Multitype Continuous Branching in Random Environment with Migration

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1 Objectives

- Define a new **continuous** multi-type branching process with migration

- Relate to queueing systems

- Obtain first two moments of the state vector of the branching process for **correlated migration** process

- Derive **expected waiting times** in polling systems with correlated vacations.

Example 1: discrete branching with migration

Queue with Vacations, Gated Regime

• $M/G/1/\infty$ queue,

• Arrival rate λ, i.i.d. service times $\{D_n\}$ with first and second moments d, $d^{(2)}$.

• Sequence of vacations: V_n. Will be assumed stationary ergodic, with first and second moments v, $v^{(2)}$.

• Gated regime: at the nth end of vacation, a gate is closed (nth polling instant). Then the server goes on serving the customers present at the queue at that polling instant:
Then the server leaves on vacation.

•We denote:

- B_n:= the number of arrivals during the nth vacation.
- $\xi_h^{(i)}$:= the number of arrivals during the service time of a customer

•Then:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n, \qquad n \geq n_0.$$

Denote

$$A_n(x) = \sum_{i=1}^{x} \xi_n^{(i)}$$

Then A_n are nonnegative and divisible:

$$A_n(x+y) = A_n^{(1)}(x) + A_n^{(2)}(y)$$

where $A_n^{(i)}$ are i.i.d.

Example 2: continuous branching with migration

Queue with Vacations, Gated Regime

•Define the time to serve N customers as:

$$\tau(N) := \sum_{i=1}^{N} D_i$$

•Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T.

•Denote by $\hat{\mathcal{A}}_n(C_n) = \tau(\mathcal{N}(C_n))$, i.e. the sum of service times of all the arrivals during C_n.

•We obtain

$$C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}. \tag{1}$$

Example 3: multitype discrete branching

Discrete time infinite server queue

•Service times are considered to be i.i.d. and independent of the arrival process.

•We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.

•The initial phase k is chosen at random according to some probability $p(k)$.

•If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.

•With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.

•P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

•Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

•The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

•Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

•Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

•B_n is a stationary ergodic sequence and has finite expectation.

•Y_n^i:= number of customers in phase i at time n. Satisfies

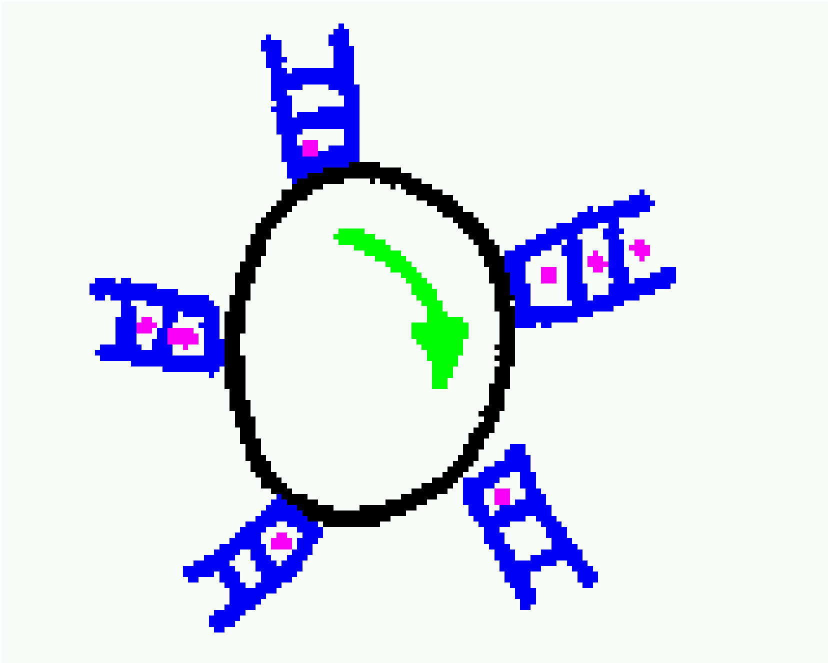
$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{2}$$

Example 4: Polling systems with N queues are special cases!

•The server moves cyclically (fixed order) between the queues $1, ..., M$.
It requires walking times (vacations) for moving from one queue to another.



•Upon arrival at a queue, some customers are served. The number to be served is determined by the "polling regime":

Globally Gated (GG) regime (Boxma, Levy, Yechiali 1992):
The cycle time satisfies a **one dimensional recursion**.
We obtained the first two moments of the cycle and the expected waiting times at all queues.

Gated and Exhaustive regimes [see e.g. book by Takagi 1986]:
satisfy M-dimensional recursive equations.

No explicit expression for 2nd moments of buffer occupancy or cycle times.

No explicit expression for the expected waiting times.

Introduction and Background on Lévy fields

Introduction

• Consider the stochastic recursive equation:

$$Y_{n+1} = A_n(Y_n) + B_n, \qquad n \geq n_0. \tag{3}$$

• Y_n is a vector in R_+^m

• $\{A_n\}_n$ are
- i.i.d., independent of B_n.
- Increasing in the arg for all n.
- nonnegative **Additive Lévy field taking values in** $\mathbb{R}_+^m$

• $\{B_n\}$ stationary ergodic taking values in $\mathbb{R}_+^m$

(3) defines a **Continuous Multitype Branching Process (BP) with Migration**

Background: Lévy processes

Lévy process taking values in $\mathbb{R}_+$:

- Example: Poisson Point Process with intensity λ,
- **Expectation** and **variance** are linear: $E[A(t)] = t\mathcal{A}$ and cov$[A(t)] = t\Gamma$.
- For random time τ independent of A,

$$E[A(\tau)] = E[\tau]\mathcal{A}\,, \qquad \mathrm{var}[A(\tau)] = \mathrm{E}[\tau]\Gamma + \mathrm{var}[\tau]\mathcal{A}^2\,,$$

- **Divisibility:** $A(\cdot)$ is divisible if the following holds.
  For any k, there exist $A^{(i)}(\cdot)$, $i = 0, ..., k$ such that for any non-negative $x_i, i = 0, ..., k$,

$$A\left(\sum_{i=0}^{k} x_i\right) = \sum_{i=0}^{k} A^{(i)}(x_i) \qquad (4)$$

  where $\{A^{(i)}(\cdot)\}_{i=0,1,2,\ldots,k}$ are i.i.d. with the same distribution as $A(\cdot)$.

Lévy process taking values in $\mathbb{R}_+^m$ (subordinators):

- Example: Poisson arrival process where the nth arrival brings a batch $B_n = (B_n^1, ..., B_n^m)$. B_n^i customers go to queue i.
- For $A(t)$ in $\mathbb{R}_+^m$, $E[A(t)] = \mathcal{A}t$ where $\mathcal{A}$ is of dimension m.
- $cov[A(t)] = \Gamma t,$ where Γ is a matrix of dimension $m \times m$.

Example of Random fields

Random field taking values in $\mathbb{R}_+$

- Example: Black and white picture.
- The level of grey is a function of two parameters: x and y.

Random field taking values in $\mathbb{R}_+^d$

- Example: color picture.
- The level of the green, red and blue as a function of the location x and y.

Background: Additive Lévy Fields

Let $A^{(1)}, ..., A^{(d)}$ be d indep. Lévy proc. on $\mathbb{R}^m$ with scalar "time" parameters.

Additive Lévy field: $A(y) = A^{(1)}(y_1) + ... + A^{(d)}(y_d)\,, \quad \forall y = (y_1, ..., y_d) \in \mathbb{R}_+^d$.

The expectation: $\mathrm{E}[A(y)] = \sum_{j=1}^{d} y^j \mathcal{A}^{(j)} = \mathcal{A}y\,,$
$\mathcal{A}$ is a matrix whose jth column equals $\mathcal{A}^{(j)}$,
$\mathcal{A}^{(j)} = \mathrm{E}[A^{(j)}(1)]$,

The covariance matrix: $\mathrm{cov}[A(y)] = \sum_{j=1}^{d} y_j \Gamma^{(j)}\,,$
where $\Gamma^{(j)} = \mathrm{cov}[A^{(j)}(1)]$ is the corresponding covariance matrix of $A^{(j)}(1)$.

Composition: If A_n and A_{n+1} are Additive Lévy processes in $\mathbb{R}_+^m$ then their composition is also an Additive Lévy process.

Properties of Lévy Fields

• Expectation and Covariance are linear in y,

• Let τ be a non-negative random variable in $\mathbb{R}_+^d$, independent of A and represented as a column vector. Then

$$E[A(\tau)] = \sum_{j=1}^{m} \mathcal{A}^{(j)} E[\tau_j] \, ,$$

and,

$$\mathrm{cov}[A(\tau)] = \sum_{j=1}^{d} \mathrm{E}[\tau_j] \Gamma^{(j)} + \mathcal{A} \, \mathrm{cov}[\tau] \mathcal{A}^T \, , \tag{5}$$

where τ_j is the jth entry of the vector τ.

Result 1: Steady State Probabilities of CBP

Iterating $Y_{n+1} = A_n(Y_n) + B_n$, we obtain from A1:

$$\begin{aligned} Y_2 &= A_1(Y_1) + B_1 \\ &= A_1(A_0(Y_0) + B_0) + B_1 \\ &= A_1^{(0)}(A_0(Y_0)) + A_1^{(1)}(B_0) + B_1 \\ &= A_1^{(0)} A_0^{(0)}(Y_0) + A_1^{(1)}(B_0) + B_1. \end{aligned}$$

$$\begin{aligned} Y_3 &= A_2(Y_2) + B_2 \\ &= A_2(A_1(Y_1) + B_1) + B_2 \\ &= A_2(A_1(A_0(Y_0) + B_0) + B_1) + B_2 \\ &= A_2^{(0)} A_1^{(0)} A_0^{(0)}(Y_0) + A_2^{(1)} A_1^{(1)}(B_0) + A_2^{(2)}(B_1) + B_2 \end{aligned}$$

In general:

$$Y_n = \sum_{j=0}^{n-1} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}) + \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (Y_0), \quad n > 0 \tag{6}$$

(we understand $\prod_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\prod_{i=n}^{k} A_i(x) = A_k A_{k-1} ... A_n$ whenever $k > n$).

Under fairly general assumptions, $\lim_{n\to\infty} \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (y) = 0$, so Y_n has a limit as $n \to \infty$ distributed like

$$Y_n^* =_d \sum_{j=0}^{\infty} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}), \qquad n \in Z, \tag{7}$$

where for each integer i, $\{A_i^{(j)}(\cdot)\}_j$ are independent of each other and have the same distribution as $A_i(\cdot)$.

Sufficient condition: stationarity plus $\|\mathcal{A}\| < 1$.

Application: Expected waiting time for a gated queue with vacations

Consider an arbitrary customer. Upon arrival, it has to wait for

1. The residual cycle time C_{res},
2. The service time of all the customers that arrived during C_{past} which is the past cycle time: $d(\lambda E[C_{past}]) = \rho E[C_{past}]$

We have from [Baccelli & Brémaud, 1994]

$$E[C_{res}] = E[C_{past}] = \frac{E[C_0^2]}{2E[C_0]}.$$

Thus the expected waiting time of an arbitrary customer is given by

$$E[W_n] = (1+\rho)\frac{E[C_0^2]}{2E[C_0]},$$

The expected number of customers in queue in stationary regime (not including service) is obtained using Little's Theorem: $\lambda E[W_n]$.

Conclusion: we need to compute $E[C_0]$ and $E[C_0^2]$!

Computing $E[C_0]$ and $E[C_0^2]$

•Dynamics: $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$.

•$\hat{\mathcal{A}}_n(c)$ is the workload that arrives during duration $[0, c)$.

•Introduce the correlation function: $r(n) = E[V_0 V_n]$.

•The first and second moments of C_n in stationary regime are given by

$$E[C_n] = \frac{v}{1-\rho},$$

$$E[C_n^2] = \frac{1}{(1-\rho^2)} \left(\frac{\lambda v d^{(2)}}{1-\rho} + r(0) + 2 \sum_{j=1}^{\infty} \rho^j r(j) \right). \tag{8}$$

Proof of expressions for $E[C_0^2]$

Useful relations: 2nd moment of workload arriving during T

- If N is a random variable independent of the sequence D_n, and $\tau(N) := \sum_{i=1}^N D_i$ then

$$E[\tau(N)^2] = E[N^2]d^2 + E[N](d^{(2)} - d^2). \tag{9}$$

- Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T. Then

$$E[\mathcal{N}(T)^2] = \lambda^2 E[T^2] + \lambda E[T]. \tag{10}$$

- If we take an arbitrary T and choose $N = \mathcal{N}(T)$, then we get from (9)-(10)

$$\begin{aligned} E[(\hat{\mathcal{A}}(T))^2] &= E[\tau(\mathcal{N}(T))^2] \\ &= d^2(\lambda^2 E[T^2] + \lambda E[T]) + \lambda E[T](d^{(2)} - d^2) \\ &= d^2\lambda^2 E[T^2] + \lambda E[T] d^{(2)}. \end{aligned} \tag{11}$$

- Also, if we take $T = \tau(N)$, then

$$E[\mathcal{N}(\tau(N))]^2 = \lambda^2 \Big[E[N^2]d^2 + E[N](d^{(2)} - d^2) \Big] + \lambda d E[N]. \tag{12}$$

•From $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$ we have

$$\begin{aligned} E[C_{n+1}^2] &= E[\hat{\mathcal{A}}_n(C_n)^2] + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] \\ &= \left(\rho^2 E[C_n^2] + \lambda E[C_n] d^{(2)}\right) + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}]. \end{aligned}$$

•To compute the last term, we now use the explicit form of C_0:

$$C_0 = \sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}).$$

•We use the fact that the processes $\{\hat{\mathcal{A}}_i^{(j)}\}$ are independent of $\{V_n\}$. We get:

$$\begin{aligned} E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] &= E[\hat{\mathcal{A}}_0(C_0)V_1] = E\left[\hat{\mathcal{A}}_0 \left(\sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}) \right) V_1 \right] \\ &= \rho \sum_{j=0}^{\infty} \rho^j E[V_{-j} V_1] = \sum_{j=1}^{\infty} \rho^j r(j). \end{aligned}$$

Substituting this, we obtain the second moment.

2nd order moments in continuous B.P.

Notation: •Auto-correlations: $\mathcal{B}(k) =_{def} \mathrm{E}[B_0(B_k)^T]$, where k is an integer
•$\hat{\mathcal{B}}(k) =_{def} \mathcal{B}(k) - \mathrm{E}[B_0]\,\mathrm{E}[B_0]^T$. (Note: $\hat{\mathcal{B}}(0)$ equals $\mathrm{cov}[B_0]$.)

Assumptions: Consider $Y_{n+1} = A_n(Y_n) + B_n,\ n \geq n_0$, where

- A_n are i.i.d. additive Lévy fields,
- A_n independent of $\{B_n\}$,
- $\{B_n\}$ are stationary ergodic,
- All eigenvalues of $\mathcal{A}$ are within the unit disk,
- the elements of B_0 have finite second order moments.

Theorem: Consider $Y_{n+1} = A_n(Y_n) + B_n$ in stationary regime. Then

(i) $\mathrm{E}[Y_0] = (\mathcal{I} - \mathcal{A})^{-1}\,\mathrm{E}[B_0]\,,$

(ii) $\mathrm{cov}(Y_0)$ is the unique solution of the linear equations:

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j\hat{\mathcal{B}}(j) + (\mathcal{A}^j\hat{\mathcal{B}}(j))^T\,, \tag{13}$$

where $\mathrm{E}[Y_0^j]$ denotes the jth element of $\mathrm{E}[Y_0]$.

Proof for first moments:
Taking expectation in $Y_{n+1} = A_n(Y_n) + B_n$ we get

$$\mathrm{E}[Y_0] = \mathcal{A}\,\mathrm{E}[Y_0] + \mathrm{E}[B_0],$$

Since the eigenvalues of $\mathcal{A}$ are within the unit disk, $(\mathcal{I} - \mathcal{A})$ is inverible.
Hence we obtain (i).

Proof of uniqueness for the second moments

- Let Z_1 and Z_2 be two solutions of

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + (\mathcal{A}^j \hat{\mathcal{B}}(j))^T\,.$$

- Define $Z = Z_1 - Z_2$. Then Z satisfies $Z = \mathcal{A}^T Z \mathcal{A}$.
- Iterating, we obtain,

$$Z = \lim_{n\to\infty} \mathcal{A}^n Z (\mathcal{A}^T)^n = 0$$

where the last equality follows from the fact that all the eigenvalues of $\mathcal{A}$ are within the unit disk.

- This implies uniqueness.

Proof for expression of second moments

•Consider $Y_{n+1} = A_n(Y_n) + B_n$.

- Multiply both sides by their transpose,
- take expectation and
- use the stationarity

we get:

$$\mathrm{E}[Y_0 Y_0^T] = \mathrm{E}[A_0(Y_0)A_0^T(Y_0)] + \mathrm{E}[B_0 B_0^T] + \mathrm{E}[A_0(Y_0)B_0^T] + \mathrm{E}[B_0 A_0^T(Y_0)] \,.$$

The covariance matrix $\mathrm{cov}[Y_0]$ therefore equals,

$$\begin{aligned} \mathrm{cov}[Y_0] \quad = \quad & \mathrm{cov}[A_0(Y_0)] + \mathrm{cov}[B_0] + \mathrm{E}\left[A_0(Y_0)B_0^T\right] \\ & -\mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T + \mathrm{E}\left[B_0 A_0(Y_0)^T\right] - \mathrm{E}[B_0](\mathcal{A}\,\mathrm{E}[Y_0])^T \,. \end{aligned} \tag{14}$$

It remains to compute the red and the blue expressions.

Red Expression: Using the convariance expression (5) of Additive Lévy processes at random "time":

$$\mathrm{cov}[A_0(Y_0)] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T\,. \tag{15}$$

Blue Expression: We use the explicit expression (7) for the stationary state process to obtain

$$\begin{aligned}
\mathrm{E}[Y_0 B_0^T] &= \sum_{j=0}^{\infty} \mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \middle| \mathbf{B_0^-} \right) \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathcal{A}^j B_{-j-1} B_0^T \right) = \sum_{j=0}^{\infty} \mathcal{A}^j \mathcal{B}(j+1)\,,
\end{aligned} \tag{16}$$

with $\mathbf{B_0^-} := (B_0, B_{-1}, B_{-2}, ...)$.

Substituting the last expression, we compute,

$$\mathrm{E}[A_0(Y_0)B_0^T] = \mathrm{E}\left[\mathrm{E}\left[A_0(Y_0)B_0^T \,\middle|\, Y_0, B_0\right]\right] = \mathcal{A}\,\mathrm{E}\left[Y_0 B_0^T\right] = \sum_{j=1}^{\infty} \mathcal{A}^j \mathcal{B}(j)\,,$$

or equivalently,

$$\begin{aligned}
\mathrm{E}[A_0(Y_0)B_0^T] &= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \sum_{j=1}^{\infty} \mathcal{A}^j\,\mathrm{E}[B_0]\,\mathrm{E}[B_0]^T \\
&= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}(\mathcal{I}-\mathcal{A})^{-1}\,\mathrm{E}[B_0]^T \\
&= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T\,. \qquad (17)
\end{aligned}$$

Substitution of expressions RED and BLUE provides the covariance equation.

2 Symmetric gated polling systems

m gated queues.

Arrivals:

- Arrival processes $\rho^i(t)$ to queue i are i.i.d. Levy processes, distributed as some $\rho(t)$, $t \in \mathbb{R}_+$.
- $\overline{\rho} = \mathrm{E}[\rho(1)]$ and $\sigma^2 = \mathrm{var}[\rho(1)]$ Itô decomposition: subordinator decomposes into a Poisson

Walking times:

- $\{V_n\}$: Stationary ergodic series of walking times, $v := \mathrm{E}[V_0]$.
- $\mathcal{V}(j) := \mathrm{E}[V_0 V_j]$ for some integer j and $\hat{\mathcal{V}}(j) := \mathrm{E}[V_0 V_j] - v^2$.

Notation:

- $I(n)$:= the queue visited at the nth polling instant
- $S(n)$:= nth polling instant (time at which the server arrives at the nth queue)
- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$

The dynamics:

$$\begin{aligned} Y_{n+1}^1 &= S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n\,, \\ Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n\,, \\ Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n\,, \\ &\vdots \\ Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n\,. \end{aligned} \tag{18}$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = the workload that arrived at queue $I(n)$ during the nth cycle.

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Y_{n+1}^1 &= S(n+1) - S(n) = \boldsymbol{\rho_n^m(Y_n^m)} + V_n \,, \qquad (18) \\
Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\
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- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$

The dynamics:

$$\begin{aligned} Y_{n+1}^1 &= S(n+1) - S(n) = \boldsymbol{\rho_n^m(Y_n^m)} + V_n\,, \\ Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n\,, \\ Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n\,, \\ &\vdots \\ Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n\,. \end{aligned} \tag{18}$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = **workload that arrived at queue $I(n)$ during the nth cycle.**

Interpretation of the other equations:

For $i > 0$, we have

$$Y_{n+1}^{i+1} = S(n+1) - S(n-i) = S(n+1) - S(n) + S(n) - S(n-i)$$

where

•by definition, $S(n) - S(n-i) = Y_n^i$, and

•$S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n$ (see previous slide).

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ with}$$

$$\begin{aligned}
Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n \,, \\
Y^2_{n+1} &= S(n+1) - S(n-1) = & Y^1_n + \rho^m_n(Y^m_n) + V_n \,, \\
Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n \,, \\
&\vdots \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n \,.
\end{aligned}$$

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \ , \text{ with}$$

$$\begin{aligned}
Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n \ , \\
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Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n \ , \\
&\vdots \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n \ .
\end{aligned}$$

where $Y_{n+1} = (Y^1_{n+1}, ..., Y^m_{n+1})^T$,

Vector notation:

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Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n\,, \\
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Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n\,, \\
&\vdots \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n\,.
\end{aligned}$$

where $B_n = V_n(1,1,1,...,1)^T$,

•in the special case that $\{B_n\}$ is i.i.d. Y_n is a Markov chain

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ with}$$

$$\begin{aligned}
Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n\,, \\
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Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n\,, \\
&\vdots & \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n\,.
\end{aligned}$$

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ where}$$

$$A_n(y) = A_n^{(1)}(y_1) + ... + A_n^{(m)}(y_m) \,, \tag{19}$$

where $y = (y_1, ..., y_m)^T \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$ and

$$\begin{aligned} A_n^{(1)}(t) &= (0, t, 0, 0, ..., 0)^T, \\ A_n^{(2)}(t) &= (0, 0, t, 0, ..., 0)^T, \\ &\vdots \\ A_n^{(m-1)}(t) &= (0, 0, 0, ..., 0, t)^T, \\ A_n^{(m)}(t) &= \rho_n^m(t)(1, 1, \dots, 1)^T \,, \end{aligned} \tag{20}$$

•For each i, $A_n^{(i)}$ is a Lévy **process** taking values in $\mathbb{R}_+^m$.

•A_n are Additive Lévy fields

Checking the stability condnition

Taking expectation we get:

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 1 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 1 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 1 & 0 & \dots & 0 & \overline{\rho} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 0 & 0 & \dots & 1 & \overline{\rho} \end{pmatrix} . \tag{21}$$

$\mathcal{A}$ is known as the *Companion matrix*.

Theorem: A sufficient and necessary condition for all eigenvalues of $\mathcal{A}$ to be in the interior of the unit circle is

$$\overline{\rho} < \frac{1}{m}.$$

Conclusions and Discussion

•We use neither the "buffer occupancy" nor the "station times" approaches.

•Advantage: one component of the state is the cycle time;
its two first moments provide the expected waiting time.

•A very similar structure is obtained in the exhaustive case.

Example 5: Discrete time infinite server queue

- Service times are geometrically distributed,
- The SRE becomes one dimensional. Y_n denotes the number of customers in the system.
- $\xi_n^{(k)}$ is the indicator that the kth customer present at the beginning of time-slot n will still be there at the end of the time-slot.
- The probability that a customer in the system finishes its service within a time slot is precisely $p = 1 - \mathsf{A} = 1 - E[\xi_n]$.
- We consider a Markov chain with two states $\{\gamma, \delta\}$ with transition probabilities given by

$$\mathcal{P} = \begin{pmatrix} 1 - \epsilon p & \epsilon p \\ \epsilon q & 1 - \epsilon q \end{pmatrix}$$

•As an example, consider the following parameters: $p = q = 1$, at a given state there is at most one arrival with prob. $p_\gamma = 1, p_\delta = 0.5$. This gives:

$$var[Y^*] = \frac{1}{(1-\mathsf{A}^2)}\left(\frac{3}{16} + \frac{2\mathsf{A}}{1-\mathsf{A}+2\epsilon\mathsf{A}} + \frac{3}{4}\mathsf{A}\right).$$

In Fig. 1 we plot the variance of the steady state number of customers, $var[Y^*]$, while varying ϵ and A.

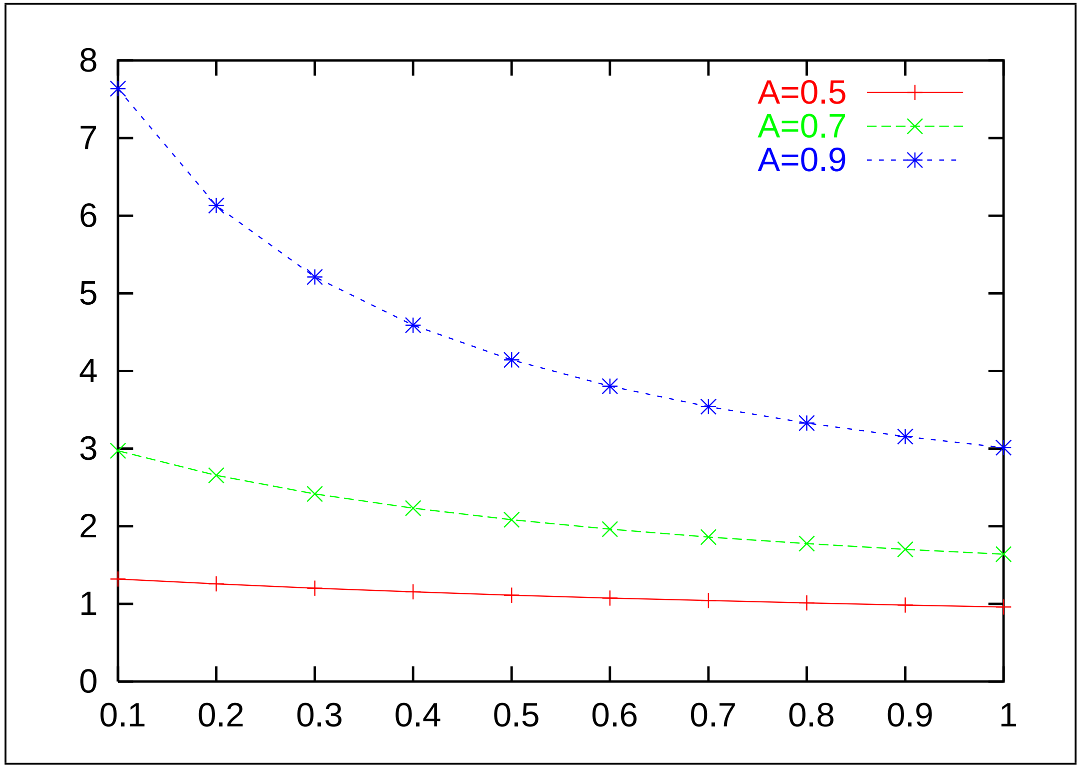

Figure 1: var$[Y*]$ as a function of ϵ and of A

Other issues:

•**No migration $Y_{n+1} = A_n(Y_n)$:** We can show using Kingman's subadditive ergodic theory that the following limit exists P-a.s.:

$$\lim_{n\to\infty} \frac{\log \|Y_n\|}{n} = \Lambda$$

•**The non contracting case:** In example 2 we have $\|A_n\| = 1$ so that $\|\mathrm{A}\| = 1$. Still the results hold.

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