(t),\mu^N(t))dt &= \sum_{u=0}^{\frac{[NT]}{n_N}-1}\int_{u\frac{n_N}{N}}^{(u+1)\frac{n_N}{N}}\mathcal{G}f(q_i^N(t),\mu^N(t))dt \\ &\quad + \int_{\frac{[NT]}{N}}^T \mathcal{G}f(q_i^N(t),\mu^N(t))dt \\ &= \frac{n_N}{N}\sum_{u=0}^{\frac{[NT]}{n_N}-1}\int_u^{u+1}\mathcal{G}f(q_i^N(\frac{tn_N}{N}),\mu^N(\frac{tn_N}{N}))dt \\ &\quad + \int_{\frac{[NT]}{N}}^T \mathcal{G}f(q_i^N(t),\mu^N(t))dt. \end{aligned}$$

Since $\sum_{s\in\mathcal{S}} \overline{F}_s(q,\alpha) = 1$, the second term is upper bounded by $2\|f\|_\infty(T - [NT]/N) \leq 2\|f\|_\infty/N$. Note also that if $t \in [u, u+1]$, arguing as in the proof of Lemma 5,

$$\begin{aligned}
&\Big| \, \mathcal{G}f(q_i^N(\frac{tn_N}{N}), \mu^N(\frac{tn_N}{N}) - \mathcal{G}f(q_i^N(\frac{un_N}{N}), \mu^N(\frac{un_N}{N}) \, \Big| \\
&\leq 2\|f\|_\infty \Big(\sum_{k=0}^{n_N} \mathbf{1}_{A_i^N(un_m+k)} \\
&\qquad + \sum_{s\in\mathcal{S}} |F_s(q_i^N(\frac{un_N}{N}), \mu^N(\frac{tn_N}{N}) - F_s(q_i^N(\frac{un_N}{N}), \mu^N(\frac{un_N}{N})| \Big) \\
&\leq 2\|f\|_\infty \Big(\sum_{k=0}^{n_N} \mathbf{1}_{A_i^N(un_m+k)} + C\|\mu^N(\frac{tn_N}{N}) - \mu^N(\frac{un_N}{N})| \Big),
\end{aligned}$$

and by Lemma 4,

$$\mathrm{E} \, \Big| \, \mathcal{G}f(q_i^N(\frac{tn_N}{N}), \mu^N(\frac{tn_N}{N}) - \mathcal{G}f(q_i^N(\frac{un_N}{N}), \mu^N(\frac{un_N}{N}) \, \Big| \leq C\|f\|_\infty \frac{n_N}{N},$$

(for a different constant C). We proceed the proof by decomposing the sum in (24) into parts of length n_N,

$$\frac{1}{N} \sum_{k=0}^{[NT]-1} \sum_{s\in\mathcal{S}} \mathcal{G}_i^{N,s} f(k) = \frac{n_N}{N} \sum_{u=0}^{\frac{[NT]}{n_N}-1} \frac{1}{n_N} \sum_{k=0}^{n_N-1} \sum_{s\in\mathcal{S}} \mathcal{G}_i^{N,s} f(un_N + k).$$

By Assumptions A1-A2, we may apply the Lemma 5 to $\varphi(x,\alpha,z) = \sum_{s\in S} f^s(x) F_s(x,\alpha,z)$ and $\varphi^N(x,\alpha,z) = \sum_{s\in S} f^s(x) F_s^N(x,\alpha,z)$, we obtain:

$$\begin{aligned}
&\mathrm{E} \, \Big| \, \frac{1}{N} \sum_{k=0}^{[NT]-1} \sum_{s\in\mathcal{S}} \mathcal{G}_i^{N,s} f(k) \\
&\qquad - \frac{n_N}{N} \sum_{u=0}^{\frac{[NT]}{n_N}-1} \frac{1}{n_N} \sum_{k=0}^{n_N-1} \varphi(q_i^N(\frac{un_N}{N}), \mu^N(\frac{un_N}{N}), Z^N(un_N + k)) \, \Big| \\
&\leq T\|f\|_\infty \delta_N,
\end{aligned} \tag{26}$$

for some sequence $(\delta_N), N \in \mathbb{N}$, with $\lim_N \delta_N = 0$.

Let $\psi_u^N(z) = \varphi(q_i^N(\frac{un_N}{N}), \mu^N(\frac{un_N}{N}), z)/\|f\|_\infty$. By Equations (24), (25), (26), it remains to prove that:

$$\lim_{N\to\infty} \mathrm{E} \, \Big| \, \frac{n_N}{NT} \sum_{u=0}^{\frac{[NT]}{n_N}-1} \Big(\frac{1}{n_N} \sum_{k=0}^{n_N-1} \psi_u^N(Z^N(un_N + k)) - \pi_{\mu^N(\frac{un_N}{N})}(\psi_u^N) \Big) \Big| = 0. \tag{27}$$

Using the notation of Assumption A6 and defining $T^N(u) = \min(\tau^N(un_N), n_N)$, we write:

$$\begin{aligned}\sum_{k=0}^{n_N-1} \psi_u^N(Z^N(un_N+k)) &= \sum_{k=0}^{T^N(u)-1} \psi_u^N(Z^N(un_N+k)) \\ &+ \sum_{k=T^N(u)}^{n_N+T^N(u)-1} \psi_u^N(Z^N(un_N+k)) \\ &- \sum_{k=n_N}^{n_N+T^N(u)-1} \psi_u^N(Z^N(un_N+k)). \end{aligned} \tag{28}$$

By Lemma 6 and the Markov property:

$$\mathrm{E}|\frac{1}{n_N}\sum_{k=T^N(u)}^{n_N+T^N(u)-1} \psi_u^N(Z^N(un_N+k)) - \pi_{\mu^N(\frac{un_N+T^N(u)}{N})}| \le \delta_N. \tag{29}$$

for some sequence (δ_N), $N \in \mathbb{N}$, with $\lim_N \delta_N = 0$ not depending on u. Hence, using Equations (28) and (29), we deduce that:

$$\begin{aligned}&\mathrm{E}\,\Big|\,\frac{1}{n_N}\sum_{k=0}^{n_N-1} \psi_u^N(Z^N(un_N+k)) - \pi_{\mu^N(\frac{un_N}{N})}(\psi_u^N)\,\Big| \\ &\le 2\|\psi\|_\infty \frac{\mathrm{E}T^N(u)}{n_N} + \delta_N + \mathrm{E}\|\pi_{\mu^N(\frac{un_N+T^N(u)}{N})}(\psi) - \pi_{\mu^N(\frac{un_N}{N})}(\psi)\| \\ &\le 4\frac{\mathrm{E}T^N(u)}{n_N} + 2\delta_N + \mathrm{E}\sup_{0\le t\le \frac{n_N}{N}} \|\pi_{\mu^N(\frac{un_N}{N}+t)} - \pi_{\mu^N(\frac{un_N}{N})}\|,\end{aligned}$$

where for the last term, we have used the fact that $T^N(u) \le n_N$ and $\|\psi\|_\infty \le 2$. Now by Assumption A7 and Lemma 4,

$$\mathrm{E}\sup_{0\le t\le \frac{n_N}{N}} \|\pi_{\mu^N(\frac{un_N}{N}+t)} - \pi_{\mu^N(\frac{un_N}{N})}\| \le C\frac{n_N}{N}.$$

By Assumption A6, $\mathrm{E}T^N(u) \le \mathrm{E}\tau^N(un_N) \le C$. It follows that for some constant C

$$\mathrm{E}\,\Big|\,\frac{1}{n_N}\sum_{k=0}^{n_N-1} \psi_u^N(Z^N(un_N+k)) - \pi_{\mu^N(\frac{un_N}{N})}(\psi_u^N)\,\Big| \le C(\frac{1}{n_N} + \delta_N + \frac{n_N}{N}).$$

Summing this last equation over all $0 \le u \le [NT]/n_N - 1$, we obtain Equation (27). This concludes the proof of Lemma 2. □

3.2 Proof of Theorem 3

Assume that $((q_i^N(0))_i, Z^N)$ represents the system of N particles in stationary regime. Then by symmetry, $(q_i^N(0))_i$ is exchangeable. Define $\Pi^N = \mathcal{L}(1/N\sum_{i=1}^N q_i^N)$. By Assumption A8, $(\Pi^N(0), Z^N(0))$ is tight, so we may consider an accumulation point $(\Pi^\infty(0), Z^\infty(0))$. If $z_1 \in \mathcal{Z}$ is in the support of $Z^\infty(0)$, $(\Pi^N(0))$ is also tight conditioned on the event $Z^N(0) = z_1$ for N

large enough. So in the sequel of proof we will assume that $Z^N(0) = z_1$ for N large enough. We cannot apply directly Theorem 1 since we do not know whether the subsequence of $\Pi^N(0)$ converges weakly toward a deterministic limit.

We now circumvent this difficulty. As in Step 1 in the proof of Theorem 1, we deduce from Sznitman [25] Proposition 2.2, that Π^N is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{X})))$. Let Q in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{X}))$ be in the support of Π^∞, an accumulation point of Π^N. We can prove similarly that Lemma 3 still holds for Q.

By Step 3 of Theorem 1, the solution of the martingale problem is unique and Q solves it with initial condition $Q(0)$. The stationarity implies that $\Pi^N(t)$ and $\Pi^N(0)$ are equal. Note also that outside a countable set D, the mapping $X \mapsto X(t)$ is continuous. So if $t \notin D$, $\Pi^\infty(t) = \Pi^\infty(0)$ and we may write $\Pi^\infty(0) = \cap_{t \notin D} \Pi^\infty(t)$. However, by Assumption A9, $\lim_{t \to +\infty} Q(t) = Q_{st}$. Therefore, $\cap_{t \geq 0} \Pi^\infty(t) = \delta_{Q_{st}}$, so $\Pi^\infty(0) = \delta_{Q_{st}}$ and $Q(0) = Q_{st}$.

Theorem 3 is then a consequence of Theorem 1.

4 A uniform domination criterion

In this section we discuss the Assumptions A1-A9 made on the particle system. Assumptions A1-A4 are natural and can be checked directly. However Assumptions A5-A7 are of a somewhat different nature since they involve finer Markov chain theory concepts: A5 corresponds to uniform rate of convergence for ergodicity, A6 to bounded mean overshoot, and A7 to the continuity of the stationary distribution with respect to the transition kernel. For Assumptions A5 and A7, a huge literature exists, see for example Fill [16] and the review paper Cho and Meyer [9]. Assumption A6 is specific to our model. The additional assumptions A8 and A9 needed to derive the mean field limit in the stationary regime may be difficult to check: A8 is a tightness assumption on the stationary measures and A9 is the global stability of a differential equation.

In this section we present a new set of assumptions, based on uniform domination of the transition kernel of the background process, that is provably sufficient to ensure that Assumptions A5-A8. The new assumptions are defined as follows:

A10 For all $\alpha \in \mathcal{P}(\mathcal{X})$ and N, the Markov chains with transition kernels K_α^N and K_α are irreducible and aperiodic.

A11 There exists a transition kernel K on $\mathcal{Z}$ which dominates the kernels K_α^N. Specifically, let $\preceq$ be a partial order on $\mathcal{Z}$, with a minimal point z_0. There exists K such that for all N, z, α,

$$K_\alpha^N(z, \cdot) \preceq_{st} K(z, \cdot),$$

where $\preceq_{st}$ is the stochastic order relation: $P \preceq_{st} P'$ if for all $z_1 \in \mathcal{Z}$: $\sum_{z \succeq z_1} P(z) \leq \sum_{z \succeq z_1} P'(z)$.

A12 The kernel K is monotonic in the sense that for all $z_1 \preceq z_2$,

$$K(z_1, \cdot) \preceq_{st} K(z_2, \cdot).$$

A13 If $Z(t)$ denotes a Markov chain with transition kernel K and $\tau = \inf\{t \geq 1 : Z(t) = z_0\}$,

$$\mathrm{E}_{z_0} \tau^2 < \infty,$$

where E_{z_0} denotes the expectation conditioned on the event $Z(0) = z_0$.

Lemma 7 *Under Assumptions A10-A13, A5 holds. Moreover the same result holds for the chains* K_α^N.

Proof. Let $\tilde{Z}$ be a Markov chain with transition kernel K_α (or K_α^N). Let $\tilde{\tau} = \inf\{k \geq 1 : \tilde{Z}(k) = z_0\}$. Since $K_\alpha(z,\cdot) \preceq_{st} K(z,\cdot)$ and $K(z_1,\cdot) \preceq_{st} K(z_2,\cdot)$ for all $z_1 \preceq z_2$, from Theorem 5.2.11 in [23] and Strassen's Theorem, we deduce that there exists a Markov Chain Z with transition kernel K such that, $Z(0) = \tilde{Z}(0)$ and $\tilde{Z}(t) \preceq Z(t)$ for all $t \geq 0$. In particular, this implies that $\tilde{\tau} \leq \tau$, hence by assumption:

$$\mathrm{E}_{z_0}\tilde{\tau} \leq \mathrm{E}_{z_0}\tau = C.$$

Since K_α is aperiodic and irreducible, we deduce immediately that $\tilde{Z}$ is an ergodic Markov chain (see for example Theorem 13.0.1 in Meyn and Tweedie [22]). Uniformity is granted by the fact that C does not depend on α (or N). □

Lemma 8 *Under Assumptions A10-A13, A6 holds.*

Proof. Since $K_\alpha^N(z,\cdot) \preceq_{st} K$ and $K(z_1,\cdot) \preceq_{st} K(z_2,\cdot)$ for all $z_1 \preceq z_2$, from Strassen's Theorem, there exists a Markov Chain Z with transition kernel K such that, $Z(0) = Z^N(0) = z_1$ and $Z^N(k) \preceq Z(k)$ for all $k \geq 0$. Let

$$\tau(t) = \inf\{k \geq 0 : Z(t+k) = z_0\}.$$

$Z^N(k) \preceq Z(k)$ implies that $\tau^N(t) \leq \tau(t)$, indeed, z_0 is the minimal element of $\preceq$ so that $Z(k) = z_0$ implies $Z^N(k) = z_0$. The Renewal Theorem and the assumption $\mathrm{E}_{z_0}\tau^2 < \infty$, implies classically (see Feller [15]) uniformly in t.

$$\mathrm{E}\tau(t) < \infty,$$

indeed, $\lim_{t\to\infty} \mathrm{E}\tau(t) = \mathrm{E}_{z_0}\tau(\tau+1)/2\mathrm{E}_{z_0}\tau$. □

The next lemma establishes the continuity of the mapping $\alpha \mapsto \pi_\alpha$.

Lemma 9 *Let* $\|K_\alpha - K_\beta\|_1 = \sup_{z\in\mathcal{Z}} \|K_\alpha(z,\cdot) - K_\beta(z,\cdot)\|$. *Under Assumptions A10-A13, A7 holds:*

$$\|\pi_\alpha - \pi_\beta\| \leq \frac{\mathrm{E}_{z_0}\tau^2}{\mathrm{E}_{z_0}\tau}\|K_\alpha - K_\beta\|_1. \tag{30}$$

Proof. Let Z_α be a Markov chain with transition kernel K_α with initial condition $Z_\alpha(0) = z_0$. We consider a Markov chain Z_β having kernel transition K_β and starting at $Z_\beta(0) = Z_\alpha(0) = z_0$. The chain Z_β is coupled with Z_α by forcing (Z_α, Z_β) to make identical transitions when they are in a common state $Z_\alpha(k) = Z_\beta(k) = z_1$ with probability

$$\begin{aligned}\sum_{z\in\mathcal{Z}} \min\big(\, K_\alpha(z_1,z), K_\beta(z_1,z)\,\big) &= 1 - \frac{1}{2}\sum_{z\in\mathcal{Z}} \big|\, K_\alpha(z_1,z) - K_\beta(z_1,z)\,\big| \\ &\geq 1 - \|K_\alpha - K_\beta\|_1,\end{aligned}$$

Consequently, if Let $Z_\alpha[0,n] = (Z_\alpha(0),\cdots,Z_\alpha(n))$, we have,

$$\begin{aligned}\mathrm{P}(Z_\alpha[0,n] \neq Z_\beta[0,n]) &\leq \sum_{k=1}^{n} \mathrm{P}(Z_\alpha(k) \neq Z_\beta(k); Z_\alpha(k-1) = Z_\beta(k-1)) \\ &\leq \sum_{k=1}^{n} \|K_\alpha - K_\beta\|_1 \\ &\leq n\|K_\alpha - K_\beta\|_1, \end{aligned} \tag{31}$$

Now, since $K_\alpha(z,\cdot) \preceq_{st} K$ (resp. $K_\beta(z,\cdot) \preceq_{st} K$) and $K(z_1,\cdot) \preceq_{st} K(z_2,\cdot)$ for all $z_1 \preceq z_2$, from Strassen's Theorem, there exists a Markov Chain Z with transition kernel K such that, $Z(0) = z_0$, $Z_\alpha(k) \preceq Z(k)$ for all $k \geq 0$ (resp. $Z_\beta(k) \preceq Z(k)$). If $\tau = \inf\{k \geq 1 : Z(k) = z_0\}$, since $Z_\alpha(\tau) = Z_\beta(\tau) = z_0$, we have:

$$\pi_\alpha(z) = \frac{\mathrm{E}_{z_0} \sum_{k=1}^{\tau} \mathbf{1}_{Z_\alpha(k)=z}}{\mathrm{E}_{z_0}\tau},$$

and respectively for β. Hence:

$$\begin{aligned}
\|\pi_\alpha - \pi_\beta\| &= \frac{1}{2}\sum_z |\pi_\alpha(z) - \pi_\beta(z)| \\
&\leq \frac{1}{2\mathrm{E}_{z_0}\tau}\mathrm{E}_{z_0}\sum_{k=1}^{\tau}\sum_z |\mathbf{1}_{Z_\alpha(k)=z} - \mathbf{1}_{Z_\beta(k)=z}| \\
&\leq \frac{1}{\mathrm{E}_{z_0}\tau}\mathrm{E}_{z_0}\sum_{k=1}^{\tau}\mathbf{1}_{Z_\alpha(k)\neq Z_\beta(k)} \\
&\leq \frac{1}{\mathrm{E}_{z_0}\tau}\mathrm{E}_{z_0}\tau\mathbf{1}_{Z_\alpha[0,\tau]\neq Z_\beta[0,\tau]}.
\end{aligned}$$

Then we may now use (31) to $n = \tau$ (indeed the bound $n\|K_\alpha - K_\beta\|_1$ does not depend of the actual states visited by the chains). We thus obtain (30). □

5 Application to random multi-access protocols

We now apply the previous analytical results to model communication networks where N users compete for the use of a common resource, a channel, to transmit data packets. We assume that users are saturated in the sense that they always have packets to transmit. They interact through interference: if two interfering users attempt to use the channel simultaneously, there is a collision, none of the transmissions is successful and the corresponding packets need to be retransmitted.

Users are assumed to share the channel in a distributed manner, meaning that they do not know when the other users may attempt to use the channel, and actually do not know that they have to share the channel with some other interfering users. This distributed architecture has played a very crucial role in the development of Local Area Networks, and has then allowed the rapid growth of the Internet. Indeed, this simplicity has ensured the scalability of networks, meaning that a new user can join the network without the need of explicitly advertising his presence. Today this distributed architecture is used in all wired and wireless LANs.

To comply with this architecture, each user has to decide when to attempt to use the channel independently of the behavior of other users. Various random multi-access algorithms were developed to achieve this. The first was Abramson's Aloha algorithm [3]. More recent algorithms are used in the IEEE802.3 Ethernet-based wired LANs [1] and in the IEEE802.11-based wireless LANs [2]. In these systems, time is divided into slots, whose duration will be taken as the unit of time in the following. For simplicity, the durations of transmissions of all packets are geometrically distributed with average equal to L slots. This assumption is not crucial, but simplifies the exposition of our theoretical model. Similarly the durations of collisions are geometrically distributed with average equal to L_c slots. The duration of collisions may be much smaller than that of successful transmissions. This is for example the case when an explicit signaling procedure allows users to determine before the end of the transmission whether the latter will be successful or not.

Usual multi-access protocols are based on back-off algorithms: when a user has a packet to transmit, it randomly chooses a back-off timer expressed in slots. The back-off timer is then decremented at the end of each slot when the user observes an idle channel, until the back-off timer reaches 0, at which time, the user attempts to send the packet. To simplify the notation, we assume that all timers are geometrically distributed, so that we can represent the state of a user by the probability it attempts to use the channel at the end of an idle slot. This probability p takes values in an at most countable set $\mathcal{B}$, and evolves as follows. In case of successful transmission, it becomes $S(p)$, and in case of collision $C(p)$, where $S : \mathcal{B} \to \mathcal{B}$ (resp. $C : \mathcal{B} \to \mathcal{B}$) is an increasing (resp. decreasing) function. In the following we denote by p_0 the supremum of $\mathcal{B}$.

The performance of networks where users run random back-off algorithms is largely unknown. This is due to the fact that the inherent interactions between users have proven to be extremely difficult to analyze. A very popular approach to circumvent this difficulty consists in decoupling the users, i.e., assuming that the (re)-transmission processes of the various users are mutually independent. This approach has been used by Bianchi [6] to capture the performance of wireless LANs with full interaction, i.e., where all users interfere with each other. In the remainder of this section we aim at justifying this approach theoretically and at explaining how to extend it to the case of networks with partial interaction, i.e., where users do not interfere with all the other users. To do so, we will apply the mean field analysis derived in the first part of the paper. In case of full interaction, the network can be modeled as a simple system of particles with no randomly varying environment [8]. However, to analyze a network with partial interactions, the introduction of this varying environment is necessary. As we will observe, the environment is going to represent the spatial *macroscopic* state of the network and have similar dynamics as those of loss networks analyzed by Kelly [21].

The remainder of this section is organized as follows. We first classify various types of random multi-access algorithms. We then present the network model and the performance metrics we are interested in. We give the performance of networks where users run non-adaptive multi-access algorithms. To analyze these networks, we do not need to represent the system as a particle system. However this analysis is useful to build and understand the behavior of the randomly varying environment, which will be useful in the performance evaluation of networks where users run adaptive multi-access algorithms. Finally we give a numerical example illustrating the problem of fairness arising in networks with partial interactions.

5.1 Random multi-access algorithms

Random multi-access algorithms are usually categorized according to their ability to adapt to the number of sources competing for the use of the channel.

- *Non-adaptive algorithms.* The first random multi-access algorithm, Aloha, was introduced by Abramson [3]. It is a non-adaptive algorithm where basically, each user transmits with a constant probability. In other words, in the above model, S and C are identical and constant functions. The problem with such an algorithm is that it does not adapt to the population of users, and as the latter grows large, the network throughput decreases and ultimately reaches 0.

- *Adaptive algorithms.* Adaptive random algorithms are more suited to the case where the environment of a user may change, which is basically always the case in practical networks.

These algorithms are designed so that a user can *discover* its environment, i.e., the number of active users competing for the use of its channel. In fact, all implemented multi-access algorithms are adaptive. They are often based on the so-called *exponential back-off algorithm*, where the transmission probability becomes p_0 after a successful transmission and divided by a factor 2 in case of collision: $S(p) = p_0$ and $C(p) = p/2$. In that case, $\mathcal{B} = \{p_0 2^{-n}, n \in \mathbb{N}\}$. The Ethernet (in wired LANs) and the IEEE802.11 DCF (in wireless LANs) protocols implement this algorithm.

5.2 Interference model and performance metrics

5.2.1 Interference model

The N users interact through interference depending on their geographical locations. The interference model is kept simple here: we assume that no transmission initiated by user i can be successful if one of the interfering users is transmitting at the same time (this model is often referred to as the *exclusion model* in the literature). Without loss of generality, users are classified according to their interference properties, i.e., two users belong to the same class if they interfere with (resp. are interfered with and by) the same set of users. Denote by $\mathcal{C}$ the set of user classes, and by μ_c the proportion of users of class c. $i \in c$ denotes the fact that user i is of class c. Interference between classes is characterized by the matrix A such that $A_{cd} = 1$ if class-c users interfere class-d links, and $A_{cd} = 0$ otherwise. Note that the matrix A is not necessarily symmetric as illustrated in example 2 below.

The above model is very general. It may be used to model mesh or ad-hoc networks. However, it may also be very useful to represent wireless LANs with access points.

Example 1: Networks with access points and overlapping cells.
Consider the uplink of a wireless LANs with two access points (APs) sharing the same channel (which means that transmissions go from users to the access points and users attach to different access points). The network is divided into 3 geographical zones: Zones 1 and 3 where users can only transmit to a single AP and do not interfere with each other; Zone 2 where users may transmit to both APs and interfere any other transmission. The network is illustrated in Figure 1(a). Links are from users to the APs. The class of a link is just the zone where the corresponding user is located. The network is clearly symmetric here.

Example 2: A simple asymmetric mesh network.
Multi-hop wireless networks are naturally asymmetric. An example is presented in Figure 1(b). Here a user is identified by the corresponding link, i.e., the corresponding transmitter and receiver. As illustrated, the receiver of link 1 is in the range of the transmitter of link 3 so link 3 interferes with link 1. However, the receiver of link 3 is not in the range of the transmitter of link 1, i.e., link 1 does not interfere with link 3.

In the following, we restrict our attention to symmetric networks although the analysis can be readily generalized to the case of asymmetric networks. We denote by $\mathcal{V}_c = \{d \in \mathcal{C} : A_{cd} = 1\}$ the set of classes of links interfering with class-c links.

5.2.2 Performance metrics

The performance of the system is measured in terms of the long-term throughputs realized by the various users. The aim of our analysis is to quantify these throughputs. Since users of the same class experience the same performance, we will only compute the global throughput of

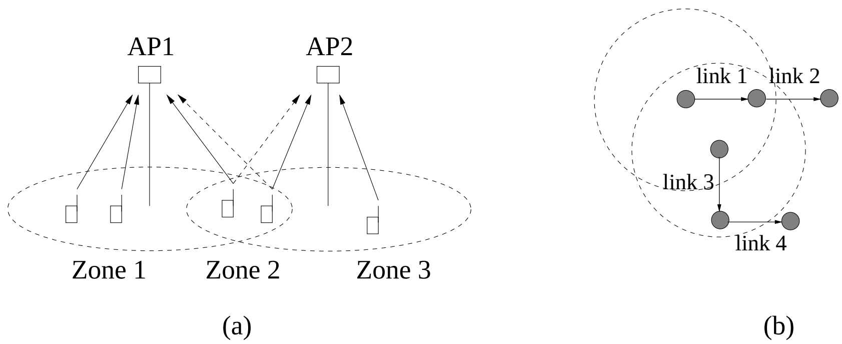

Figure 1: (a) A symmetric cellular network, two overlapping cells; (b) An asymmetric multi-hop network.

users of the different classes. We denote by γ_c the throughput of class-c links. It is by definition the proportion of time users of class c transmit successfully.

The system considered here may have natural spatial heterogeneity as some users may suffer interference from more users than others. This may create substantial unfairness in terms of user throughput. We will then be interested in quantifying the basic trade-off between the total network throughput and fairness realized by the random access algorithms considered.

5.3 Performance analysis of non-adaptive algorithms

In this section, we analyze the performance of networks under a non-adaptive multi-access algorithm. The analysis is simple since the system can be modeled as a Kelly loss network [20, 21] for which the stationary distribution is explicit. Kelly's model has been recently revisited by Durvy-Thiran [13] and applied in the context of wireless networks with partial interaction. In the following, we basically quote the results of [13], but give a different presentation that will be useful when evaluating the performance of adaptive algorithms. The user i attempts to use the channel at the end of an empty slot with constant probability p_i. To simplify, we assume that the average durations of transmission and collisions are identical ($L = L_c$); this assumption can be readily removed.

To partially capture the network dynamics, we define a process $Z = \{Z(k), k \geq 0\}$ representing the classes for which there exists at least one transmitting user during slot k: $Z(k) \in \mathcal{Z} = \{0,1\}^{|\mathcal{C}|}$, where $Z_c(k) = 0$ if and only if there is no transmitting user of class c. Note that if $d \in \mathcal{V}_c$ and $Z_c(k) = 1 = Z_d(k)$, there is necessarily a collision between users of classes c, d. Given that $Z_c(k) = 1$, and that for all $d \in \mathcal{V}_c$, $Z_d(k) = 0$, there is either a successful transmission or a collision involving users of class c only. We also introduce the *clear-to-send* functions C_c as follows. If $Z(k) = z$, a class-c link is clear to send at the end of slot k if $C_c(z) = 1$ where $C_c(z) = 1$ if $z_d = 0$ for $d \in \mathcal{V}_c$, otherwise $C_c(z) = 0$.

The evolution of Z is driven by a process describing the number of clients in a certain loss network. The loss network considered is composed of a set $\mathcal{R}$ of routes, where a route r can be any set of consecutive classes. Two classes c, d are said to be consecutive if the corresponding users interfere, i.e., $A_{cd} = 1$. We also introduce the process $n = \{n(k), k \geq 0\}$, where $n(k) = (n_r(k), r \in \mathcal{R})$ and $n_r(k)$ is the number of clients on route r during slot k. This number can be 0 or 1, and the possible states of the loss network are included in

$$\mathcal{Y} = \{n : \forall c \in \mathcal{C}, \sum_{r:c\in r} n_r \leq 1\}.$$

The relationship between the processes Z and n is that for all $c \in \mathcal{C}$, $Z_c(k) = \sum_{r:c\in r} n_r(k)$. More precisely, for all z there exists a unique corresponding state n^z for the process n: for $c \in \mathcal{C}$, if $z_c = 1$, then $n^z_r = 1$ if r denotes the largest set of consecutive classes containing c. Recall that we assume that the durations of the transmissions and of the collisions are geometrically distributed. Then the process n is a Markov chain with the following transition rates. Assume the network is in state n (or equivalently that the process Z is in state z):

- Arrivals: a connection on route r is generated with probability λ_r at time k if for all $c \in r$ and all $d \in \mathcal{V}_c$, $\sum_{r:d\in r} n_r(k) = 0$ (or equivalently if $C_c(z) = 1$), where:

$$\lambda_r = \prod_{c\in r}(1 - \prod_{i\in c}(1 - p_i)). \tag{32}$$

- Departures: a connection on route r leaves the network with probability $1/L$ at the end of each slot.

As a consequence, the process n is reversible [20], and its stationary distribution depends on the distributions of the durations of transmissions/collisions. This insensitivity allows us to relax the assumptions that these durations are geometrically distributed. The respective stationary distributions of processes n and Z are given by:

$$\xi(n) = \frac{\prod_{r\in\mathcal{R}}(\lambda_r L)^{n_r}}{\sum_{m\in\mathcal{Y}}\prod_{r\in\mathcal{R}}(\lambda_r L)^{m_r}}, \quad \pi(z) = \xi(n^z).$$

From the latter distribution, we can deduce the mean throughput γ_c obtained by users of class c. Consider the point process of returns to the set $\mathcal{A} = \{z : C_c(z) = 1\}$. Let T_1 denote the first return time after time zero. By the cycle formula (see (1.3.2) in [5]) we may express the steady state probability of a user in c successfully transmitting a packet by the mean time spent in the transmission state per cycle divided by the mean cycle length. The expectation is calculated with respect to the Palm measure of the point process of returns to $\mathcal{A}$ but in this Markovian case this just means starting on $\mathcal{A}$ with probability $\pi^{\mathcal{A}}$ which is π renormalized to be a probability on $\mathcal{A}$.

A user in c can only go into a successful transmission state once per cycle; i.e. no other user in c transmits and other users in $\mathcal{V}_c$ are either blocked or remain silent. Hence the mean time per cycle spent in a transmission state is $\sum_{z\in\mathcal{A}} \pi^{\mathcal{A}}(z)Lg(z)$ where

$$g(z) = \sum_{i\in c} p_i \prod_{j\in c, j\neq i}(1 - p_j) \prod_{d\in\mathcal{V}_c, d\neq c}(C_d(z)(\prod_{j\in d}(1 - p_j) - 1) + 1).$$

Moreover, $\sum_{z\in\mathcal{A}} \pi^{\mathcal{A}}(z)E_z[T_1] = \frac{1}{\pi(A)}$; i.e. the intensity of the point process of visits to $\mathcal{A}$. Finally the throughput is given by:

$$\gamma_c = \sum_{z:C_c(z)=1} \pi(z)L\left(\sum_{i\in c}\frac{p_i}{1-p_i}\prod_{d\in\mathcal{V}_c}\left(C_d(z)(\prod_{j\in d}(1-p_j) - 1) + 1\right)\right). \tag{33}$$

5.4 Performance analysis of adaptive algorithms

We now extend the results to networks where users run adaptive multi-access algorithm. We analyze the system at the beginning of each slot. Denote by $p_i^N(k)/N$ the probability user

i becomes active at the end of the k-th slot, if idle (note that we already renormalized this probability by $1/N$ to be able to conduct the asymptotic analysis when N grows large). For all i, k, N, $p_i^N(k) \in \mathcal{B}$. For simplicity, we assume $L = L_c$ (this assumption can be relaxed by extending the state space of the process Z^N).

5.4.1 Model Analysis

We first show how to model the network as a set of interacting particles as described in Section 2.

- The particles: the i-th user corresponds to the i-th particle with state describing the class of the user and the transmission probability at the end of the next idle slot $X_i^N(k) = (c_i, p_i^N(k)) \in \mathcal{X} = \mathcal{C} \times \mathcal{B}$.
- The background process: it represents the classes for which there exists at least one active user: $Z^N(k) \in \mathcal{Z} = \{0,1\}^{|\mathcal{C}|}$, where $Z_c^N(k) = 0$ if and only if there is no active user of class c. As in case of non-adaptive multi-access algorithms (see Section 5.3), we introduce the loss network process $n^N(k)$, and the clear to send functions C_c.

Particle transitions We first compute the transition probabilities for the various particles. The set $\mathcal{S}$ of possible transitions is composed by two functions, the first one representing a successful transmission $p \mapsto S(p)$ and the other one collisions $p \mapsto C(p)$. Note that the class of a particle / user does not change. Let $\nu_c^N(k) = \frac{1}{N}\sum_{i=1}^N \delta_{p_i^N(k)} \mathbf{1}_{c(i)=c}$ and $\nu^N(k) = (\nu_c^N(k))_{c\in\mathcal{C}}$.

Assume that at some slot k, the system is in state

$$((c_i^N(k), p_i^N(k))_{i=1,\dots N}, \nu^N(k), Z^N(k))) = ((c_i, p_i)_{i=1,\dots,N}, \alpha, z).$$

A class-c user i may have a transition at the end of slot k only if $C_c(z) = 1$. In this case it can either initiate a successful transmission or experience a collision. If $C_c(z) = 1$, the event that none of the users in c transmits at the end of slot k is given by $D_c^N = \prod_{i\in c} 1_{(NU_i > p_i)}$, where the U_i's are i.i.d. r.v. uniformly distributed on $[0,1]$. The event that user $i \in c$ accesses the channel with success at the end of slot k is given by the indicator:

$$1_{NU_i \le p_i} C_c(z) \prod_{j\in c, j\neq i} (1_{NU_j > p_j}) \prod_{d\in\mathcal{V}_c, d\neq c} \left(\mathbf{1}_{C_d(z)=1} D_d^N + \mathbf{1}_{C_d(z)=0} \right).$$

Averaging the above quantity gives the transition probability $F_S^N((c,p_i),\alpha,z)/N$ corresponding to a successful transmission. For all $\alpha \in \mathcal{P}(\mathcal{B})$ and all f $\mathcal{B}$-valued function, define $\langle f, \alpha\rangle = \sum_p f(p)\alpha(p)$. Moreover let α_c denote the restriction of α to users of class c. Let I denote the identity function. One can readily see that we have:

$$F_S^N((c,p_i),\alpha,z) = \frac{p_i}{1-p_i/N} C_c(z) \prod_{d\in\mathcal{V}_c} \left(C_d(z)(e^{\langle N\log(1-\frac{I}{N}),\alpha_d\rangle} - 1) + 1 \right) \tag{34}$$

Similarly, the event that user $i \in c$ experiences a collision at the end of slot k is given by the indicator:

$$1_{NU_i \le p_i} C_c(z) \left(1 - \prod_{j\in c, j\neq i} (1_{NU_j > p_j}) \prod_{d\in\mathcal{V}_c, d\neq c} \left(\mathbf{1}_{C_d(z)=1} D_d^N + \mathbf{1}_{C_d(z)=0} \right) \right),$$

and the transition probability $F_C^N((c,p_i),\alpha,z)/N$ corresponding to a collision reads:

$$F_C^N((c,p_i),\alpha,z) = p_i C_c(z)\left(1 - \frac{1}{1-p_i/N}\prod_{d\in\mathcal{V}_c}\Big(C_d(z)(e^{\langle N\log(1-\frac{I}{N}),\alpha_d\rangle} - 1) + 1 \Big)\right). \quad (35)$$

In order to fit into the scheme to the particle system of Section 2, we need to introduce a virtual transition from (c,p) to (c,p) with transition rate $F_\emptyset^N((c,p_i),\alpha,z) = 1 - p_i C_c(z)$. With this virtual transition the sum of the transition rates sums to 1. Since $N\log(1-x/N)$ converges to $-x$, we obtain the following expressions for the asymptotic transition rates, $F_\emptyset((c,p_i),\alpha,z) = 1 - p_i C_c(z)$,

$$F_S((c,p_i),\alpha,z) = p_i C_c(z)\prod_{d\in\mathcal{V}_c}\Big(C_d(z)(e^{-\langle I,\alpha_d\rangle} - 1) + 1 \Big), \quad (36)$$

$$F_C((c,p_i),\alpha,z) = p_i C_c(z)\left(1 - \prod_{d\in\mathcal{V}_c}\Big(C_d(z)(e^{-\langle I,\alpha_d\rangle} - 1) + 1 \Big)\right). \quad (37)$$

The convergence of F_S^N (resp. F_C^N) to F_s (resp. F_C)is uniform in α and z, so that Assumption A1 is satisfied.It is also easy to check that the functions F_S and F_C are uniformly Lipschitz, which ensures Assumption A2.

Transitions of the background process The kernel of process Z^N is determined by that of the process n^N. The latter is then obtained precisely as in Section 5.3. Assume that the system is in state $((c_i,p_i)_{i=1,\dots N},\alpha,z)$. An arrival on route r may occur only if $C_d(z) = 1$, for all $d \in \mathcal{V}_c$, in which case it occurs with probability:

$$\lambda_r^N(\alpha,z) = \prod_{c\in r} C_c(z)(1 - \prod_{i\in c}(1 - \frac{p_i}{N})) = \prod_{c\in r} C_c(z)(1 - e^{\langle N\log(1-\frac{I}{N}),\alpha_c\rangle}).$$

A connection on route r leaves the network with probability $1/L$ at the end of each slot. The limit kernel of Z^N is obtained considering the limit kernel of n^N, defined as follows. The limit arrival rate on route r is:

$$\lambda_r(\alpha,z) = \prod_{c\in r} C_c(z)(1 - e^{-\langle I,\alpha_c\rangle}).$$

The Assumptions A3 and A4 can then be easily verified.

Mean field asymptotics We now verify that Assumptions A10-A13 are satisfied, implying that Assumptions A5-A7 also hold. A10 is straightforward. Now let us build a transition kernel K, corresponding to a process Z with values in $\mathcal{Z} = \{0,1\}^{|\mathcal{C}|}$. When equal to 0, a component of Z almost surely becomes 1 at the next slot, and whatever the state of the system is. The transition probabilities from 1 to 0 of the various components are those corresponding to the kernel K_α^N. The kernel K then corresponds to a system where there are always users of each class attempting to use the channel at each slot. One can easily verify that Assumption A11-A13 are satisfied for this kernel K, for the usual partial order $\preceq$ on $\mathcal{Z}$ and for $z_0 = 0$. For instance, A13 holds because there is a positive probability that all the components of Z jump to zero simultaneously; i.e. the probability of jumping to z_0 is bounded away from zero uniformly over all states. The return time to z_0 is therefore exponential and consequentially has a finite second moment.

We rescale time and define $q_i^N(t) = p_i^N([Nt])$. Since the set of transitions is finite, the tightness of $\mathcal{L}(q_1^N(\cdot))$ follows easily from Theorem 7.2 in Ethier-Kurtz [14] p 128. (see the comment after Theorem 1). It follows that Theorem 1 applies. Assume that the class of the particle i is a r.v. fixed at the time 0 such that the vector $(c_1, \cdots, c_N)$ is an exchangeable random vector (for example the c_i's may be i.i.d. and equal to c with probability μ_c). Theorem 1 asserts that as $N \to \infty$, the q_i^N's become independent and evolve according to a measure $Q = (Q(t))_{t \in \mathbb{R}^+}$.

5.4.2 Stationary throughputs

Assume that Assumptions A8-A9 hold, so that Theorem 3 applies. These assumptions will be partly justified below for the case of the exponential back-off algorithm. We are interested in deriving the stationary throughputs achieved by users of various classes. To do so, we derive the stationary distribution Q_{st} and $\pi_{Q_{st}}$ of the particles and the background process. To simplify the notation we write $Q_{st} = Q$ and $\pi_Q = \pi$. Also denote $Q_c^p = Q(\{c, p\})$ the stationary proportion of users of class c transmitting with probability p.

Following the same reasoning leading to (33) in Section 5.3, the total throughput of the users of class c is

$$\gamma_c = \sum_{z: C_c(z)=1} \pi(z) L \rho_c \prod_{d \in \mathcal{V}_c} \left(C_d(z)(e^{-\rho_d} - 1) + 1 \right), \tag{38}$$

where

$$\rho_c = \sum_{p \in \mathcal{B}} p Q_c^p, \tag{39}$$

which can be interpreted as the probability that a user of class c attempts to use the channel at the end of an empty slot. We now evaluate Q and π. Note that π depends on Q through the ρ_c's only, because the limit kernel of process n or Z involves terms like $\lambda_r(Q, z) = \prod_{c \in r} C_c(z)(1 - e^{-\rho_c})$. π is given by: for all $z \in \mathcal{Z}$,

$$\pi(z) = \pi(0) L^{r(z)} \prod_{c \in \mathcal{C}} (1 - e^{-\rho_c})^{z_c}, \tag{40}$$

where $r(z)$ denotes the number of active routes for the loss process n in state n^z.

Now define G_c, H_c and I_c as follows:

$$G_c = \sum_z \pi(z) C_c(z) \prod_{d \in \mathcal{V}_c} \Big(C_d(z)(e^{-\rho_d} - 1) + 1 \Big), \tag{41}$$

$$H_c = \sum_z \pi(z) C_c(z) \left(1 - \prod_{d \in \mathcal{V}_c} \Big(C_d(z)(e^{-\rho_d} - 1) + 1 \Big) \right), \tag{42}$$

$$I_c = G_c + H_c = \sum_z \pi(z) C_c(z). \tag{43}$$

G_c, H_C, I_c depend on Q through the ρ_c's only. We have for all c, p: $pG_c = \overline{F}_S((c,p), Q)$, $pH_c = \overline{F}_C((p,c), Q)$. The marginals Q_c^p satisfy the balance equations (5), i.e., for all c, p,

$$G_c \left(\sum_{p' \in \mathcal{B}: S(p')=p} p' Q_c^{p'} - p Q_c^p \right) + H_c \left(\sum_{p' \in \mathcal{B}: C(p')=p} p' Q_c^{p'} - p Q_c^p \right) = 0. \tag{44}$$

They also satisfy:

$$\forall c \in \mathcal{C}, \quad \sum_{p \in \mathcal{B}} Q_c^p = \mu_c. \tag{45}$$

Summarizing the above analysis, we have:

Theorem 4 *The stationary distribution Q is characterized by the set of equations (39), (40), (41), (42), (44), (45).*

5.4.3 The exponential back-off algorithm

We now examine the specific case of the exponential back-off algorithm. We first justify Assumption A8.

Lemma 10 *In case of the exponential back-off algorithm, there exists a $p^* > 0$, such that for any $0 < p_0 < p^*$, the Markov process $(X_i^N(k), Z^N(k))_{k \in \mathbb{N}}$ is positive recurrent for all N and the family of stationary distributions $\mathcal{L}_{st}(X_1^N(0))$ is tight.*

Deriving a tight bound for p^* would involve technical details which are beyond the scope of this paper. We will only sketch the main idea and prove $p^* > 0$. Along the proof of Lemma 10, we may check that the statement of Lemma 10 holds for $p^* = \frac{\ln 2}{L\overline{\mu}}$, where $\overline{\mu} = \max_{c \in \mathcal{C}} \overline{\mu}_c$ and $\overline{\mu}_c = \sum_{d \in \mathcal{V}_c} \mu_d$ is the mean proportion of particles which are in interaction with particles of class c.

Proof. To prove the recurrence we introduce a fictive system which stochastically bounds $p_1^N(k)$.

In the fictive system, the states of the particles $i \geq 2$ are independent, a particle $i \geq 2$ has two states: active or inactive. If the particle $i \geq 2$, is active, it remains active for the next slot with probability $1 - 1/L$, if it is inactive, it becomes active with probability p_0/N. The stationary probability that the particle i is active is $L/(L+N/p_0)$ and the stationary probability that at least one is active is $a_N = 1-(1-L/(L+N/p_0))^{N-1}$ which converges to $a = 1 - e^{-Lp_0}$.

The particle 1 tries to become active at slot k with probability $p_1^N(k)/N$. If it remains inactive, $p_1^N(k) = p_1^N(k+1)$. If it is active and if another particle is also active, then the particle 1 encounters a collision and $p_1^N(k+1) = p_1^N(k)/2$. Otherwise $p_1^N(k+1) = p_0$.

Clearly, this virtual system is stochastically less than or equal to $p_1^N(k)$ in the exponential back-off case.

Let $b^N(k) = p_0/p_1^N(k)$, $b^N(k) \in \{2^n\}_{n \in \mathbb{N}}$, the lemma will follow if we prove that for p_0 small enough,

$$\sup_{N,k} \mathrm{E}[b^N(k) \,|\, b^N(0) = 1] < \infty. \tag{46}$$

In the remaining part of the proof, using elements of queueing theory, we justify (46).

We first analyze the sequence of slots such that none of the particles $i \geq 2$ is active. If the particle $i \geq 2$ is active at time k, let $l_i(k)$ be the number of slots the particle remains active. $l_i(k)$ is a geometric distribution with parameter $1/L$. Now, let

$$W^N(k) = \max_{2 \leq i \leq N} \mathbf{1}(i \text{ active}) l_i(k).$$

If $W^N(k) = 0$ none of the particles $i \geq 2$ is active at time k. W^N satisfies the recursion:

$$W^N(k+1) = \max \left(W^N(k) - 1, \max_{2 \leq i \leq N} \mathbf{1}(i \text{ active at } k+1, \text{ inactive at } k) l_i(k+1) \right).$$

W^N is thus the workload in a $G/G/\infty$ queue with inter-arrival time 1 and service time requirement $\sigma^N(k+1) = \max_{i\geq 2} \mathbf{1}(i \text{ active at } k+1, \text{ inactive at } k) l_i(k+1)$. Independently of the past, $\sigma^N(k+1)$ is easily bounded stochastically; indeed, let $0 < s < \ln L$,

$$\begin{aligned}
\mathrm{E} e^{s\sigma^N(k+1)} &\leq 1 + \sum_{i=2}^{N} \mathrm{E}\mathbf{1}(i \text{ active at } k+1, \text{ inactive at } k) e^{s l_i(k+1)} \\
&\leq 1 + (N-1)\frac{p_0}{N} \mathrm{E} e^{s l_i(k+1)} \\
&\leq 1 + p_0 \frac{e^s/L}{1-(1/L)e^s}
\end{aligned}$$

Note that this last bound is uniform in N and k. Let $\theta_0 = 0$, $\theta_{n+1} = \inf\{k > \theta_n : W^N(k) = 0\}$, and $\Theta^N = \{\theta_n\}_{n\in\mathbb{N}}$. Classically, there exists $C > 0$ such that for all N:

$$\mathrm{E}[e^{C(\theta_{n+1}-\theta_n)} \,|\, W^N(0) = 0] < \infty,$$

see for example Appendix A.4 in [4]. By the renewal theorem, we deduce, uniformly in N, $\lim_{k\to\infty} \mathrm{P}(k \in \Theta^N) = \frac{1}{\mathrm{E}\theta_1} = 1 - a_N$. Moreover, the monotonicity of $W^N(k)$ with respect to the initial condition implies easily that $\mathrm{P}(k \in \Theta^N | W^N(0) = 0) \geq \lim_{k\to\infty} \mathrm{P}(k \in \Theta^N) = 1 - a_N$. Since $1 - a_N$ converges to e^{-Lp_0}, it follows that

$$\lim_{p_0\to 0} \inf_{k,N} \mathrm{P}(k \in \Theta^N | W^N(0) = 0) = 1. \tag{47}$$

We now turn back to the process b^N and prove (46). Let $U(k)$ be a sequence of independent and uniformly distributed variables on $[0,1]$. We may write

$$b^N(k+1) = b^N(k)\mathbf{1}_{U(k+1)>\frac{p_0}{b^N(k)N}} + 2b^N(k)\mathbf{1}_{U(k+1)\leq\frac{p_0}{b^N(k)N}}\mathbf{1}_{k\notin\Theta^N} + \mathbf{1}_{U(k+1)\leq\frac{p_0}{b^N(k)N}}\mathbf{1}_{k\in\Theta^N}$$

In particular

$$b^N(k+1)\mathbf{1}_{b^N(k)\geq 2} \leq b^N(k)\mathbf{1}_{U(k+1)>\frac{p_0}{b^N(k)N}} + 2b^N(k)\mathbf{1}_{U(k+1)\leq\frac{p_0}{b^N(k)N}}\mathbf{1}_{k\notin\Theta^N} + \mathbf{1}_{U(k+1)\leq\frac{p_0}{2N}}\mathbf{1}_{k\in\Theta^N}$$

Taking expectation, we obtain

$$\begin{aligned}
\mathrm{E}b^N(k+1)\mathbf{1}_{b^N(k)\geq 2} &\leq \mathrm{E}b^N(k) - \frac{p_0}{N} + 2\frac{p_0}{N}\mathrm{P}(k \notin \Theta^N) + \frac{p_0}{2N}\mathrm{P}(k \in \Theta^N) \\
&\leq \mathrm{E}b^N(k) - \frac{p_0}{N}(\frac{3}{2}\mathrm{P}(k \in \Theta^N) - 1).
\end{aligned}$$

Similarly, since $b^N(k+1)\mathbf{1}_{b^N(k)=1} \leq 2$, we have:

$$\mathrm{E}b^N(k+1) \leq \max\left(2, \mathrm{E}b^N(k) - \frac{p_0}{N}(\frac{3}{2}\mathrm{P}(k \in \Theta^N) - 1)\right).$$

From (47), for p_0 small enough, for all N and $k \geq 0$, $\mathrm{P}(k \in \Theta^N) > 2/3$. We deduce by recursion that $\mathrm{E}[b^N(k)|b^N(0) = 1] \leq 2$ and (46) holds. □

Now Lemma 10 implies that Assumption A8 holds. It remains to check Assumption A9. We only provide a characterization of the equilibrium point of the dynamical system (4). We leave the study of its global stability for future work.

So let us assume that an equilibrium point exists, and denote by Q this point. Further define $Q_c^n = Q(\{c, p_0 2^{-n}\})$ for all $n \in \mathbb{N}$. Then we have:

$$Q_c^{n-1}\overline{F}_C((p_0 2^{-n+1}, c), Q) = Q_c^n(\overline{F}_S((p_0 2^{-n}, c), Q) + \overline{F}_C((p_0 2^{-n}, c), Q)),$$

or equivalently

$$2Q_c^{n-1}H_c = Q_c^n I_c, \tag{48}$$

and

$$\sum_{n \geq 0} Q_c^n \overline{F}_S((p_0 2^{-n}, c), Q) = Q_c^0 \overline{F}_C((p_0, c), Q)),$$

or equivalently

$$\rho_c G_c = Q_c^0 H_c. \tag{49}$$

Solving (48) and (49) leads to a solution of the form $Q_c^n = \beta_c(2H_c/I_c)^n$. Since $\sum_n Q_c^n = \mu_c$, we have $\beta_c = Q_c^0 = \mu_c(1 - 2H_c/I_c) = \mu_c(G_c - H_c)/I_c$. We require that $H_c < G_c$ or equivalently, $G_c/I_c > 1/2$. G_c/I_c may be interpreted as the probability in steady state that no user of class in $\mathcal{V}_c$ tries to access the channel given that no user of class in $\mathcal{V}_c$ are currently sending. Next, $\rho_c = \sum_{n \geq 0} p_0 2^{-n} Q_c^n$, which implies that:

$$\rho_c = p_0 \mu_c \frac{G_c - H_c}{G_c}. \tag{50}$$

Now the following corollary summarizes the above analysis and then it characterizes the system behavior in steady state and in case of exponential back-off algorithms.

Corollary 1 *The stationary distribution Q is given by: for all $c \in \mathcal{C}$,*

$$Q_c^n = \mu_c \frac{G_c - H_c}{G_c + Hc} \left(\frac{2H_c}{G_c + H_c} \right)^n,$$

where the G_c's, H_s's, and ρ_c's are the unique solutions of the system of equations (40), (41), (42), (50).

5.5 A numerical example

We now illustrate our analytical results on the simple network of Example 1. Here the state space of the background process is $\mathcal{Z} = \{0,1\}^3$ and its stationary distribution is given by:

$$\begin{aligned}
&\pi(100) = \pi_0 L(1 - e^{-\rho_1}),\ \pi(010) = \pi_0 L(1 - e^{-\rho_2}),\ \pi(001) = \pi_0 L(1 - e^{-\rho_3}),\\
&\pi(110) = \pi_0 L(1 - e^{-\rho_1})(1 - e^{-\rho_2}),\ \pi(101) = \pi_0 L^2(1 - e^{-\rho_1})(1 - e^{-\rho_3}),\\
&\pi(011) = \pi_0 L(1 - e^{-\rho_2})(1 - e^{-\rho_3}),\ \pi(111) = \pi_0 L(1 - e^{-\rho_1})(1 - e^{-\rho_2})(1 - e^{-\rho_3}).
\end{aligned}$$

$\pi_0 = \pi(000)$ is the normalization constant. The terms G_c and H_c are:

$$\begin{aligned}
&G_1 = \pi_0 e^{-\rho_1}(e^{-\rho_2} + 1 - e^{-\rho_3}),\ G_2 = \pi_0 e^{-\rho_1} e^{-\rho_2} e^{-\rho_3},\\
&H_1 = \pi_0((1 - e^{-\rho_1} e^{-\rho_2}) + (1 - e^{-\rho_1})(1 - e^{-\rho_3})),\ H_2 = \pi_0(1 - e^{-\rho_1} e^{-\rho_2} e^{-\rho_3}).
\end{aligned}$$

The terms G_3, H_3 are obtained by symmetry. Now solving the above equations together with (50), we may compute the throughputs of the users of various classes. Applying (38):

$$\begin{aligned}
\gamma_1 &= L\rho_1 e^{-\rho_1}\left(\pi_0 e^{-\rho 2} + \pi(001)\right),\\
\gamma_2 &= L\rho_2 e^{-\rho_2} e^{-\rho_1} e^{-\rho_3} \pi_0,\\
\gamma_3 &= L\rho_3 e^{-\rho_3}\left(\pi_0 e^{-\rho 2} + \pi(100)\right),
\end{aligned}$$

In Figure 2, the throughputs of the various user classes are presented assuming that the proportions of users of class 1 and 3 are identical, $\mu_1 = \mu_3$. On the left side we give the throughputs as a function a the proportion of users of class 2 - Here the packet duration is fixed and equal to $L = 100$ slots, which roughly corresponds to the case of packets of size 1000 bytes transmitted in a IEEE802.11g-based network (this duration depends of course on the physical transmission rate). The total network throughput decreases when the proportion of class-2 users increases, which illustrates the loss of efficiency due to the network spatial heterogeneity. On the left side of Figure 2, we assume a uniform user distribution among the 3 classes, $\mu_1 = \mu_2 = \mu_3$, and we give the throughputs as a function of the packet duration L. First note that whatever the value of L, the network is highly unfair: for example when $L = 100$ slots, the throughput of a user of class 1 is almost 5 times greater than that of a user of class 2. This unfairness increases with L and ultimately when L is very large, users of class 2 never access the channel successfully.

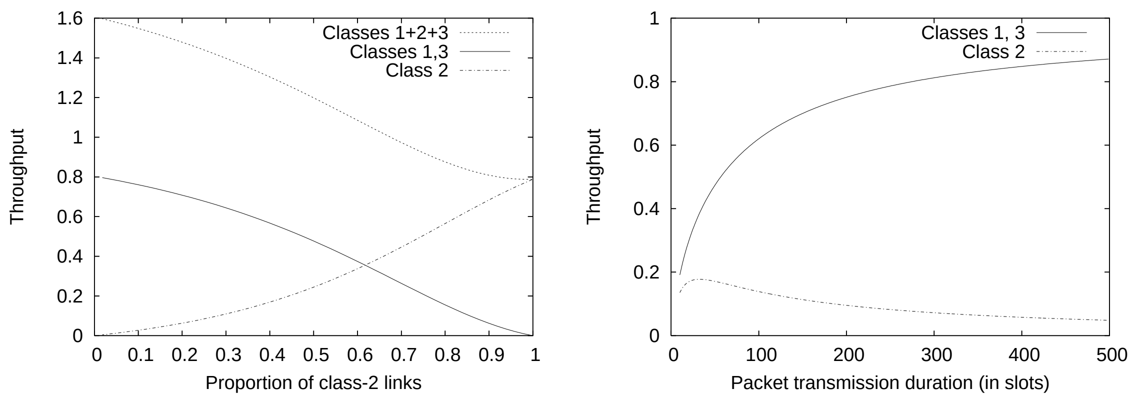

Figure 2: Throughputs of users, (left) as a function of μ_2, $L = 100$ slots - (right) as a function of L, when $\mu_1 = \mu_2 = \mu_3$.

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