The Mathematics of Routing in Massively Dense Ad-Hoc Networks

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Statement Problem

- Study the global and the non-cooperative optimal solution for the routing problem among a large quantity of nodes.
- Find a general optimization framework for handling minimum cost paths in massively dense ad-hoc networks.

Previous Works

- Geometrical Optics
  P. Jacquet studies the routing problem as a parallel to an optics problem.
  Drawback: It doesn't consider interaction between each user's decision.
- Electrostatics
  S. Toumpis studies the problem of the optimal deployment of wireless sensor networks.
  Drawback: The local cost assumed is too particular ($\mathrm{cost}(f) = |f|^2$ where f is the flow).

Previous Works

- Road Traffic

  S. Dafermos studies the user-optimizing and the system-optimizing pattern.

  Beckmann (1956) studies the system-optimizing pattern.

  Drawback: The present mathematical tools from Optimization and Control Theory were not available.

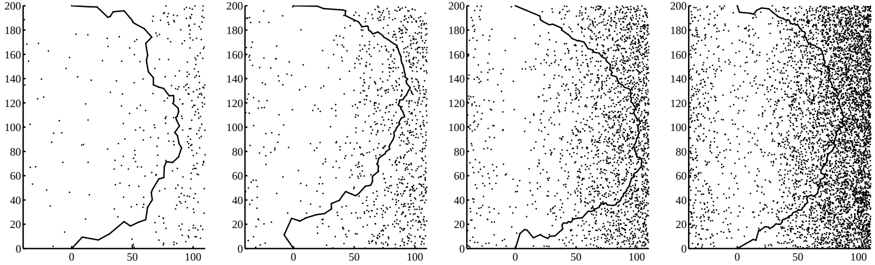

Figure: Minimum cost routes ($\mathrm{cost} = \mathrm{distance}^2$) where relay nodes are placed according to a spatial Poisson process of density $\lambda(x, y) = a \cdot (10^{-4}x^2 + 0.05)\ \mathrm{nodes/m}^2$, for four increasing values of a.

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Let Ω be an open and bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\Gamma = \partial\Omega$, densely covered by potential routers.
Messages flow from $\Gamma_{\mathcal{S}} \subseteq \Gamma$ to $\Gamma_{\mathcal{R}} \subseteq \Gamma$ (with $\Gamma_{\mathcal{S}} \cap \Gamma_{\mathcal{R}} = \emptyset$).
On the rest $\Gamma_{\mathcal{T}}$ of the boundary, no message should enter nor leave Ω.

Figure: Description of the domain

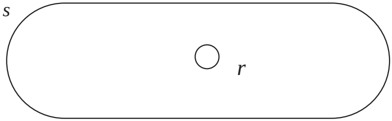

Figure: Description of domain with sensor networks

Assumptions:

- The intensity of message generation $\sigma|_{\Gamma_{\mathcal{S}}} \in L^2(\Gamma_{\mathcal{S}})$ is known.
- The intensity of message reception $\sigma|_{\Gamma_{\mathcal{R}}}$ is unknown.

The total flow of messages emitted and received are equal.

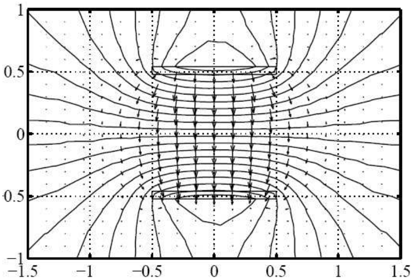

Figure: The function f.

Let the vector field $f = (f_1(x), f_2(x)) \in (H^1(\Omega))^2$ $[\mathrm{bps/m}]$ represent the flow of messages, and $\phi(x) = \|f(x)\|$ be its intensity.

Let $\Gamma_1 = \Gamma_{\mathcal{S}} \cup \Gamma_{\mathcal{T}}$.
Extend the function σ to Γ_1 by $\sigma(x) = 0$ on $\Gamma_{\mathcal{T}}$.
We modelize the conditions on the boundary as

$$\boxed{\forall x \in \Gamma_1 \quad \langle f(x), n(x) \rangle = -\sigma(x)}$$

The Conservation Equation

Suppose there is no source nor sink of messages in Ω. Over a surface $\Phi_0 \subseteq \Omega$ of arbitrary shape,

$$\oint_{\partial\Phi_0} \langle f(x), n(x)\rangle \mathrm{d}\Phi_0 = 0,$$

where n is the unit normal vector.
Last equation holding for any smooth domain, then

$$\boxed{\forall x \in \Omega \quad \mathrm{div} f(x) = 0.}$$

Let the congestion cost per packet $c = c(x, \phi) \in \mathcal{C}^1(\Omega \times \mathbb{R}_+, \mathbb{R}_+)$ be a strictly positive function, increasing and convex in ϕ for each x.
The total cost of congestion will be taken as

$$G(f(\cdot)) = \int_\Omega c(x, \|f(x)\|)\|f(x)\| \,\mathrm{d}x.$$

The path followed by a packet is specifed by its direction of travel $e_\theta = (\cos\theta, \sin\theta)$ along its path, according to $\dot{x} = e_\theta$. The cost incurred by one packet travelling from $x_0 \in \Gamma_\mathcal{S}$ at time t_0 to $x_1 \in \Gamma_\mathcal{R}$ reached at time t_1 is

$$J(e_\theta(\cdot)) = \int_{x_0}^{x_1} c(x, \|f(x)\|)\sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2} = \int_{t_0}^{t_1} c(x(t), \|f(x(t))\|) \,\mathrm{d}t,$$

Global Optimum

We seek here the vector field $f^* \in (L^2(\Omega))^2$ minimizing $G(f)$ under the constraints:

$$\begin{aligned} \forall x \in \Gamma_1 \quad & \langle f(x), n(x) \rangle = -\sigma(x) \\ \forall x \in \Omega \quad & \mathrm{div} f(x) = 0. \end{aligned}$$

Let $C(\mathbf{x}, \phi) = c(\mathbf{x}, \phi)\phi$. It is convex in ϕ and coercive (i.e. goes to infinity with ϕ).
Then $f(\cdot) \mapsto G(f(\cdot))$ is continuous, convex and coercive. Moreover, the constraints are linear.

We dualize only the constraint of the divergence and look for f satisfying the other constraint.
Let therefore $p(\cdot) \in L^2(\Omega)$ be the dual variable, we let

$$\mathcal{L}(f,p) = \int_\Omega \Big(C(x, \|f(x)\|) + p(x) \mathrm{div} f(x) \Big) \mathrm{d}x \,.$$

Kuhn-Tucker conditions implies that for $f^*(\cdot)$ to be optimal, there must exist a $p(\cdot) : \Omega \to \mathbb{R}$ such that

$$\begin{array}{ll} \forall x \in \Omega : f^*(x) \neq 0\,, & \nabla p(x) = \mathcal{D}_2 C(x, \|f(x)^\star\|) \frac{1}{\|f^*(x)\|} f^*(x), \\ \forall x \in \Omega : f^*(x) = 0\,, & \|\nabla p(x)\| \leq \mathcal{D}_2 C(x, 0), \\ \forall x \in \Gamma_{\mathcal{R}}\,, & p(x) = 0\,. \end{array}$$

User Optimum

The optimization of the criterion

$$J(e_\theta(\cdot)) = \int_{x_0}^{\mathbf{x}_1} c(x, \|f(x)\|)\sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2} = \int_{t_0}^{t_1} c(x(t), \|f(x(t))\|)\,\mathrm{d}t,$$

via its Hamilton-Jacobi-Bellman equation:
Let $V(x)$ be the return function, it must be a viscosity solution of

$$\begin{array}{ll} \forall x \in \Omega, & \min_\theta \langle e_\theta, \nabla V(x)\rangle + c(x, \|f^*(x)\|) = 0, \\ \forall x \in \mathcal{R}, & V(x) = 0. \end{array}$$

Hence

$$\forall x \in \Omega\,, \quad -\|\nabla V(x)\| + c(x, \|f^*(x)\|) = 0\,,$$
$$\forall x \in \mathcal{R}, \quad V(x) = 0\,.$$

The optimal direction of travel is opposite to $\nabla V(x)$, i.e. $e_\theta = -\nabla V(x)/\|\nabla V(x)\|$.
This is the same system of equations as previously, upon replacing $p(x)$ by $-V(x)$, and $\mathcal{D}_2 C(x,\phi)$ by $c(x,\phi)$.
Conclusion The Wardrop equilibrium can be obtained by solving the globally optimal problem in which the cost density is replaced by $\int_0^\phi c(x,\phi)\mathrm{d}\phi$.

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Example : Linear Congestion Cost

If the cost of congestion is linear : $c(x,\phi) = \frac{1}{2}c(x)\phi$, so that

$$C(x,\phi) = \frac{1}{2}c(x)\phi^2 .$$

Then, $\mathcal{L}$ is differentiable everywhere, and the necessary condition of optimality is just that there should exist $p : \Omega \rightarrow \mathbb{R}^2$ such that $\nabla p(x) = c(x)f^*(x)$.

Placing this into the divergence equation and the boundary equation, we see that we end up with a simple elliptic equation with mixed Dirichlet - (non-homogeneous) Neuman boundary conditions :

$$\left.\begin{array}{ll} \forall x \in \Omega,, & \mathrm{div}(\frac{1}{c(x)}\nabla p(x)) = 0\,, \\ \forall x \in \Gamma_1, & \dfrac{\partial p}{\partial n}(x) = c(x)\sigma(x), \\ \forall x \in \Gamma_{\mathcal{R}}\,, & p(x) = 0\,, \end{array}\right\}$$

for which we get existence and uniqueness of the solution (Lax-Milgram Theorem $p \in H^1_{\Gamma_{\mathcal{R}}}$).

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Conclusions

- We study a setting to describe the network in terms of macroscopic parameters rather than in terms of microscopic parameters.
- We solve the routing problem for the affine cost per packet obtaining existence and uniqueness of the solution.

Future Works

1. Numerical solution using Finite Element Method for the affine cost per packet.
2. Investigate the quantity of nodes required so this approch to be a good aproximation for the routing problem.

Thank you !