Branching type processes

with Stationary Ergodic Immigration

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1 Background

- Most queueing Theory is Markovian

- Some results are insensitive to correlations, only depend on the the first moment. Example: MG1 PS queue.

- **Objective:** Develop tools for handling non Markovian queues.

- Examples of tools: Stochastic linear difference equations, branching processes.

Background on Branching

•19th centuty: concern among Victorians about possible extinction of aristocratic surnames.

•Galton posed this question in the *Educational Times* of 1873. The Reverand Watson replied with a solution. Joint publication of the solution in 1874.

•The G-W process: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)}$.

•The G-W process with immigratioon: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n$.

•Multitype: Y_n is a vector, $Y_{n+1} = A_n(Y_n) + B_n$, where

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n)$$

Example 1: discrete branching with migration

Queue with Vacations, Gated Regime

• $M/G/1/\infty$ queue,

• Arrival rate λ, i.i.d. service times $\{D_n\}$ with first and second moments d, $d^{(2)}$.

• Sequence of vacations: V_n. Will be assumed stationary ergodic, with first and second moments v, $v^{(2)}$.

• Gated regime: at the nth end of vacation, a gate is closed (nth polling instant). Then the server goes on serving the customers present at the queue at that polling instant:
Then the server leaves on vacation.

•We denote:

- B_n:= the number of arrivals during the nth vacation.
- $\xi_h^{(i)}$:= the number of arrivals during the service time of a customer

•Then:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n, \qquad n \geq n_0.$$

•**Divisibility property:** Denote

$$A_n(x) = \sum_{i=1}^{x} \xi_n^{(i)}$$

Then A_n are nonnegative and divisible:

$$A_n(x+y) = A_n^{(1)}(x) + A_n^{(2)}(y)$$

where $A_n^{(i)}$ are i.i.d.

Example 2: continuous branching with migration

Queue with Vacations, Gated Regime

- Define the time to serve N customers as:

$$\tau(N) := \sum_{i=1}^{N} D_i$$

- Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T.

- Denote by $\hat{\mathcal{A}}_n(C_n) = \tau(\mathcal{N}(C_n))$, i.e. the sum of service times of all the arrivals during C_n.

- We obtain

$$C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}. \tag{1}$$

Example 3: multitype discrete branching

Discrete time infinite server queue

• Service times are considered to be i.i.d. and independent of the arrival process.

• We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.

• The initial phase k is chosen at random according to some probability $p(k)$.

• If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.

• With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.

• P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

•Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

•The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

•Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

•Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

•B_n is a stationary ergodic sequence and has finite expectation.

•Y_n^i:= number of customers in phase i at time n. Satisfies

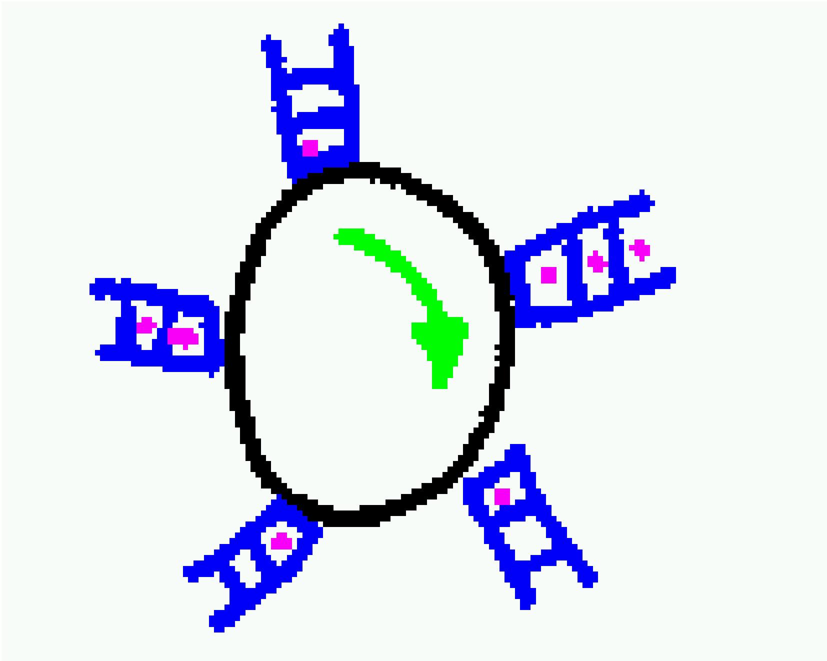
$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{2}$$

Example 4: Polling systems with N queues are special cases!

•The server moves cyclically (fixed order) between the queues $1, ..., M$.
It requires walking times (vacations) for moving from one queue to another.



•Upon arrival at a queue, some customers are served. The number to be served is determined by the "polling regime":

Globally Gated (GG) regime (Boxma, Levy, Yechiali 1992):
The cycle time satisfies a **one dimensional recursion**.
We obtained the first two moments of the cycle and the expected waiting times at all queues.

Gated and Exhaustive regimes [see e.g. book by Takagi 1986]:
satisfy M-dimensional recursive equations.

No explicit expression for 2nd moments of buffer occupancy or cycle times.

No explicit expression for the expected waiting times.

2 Introduction and Background on Lévy fields

Introduction

•Consider the stochastic recursive equation:

$$Y_{n+1} = A_n(Y_n) + B_n, \qquad n \geq n_0. \tag{3}$$

•Y_n is a vector in R_+^m

•$\{A_n\}_n$ are

- i.i.d., independent of B_n.
- Increasing in the arg for all n.
- nonnegative **Additive Lévy field taking values in** $\mathbb{R}_+^m$

•$\{B_n\}$ stationary ergodic taking values in $\mathbb{R}_+^m$

(3) defines a **Continuous Multitype Branching Process (BP) with Migration**

Background: Lévy processes

Lévy process taking values in $\mathbb{R}_+$:

- Example: Poisson Point Process with intensity λ,
- **Expectation** and **variance** are linear: $E[A(t)] = t\mathcal{A}$ and cov$[A(t)] = t\Gamma$.
- For random time τ independent of A,

$$E[A(\tau)] = E[\tau]\mathcal{A}\,, \qquad \mathrm{var}[A(\tau)] = \mathrm{E}[\tau]\Gamma + \mathrm{var}[\tau]\mathcal{A}^2\,,$$

- **Divisibility:** $A(\cdot)$ is divisible if the following holds.
  For any k, there exist $A^{(i)}(\cdot)$, $i = 0, ..., k$ such that for any non-negative $x_i, i = 0, ..., k$,

$$A\left(\sum_{i=0}^{k} x_i\right) = \sum_{i=0}^{k} A^{(i)}\left(x_i\right) \tag{4}$$

  where $\{A^{(i)}(\cdot)\}_{i=0,1,2,...,k}$ are i.i.d. with the same distribution as $A(\cdot)$.

Lévy process taking values in $\mathbb{R}_+^m$ (subordinators):

- Example: Poisson arrival process where the nth arrival brings a batch $B_n = (B_n^1, ..., B_n^m)$. B_n^i customers go to queue i.
- For $A(t)$ in $\mathbb{R}_+^m$, $E[A(t)] = \mathcal{A}t$ where $\mathcal{A}$ is of dimension m.
- $cov[A(t)] = \Gamma t,$ where Γ is a matrix of dimension $m \times m$.

Example of Random fields

Random field taking values in $\mathbb{R}_+$

- Example: Black and white picture.
- The level of grey is a function of two parameters: x and y.

Random field taking values in $\mathbb{R}_+^d$

- Example: color picture.
- The level of the green, red and blue as a function of the location x and y.

Background: Additive Lévy Fields

Let $A^{(1)},...,A^{(d)}$ be d indep. Lévy proc. on $\mathbb{R}^m$ with scalar "time" parameters.

Additive Lévy field: $A(y) = A^{(1)}(y_1) + ... + A^{(d)}(y_d)\,, \quad \forall y = (y_1,...,y_d) \in \mathbb{R}_+^d$.

The expectation: $\mathrm{E}[A(y)] = \sum_{j=1}^d y^j \mathcal{A}^{(j)} = \mathcal{A}y\,,$
$\mathcal{A}$ is a matrix whose jth column equals $\mathcal{A}^{(j)}$,
$\mathcal{A}^{(j)} = \mathrm{E}[A^{(j)}(1)]$,

The covariance matrix: $\mathrm{cov}[A(y)] = \sum_{j=1}^d y_j \Gamma^{(j)}\,,$
where $\Gamma^{(j)} = \mathrm{cov}[A^{(j)}(1)]$ is the corresponding covariance matrix of $A^{(j)}(1)$.

Composition: If A_n and A_{n+1} are Additive Lévy processes in $\mathbb{R}_+^m$ then their composition is also an Additive Lévy process.

Properties of Lévy Fields

•Expectation and Covariance are linear in y,

•Let τ be a non-negative random variable in $\mathbb{R}_+^d$, independent of A and represented as a column vector. Then

$$E[A(\tau)] = \sum_{j=1}^{m} \mathcal{A}^{(j)} E[\tau_j]\,,$$

and,

$$\mathrm{cov}[A(\tau)] = \sum_{j=1}^{d} \mathrm{E}[\tau_j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[\tau]\mathcal{A}^T\,, \tag{5}$$

where τ_j is the jth entry of the vector τ.

Result 1: Steady State Probabilities of CBP

Iterating $Y_{n+1} = A_n(Y_n) + B_n$, we obtain from A1:

$$\begin{aligned} Y_2 &= A_1(Y_1) + B_1 \\ &= A_1(A_0(Y_0) + B_0) + B_1 \\ &= A_1^{(0)}(A_0(Y_0)) + A_1^{(1)}(B_0) + B_1 \\ &= A_1^{(0)} A_0^{(0)}(Y_0) + A_1^{(1)}(B_0) + B_1. \end{aligned}$$

$$\begin{aligned} Y_3 &= A_2(Y_2) + B_2 \\ &= A_2(A_1(Y_1) + B_1) + B_2 \\ &= A_2(A_1(A_0(Y_0) + B_0) + B_1) + B_2 \\ &= A_2^{(0)} A_1^{(0)} A_0^{(0)}(Y_0) + A_2^{(1)} A_1^{(1)}(B_0) + A_2^{(2)}(B_1) + B_2 \end{aligned}$$

In general:

$$Y_n = \sum_{j=0}^{n-1} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}) + \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (Y_0), \quad n > 0 \qquad (6)$$

(we understand $\prod_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\prod_{i=n}^{k} A_i(x) = A_k A_{k-1} ... A_n$ whenever $k > n$).

•Under fairly general assumptions, $\lim_{n\to\infty}\left(\prod_{i=0}^{n-1} A_i^{(0)}\right)(y) = 0$, so Y_n has a limit as $n \to \infty$ distributed like

$$Y_n^* =_d \sum_{j=0}^{\infty}\left(\prod_{i=n-j}^{n-1} A_i^{(n-j)}\right)(B_{n-j-1}), \qquad n \in Z, \tag{7}$$

where for each integer i, $\{A_i^{(j)}(\cdot)\}_j$ have the same distribution as $A_i(\cdot)$.

•Sufficient condition: stationarity plus $\|\mathcal{A}\| < 1$.

•Branching processes: $\{A_i^{(j)}(\cdot)\}_j$ are i.i.d.

•Stochastic differential equations: they are equal.

•The representation holds for general dependence: Semi linear processes.

Application: Expected waiting time for a gated queue with vacations

Consider an arbitrary customer. Upon arrival, it has to wait for

1. The residual cycle time C_{res},
2. The service time of all the customers that arrived during C_{past} which is the past cycle time: $d(\lambda E[C_{past}]) = \rho E[C_{past}]$

We have from [Baccelli & Brémaud, 1994]

$$E[C_{res}] = E[C_{past}] = \frac{E[C_0^2]}{2E[C_0]}.$$

Thus the expected waiting time of an arbitrary customer is given by

$$E[W_n] = (1+\rho)\frac{E[C_0^2]}{2E[C_0]},$$

The expected number of customers in queue in stationary regime (not including service) is obtained using Little's Theorem: $\lambda E[W_n]$.

Conclusion: we need to compute $E[C_0]$ and $E[C_0^2]$!

Computing $E[C_0]$ and $E[C_0^2]$

•Dynamics: $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$.

•$\hat{\mathcal{A}}_n(c)$ is the workload that arrives during duration $[0, c)$.

•Introduce the correlation function: $r(n) = E[V_0 V_n]$.

•The first and second moments of C_n in stationary regime are given by

$$E[C_n] = \frac{v}{1-\rho},$$

$$E[C_n^2] = \frac{1}{(1-\rho^2)} \left(\frac{\lambda v d^{(2)}}{1-\rho} + r(0) + 2 \sum_{j=1}^{\infty} \rho^j r(j) \right). \qquad (8)$$

Proof of expressions for $E[C_0^2]$

Useful relations: 2nd moment of workload arriving during T

•If N is a random variable independent of the sequence D_n, and $\tau(N) := \sum_{i=1}^{N} D_i$ then

$$E[\tau(N)^2] = E[N^2]d^2 + E[N](d^{(2)} - d^2). \tag{9}$$

•Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T. Then

$$E[\mathcal{N}(T)^2] = \lambda^2 E[T^2] + \lambda E[T]. \tag{10}$$

•If we take an arbitrary T and choose $N = \mathcal{N}(T)$, then we get from (9)-(10)

$$\begin{aligned} E[(\hat{\mathcal{A}}(T))^2] &= E[\tau(\mathcal{N}(T))^2] \\ &= d^2(\lambda^2 E[T^2] + \lambda E[T]) + \lambda E[T](d^{(2)} - d^2) \\ &= d^2\lambda^2 E[T^2] + \lambda E[T] d^{(2)}. \end{aligned} \tag{11}$$

•Also, if we take $T = \tau(N)$, then

$$E[\mathcal{N}(\tau(N))]^2 = \lambda^2 \Big[E[N^2]d^2 + E[N](d^{(2)} - d^2) \Big] + \lambda d E[N]. \tag{12}$$

• From $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$ we have

$$\begin{aligned} E[C_{n+1}^2] &= E[\hat{\mathcal{A}}_n(C_n)^2] + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] \\ &= \left(\rho^2 E[C_n^2] + \lambda E[C_n] d^{(2)}\right) + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}]. \end{aligned}$$

• To compute the last term, we now use the explicit form of C_0:

$$C_0 = \sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}).$$

• We use the fact that the processes $\{\hat{\mathcal{A}}_i^{(j)}\}$ are independent of $\{V_n\}$. We get:

$$\begin{aligned} E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] &= E[\hat{\mathcal{A}}_0(C_0)V_1] = E\left[\hat{\mathcal{A}}_0 \left(\sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}) \right) V_1 \right] \\ &= \rho \sum_{j=0}^{\infty} \rho^j E[V_{-j} V_1] = \sum_{j=1}^{\infty} \rho^j r(j). \end{aligned}$$

Substituting this, we obtain the second moment.

3 2nd order moments in continuous B.P.

Joint work with Dieter Fiems

Notation: •Auto-correlations: $\mathcal{B}(k) =_{def} \mathrm{E}[B_0(B_k)^T]$, where k is an integer
•$\hat{\mathcal{B}}(k) =_{def} \mathcal{B}(k) - \mathrm{E}[B_0]\,\mathrm{E}[B_0]^T$. (Note: $\hat{\mathcal{B}}(0)$ equals $\mathrm{cov}[B_0]$.)

Assumptions: Consider $Y_{n+1} = A_n(Y_n) + B_n,\ n \geq n_0$, where

- A_n are i.i.d. additive Lévy fields,
- A_n independent of $\{B_n\}$,
- $\{B_n\}$ are stationary ergodic,
- All eigenvalues of $\mathcal{A}$ are within the unit disk,
- the elements of B_0 have finite second order moments.

Theorem: Consider $Y_{n+1} = A_n(Y_n) + B_n$ in stationary regime. Then

(i) $\mathrm{E}[Y_0] = (\mathcal{I} - \mathcal{A})^{-1}\,\mathrm{E}[B_0]\,,$

(ii) $\mathrm{cov}(Y_0)$ is the unique solution of the linear equations:

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j\hat{\mathcal{B}}(j) + (\mathcal{A}^j\hat{\mathcal{B}}(j))^T\,, \tag{13}$$

where $\mathrm{E}[Y_0^j]$ denotes the jth element of $\mathrm{E}[Y_0]$.

Proof for first moments:

Taking expectation in $Y_{n+1} = A_n(Y_n) + B_n$ we get

$$\mathrm{E}[Y_0] = \mathcal{A}\,\mathrm{E}[Y_0] + \mathrm{E}[B_0],$$

Since the eigenvalues of $\mathcal{A}$ are within the unit disk, $(\mathcal{I} - \mathcal{A})$ is inverible.
Hence we obtain (i).

Proof of uniqueness for the second moments

•Let Z_1 and Z_2 be two solutions of

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + (\mathcal{A}^j \hat{\mathcal{B}}(j))^T \,.$$

•Define $Z = Z_1 - Z_2$. Then Z satisfies $Z = \mathcal{A}^T Z \mathcal{A}$.

•Iterating, we obtain,

$$Z = \lim_{n \to \infty} \mathcal{A}^n Z (\mathcal{A}^T)^n = 0$$

where the last equality follows from the fact that all the eigenvalues of $\mathcal{A}$ are within the unit disk.

•This implies uniqueness.

Proof for expression of second moments

•Consider $Y_{n+1} = A_n(Y_n) + B_n$.

- Multiply both sides by their transpose,
- take expectation and
- use the stationarity

we get:

$$\mathrm{E}[Y_0 Y_0^T] = \mathrm{E}[A_0(Y_0) A_0^T(Y_0)] + \mathrm{E}[B_0 B_0^T] + \mathrm{E}[A_0(Y_0) B_0^T] + \mathrm{E}[B_0 A_0^T(Y_0)] \,.$$

The covariance matrix $\mathrm{cov}[Y_0]$ therefore equals,

$$\begin{aligned} \mathrm{cov}[Y_0] \quad = \quad & \mathrm{cov}[A_0(Y_0)] + \mathrm{cov}[B_0] + \mathrm{E}\left[A_0(Y_0) B_0^T\right] \\ & - \mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T + \mathrm{E}\left[B_0 A_0(Y_0)^T\right] - \mathrm{E}[B_0](\mathcal{A}\,\mathrm{E}[Y_0])^T \,. \qquad (14) \end{aligned}$$

It remains to compute the red and the blue expressions.

Red Expression: Using the convariance expression (5) of Additive Lévy processes at random "time":

$$\mathrm{cov}[A_0(Y_0)] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T\,. \tag{15}$$

Blue Expression: We use the explicit expression (7) for the stationary state process to obtain

$$\begin{aligned}
\mathrm{E}[Y_0 B_0^T] &= \sum_{j=0}^{\infty} \mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \middle| \mathbf{B_0^-} \right) \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathcal{A}^j B_{-j-1} B_0^T \right) = \sum_{j=0}^{\infty} \mathcal{A}^j \mathcal{B}(j+1)\,,
\end{aligned} \tag{16}$$

with $\mathbf{B_0^-} := (B_0, B_{-1}, B_{-2}, ...)$.

Substituting the last expression, we compute,

$$\mathrm{E}[A_0(Y_0)B_0^T] = \mathrm{E}\left[\mathrm{E}\left[A_0(Y_0)B_0^T \,\middle|\, Y_0, B_0\right]\right] = \mathcal{A}\,\mathrm{E}\left[Y_0 B_0^T\right] = \sum_{j=1}^{\infty} \mathcal{A}^j \mathcal{B}(j)\,,$$

or equivalently,

$$\begin{aligned}
\mathrm{E}[A_0(Y_0)B_0^T] &= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \sum_{j=1}^{\infty} \mathcal{A}^j\,\mathrm{E}[B_0]\,\mathrm{E}[B_0]^T \\
&= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}(\mathcal{I}-\mathcal{A})^{-1}\,\mathrm{E}[B_0]^T \\
&= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T\,. \qquad (17)
\end{aligned}$$

Substitution of expressions RED and BLUE provides the covariance equation.

4 Symmetric gated polling systems

m gated queues.

Arrivals:

- Arrival processes $\rho^i(t)$ to queue i are i.i.d. Levy processes, distributed as some $\rho(t)$, $t \in \mathbb{R}_+$.
- $\overline{\rho} = \mathrm{E}[\rho(1)]$ and $\sigma^2 = \mathrm{var}[\rho(1)]$

Walking times:

- $\{V_n\}$: Stationary ergodic series of walking times, $v := \mathrm{E}[V_0]$.
- $\mathcal{V}(j) := \mathrm{E}[V_0 V_j]$ for some integer j and $\hat{\mathcal{V}}(j) := \mathrm{E}[V_0 V_j] - v^2$.

Notation:

- $I(n)$:= the queue visited at the nth polling instant
- $S(n)$:= nth polling instant (time at which the server arrives at the nth queue)
- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$

The dynamics:

$$
\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n \,, \qquad (18)\\
Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n \,,\\
Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,,\\
&\vdots\\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,.
\end{aligned}
$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = the workload that arrived at queue $I(n)$ during the nth cycle.

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Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n\,,\\
&\ \vdots\\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n\,.
\end{aligned}
$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus **the nth vacation time;**

•The busy period = the workload that arrived at queue $I(n)$ during the nth cycle.

Notation:

- $I(n)$:= the queue visited at the nth polling instant
- $S(n)$:= nth polling instant (time at which the server arrives at the nth queue)
- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$

The dynamics:

$$\begin{aligned} Y_{n+1}^1 &= S(n+1) - S(n) = \boldsymbol{\rho_n^m(Y_n^m)} + V_n \,, \qquad (18) \\ Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\ Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\ &\vdots \\ Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,. \end{aligned}$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = **workload that arrived at queue $I(n)$ during the nth cycle.**

Interpretation of the other equations:

For $i > 0$, we have

$$Y_{n+1}^{i+1} = S(n+1) - S(n-i) = S(n+1) - S(n) + S(n) - S(n-i)$$

where

•by definition, $S(n) - S(n-i) = Y_n^i$, and

•$S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n$ (see previous slide).

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ with}$$

$$\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = & \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^2 &= S(n+1) - S(n-1) = & Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^3 &= S(n+1) - S(n-2) = & Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\
&\vdots \\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = & Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,.
\end{aligned}$$

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ with}$$

$$\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = & \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^2 &= S(n+1) - S(n-1) = & Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^3 &= S(n+1) - S(n-2) = & Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\
&\vdots \\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = & Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,.
\end{aligned}$$

where $Y_{n+1} = (Y_{n+1}^1, ..., Y_{n+1}^m)^T$,

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ with}$$

$$\begin{aligned}
Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n\,,\\
Y^2_{n+1} &= S(n+1) - S(n-1) = & Y^1_n + \rho^m_n(Y^m_n) + V_n\,,\\
Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n\,,\\
&\vdots \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n\,.
\end{aligned}$$

where $B_n = V_n(1,1,1,...,1)^T$,

•in the special case that $\{B_n\}$ is i.i.d. Y_n is a Markov chain

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ with}$$

$$\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = & \rho_n^m(Y_n^m) + V_n\,,\\
Y_{n+1}^2 &= S(n+1) - S(n-1) = & Y_n^1 + \rho_n^m(Y_n^m) + V_n\,,\\
Y_{n+1}^3 &= S(n+1) - S(n-2) = & Y_n^2 + \rho_n^m(Y_n^m) + V_n\,,\\
&\vdots & \\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = & Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n\,.
\end{aligned}$$

$$Y_{n+1} = Y_n^1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + Y_n^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + Y_n^{m-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + \rho_n^m(Y_n^m) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + B_n$$

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ where}$$

$$A_n(y) = A_n^{(1)}(y_1) + ... + A_n^{(m)}(y_m)\,, \tag{19}$$

where $y = (y_1, ..., y_m)^T \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$ and

$$\begin{aligned} A_n^{(1)}(t) &= (0, t, 0, 0, ..., 0)^T, \\ A_n^{(2)}(t) &= (0, 0, t, 0, ..., 0)^T, \\ &\vdots \\ A_n^{(m-1)}(t) &= (0, 0, 0, ..., 0, t)^T, \\ A_n^{(m)}(t) &= \rho_n^m(t)(1, 1, \ldots, 1)^T\,, \end{aligned} \tag{20}$$

•For each i, $A_n^{(i)}$ is a Lévy **process** taking values in $\mathbb{R}_+^m$.

•A_n are Additive Lévy fields

Checking the stability condnition

Taking expectation we get:

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 1 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 1 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 1 & 0 & \dots & 0 & \overline{\rho} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 0 & 0 & \dots & 1 & \overline{\rho} \end{pmatrix}. \tag{21}$$

$\mathcal{A}$ is known as the *Companion matrix*.

Theorem: A sufficient and necessary condition for all eigenvalues of $\mathcal{A}$ to be in the interior of the unit circle is

$$\overline{\rho} < \frac{1}{m}.$$

Conclusions and Discussion

• We use neither the "buffer occupancy" nor the "station times" approaches.

• Advantage: one component of the state is the cycle time;
its two first moments provide the expected waiting time.

• A very similar structure is obtained in the exhaustive case.

5 Semi linear processes

We shall assume that A_n satisfy the following conditions:

A1: $A_n(y)$ has the following **divisibility property**: if for some k, $y = y^0 + y^1 + ... + y^k$ where y^m are vectors, then $A_n(y)$ can be represented as

$$A_n(y) = \sum_{i=0}^{k} \widehat{A}_n^{(i)}(y^i)$$

where $\{\widehat{A}_n^{(i)}\}_{i=0,1,2,\dots,k}$ are identically distributed with the same distribution as $A_n(\cdot)$.

A2: (i) There is some matrix $\mathcal{A}$ such that for every y,

$$E[A_n(y)] = \mathcal{A}y.$$

(ii) The correlation matrix of $A_n(y)$ is linear in yy^T and in y. We shall represent it as

$$E[A_n(y)A_n(y)^T] = F(yy^T) + \sum_{j=1}^{d} y_j \Gamma^{(j)}\,, \tag{22}$$

where F is a linear operator that maps $d \times d$ nonnegative definite matrices to other $d \times d$ nonnegative definite matrices and satisfies $F(0) = 0$.

Moments:

•(i) The first moment of X_n^* is given by

$$E[X_0^*] = (I - \mathcal{A})^{-1} b. \tag{23}$$

•(ii) Assume that the first and second moments b_i and $b_i^{(2)}$'s are finite and that F satisfies

$$\lim_{n \to \infty} F^n = 0. \tag{24}$$

Define Q to be the matrix whose ijth entry is $Q_{ij} = \sum_{k=1}^{d} \overline{y}_k \Gamma^{(k)}$. Then the matrix $cov(X^*)$ is the unique solution of the set of linear equations:

$$cov(X) \quad = \quad cov(B) + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \widehat{\mathcal{B}}(r) + \left[\mathcal{A}^r \widehat{\mathcal{B}}(r) \right]^T \right) + F(cov[X]) + Q. \tag{25}$$

The second moment matrix $E[XX^T]$ in steady state is the unique solution of the set of linear equations:

$$E[XX^T] \quad = \quad E[B_0 B_0^T] + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \mathcal{B}(r) + \left[\mathcal{A}^r \mathcal{B}(r) \right]^T \right) + F(E[XX^T]) + Q \tag{26}$$

6 Example: Discrete time infinite server queue

Example 5: Discrete time infinite server queue

- Service times are considered to be i.i.d. and independent of the arrival process.
- We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.
- The initial phase k is chosen at random according to some probability $p(k)$.
- If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.
- With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.
- P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

• Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

• The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

• Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

• Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

• B_n is a stationary ergodic sequence and has finite expectation.

• $Y_n^i :=$ number of customers in phase i at time n. Satisfies

$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{27}$$

•Numerical example: Service times are geometrically distributed,

•The SRE becomes one dimensional. Y_n denotes the number of customers in the system.

•$\xi_n^{(k)}$ is the indicator that the kth customer present at the beginning of time-slot n will still be there at the end of the time-slot.

•The probability that a customer in the system finishes its service within a time slot is precisely $p = 1 - \mathsf{A} = 1 - E[\xi_n]$.

•We consider a Markov chain with two states $\{\gamma, \delta\}$ with transition probabilities given by

$$\mathcal{P} = \begin{pmatrix} 1 - \epsilon p & \epsilon p \\ \epsilon q & 1 - \epsilon q \end{pmatrix}$$

•As an example, consider the following parameters: $p = q = 1$, at a given state there is at most one arrival with prob. $p_\gamma = 1, p_\delta = 0.5$. This gives:

$$var[Y^*] = \frac{1}{(1 - \mathsf{A}^2)} \left(\frac{3}{16} + \frac{2\mathsf{A}}{1 - \mathsf{A} + 2\epsilon \mathsf{A}} + \frac{3}{4}\mathsf{A} \right).$$

In Fig. 1 we plot the variance of the steady state number of customers, $var[Y^*]$, while varying ϵ and A.

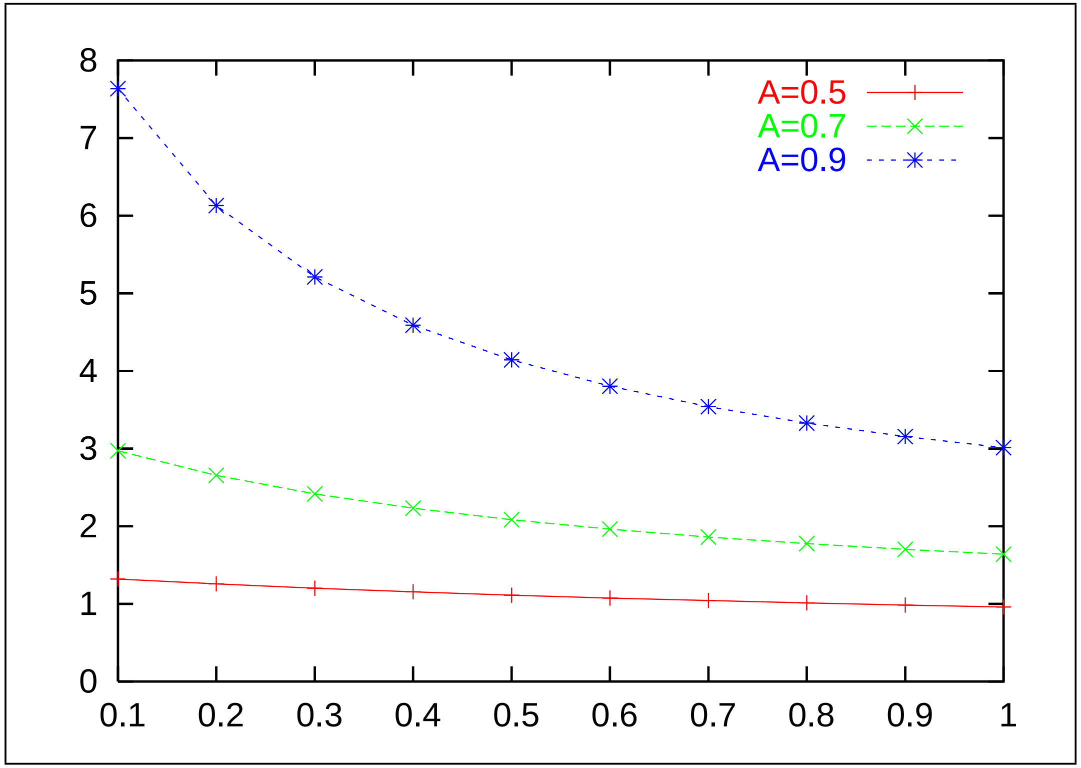

Figure 1: var$[Y*]$ as a function of ϵ and of A

7 Example: Delay Tolerant Ad-hoc Networks

•Delay tolerant Ad-hoc Networks make use of nodes' mobility to compensate for lack of instantaneous connectivity.

•Information sent by a source to a disconnected destination can be forwarded and relayed by other mobile nodes.

•Let X_n^+ be the number of nodes that have a copy of the packet at time n,

•Let X_n^- be the number of nodes that do not have a copy of the packet at time n.

•Mobility: a mobile present at time n may leave and other may arrive. Let B_n be the number of new arrivals.

• Let $\rho_n^{(i)}$ and $\hat{\rho}_n^{(i)}$ be the indicator that node i remains in the system for the next slot. ρ is used for nodes that have the packet and $\hat{\rho}$ for the others.

• Let $\xi_n^{(i)}$ be the indicator that the source meats mobile i at time slot n. These are i.i.d. Then

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

Controling the Energy

- Assume that the source limits the transmissions in order to save energy
- Let ζ_n be the indicator that the source intends to transmit a packet at time n. Assume ζ_n are i.i.d.

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \zeta_n \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \zeta_n \xi_n^{(i)}) + B_n$$

- This is a semi-linear process, not a branching process

Partial Information

•Observations: Assume that each node is "sampled" each time unit with some small probability.

•Let $\alpha_n^{(i)}$ be the indicator that node i is sampled at time n.

$$X_{n+1}^{+} = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^{-} = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

$$Y_{n+1} = \sum_{i=1}^{X_n^+} \alpha_n^{(i)}$$

Filtering

- Objective: monitoring the number of packets.
- Example: first order linear filter
- Let $\widehat{X}_n^+$ be the estimator of X_n^+, The estimation error is $\epsilon_n = \hat{X}_n^+ - X_n^+$.

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

$$Y_{n+1} = \sum_{i=1}^{X_n^+} \alpha_n^{(i)}$$

$$\hat{X}_{n+1} = K(Y_n - \hat{X}_n) + (1-K)\hat{X}_n$$

$$\epsilon_n = \hat{X}_n - X_n$$

- Semi-linear process. We can compute $E[\epsilon_n^2]$ and compute K that minimizes it.

8 Evolving files in a mobile environment

by E Altman, P Nain, JC Bermond

- N+1 mobile nodes (called nodes) including one source node (called source)

- q = prob. any node meets source in a time-slot state of a node : age of file it carries (if any). $\zeta_{n,j}^{(i)}$ is the indicator.

- File of age K+1 has no utility anymore (state 0)

- File of age j = state j, j = 1, ... ,K

- When a node in state j meets source, latest version of file transmitted to node n with prob. a_j. $\xi_{n,j}^{(i)}$ is the indicator.

- X_n^j is the number of mobiles with a file with age j at generation n

- For $0 < j < K$:

$$X_{n+1}^{j+1} = \sum_{i=1}^{X_n^j} (1 - \zeta_{n,j}^{(i)} \xi_{n,j}^{(i)})$$

- The number of nodes without a file:

$$X_{n+1}^{0} = \sum_{i=1}^{X_n^0} (1 - \xi_{n,0}^{(i)} \zeta_{n,0}^{(i)}) + \sum_{i=1}^{X_n^K} (1 - \xi_{n,K}^{(i)} \zeta_{n,K}^{(i)})$$

- The number of nodes with file of age 1:

$$X_{n+1}^{1} = \sum_{j=0}^{K-1} \sum_{i=1}^{X_n^j} \xi_{n,j}^{(i)} \zeta_{n,j}^{(i)}$$

Open network

- $\rho_{n,j}^{(i)}$ is the indicator that the node will remain till next slot.

- B_n is the number of arrivals of mobiles at slot n.

- For $0 < j < K$:

$$X_{n+1}^{j+1} = \sum_{i=1}^{X_n^j} (1 - \zeta_{n,j}^{(i)} \xi_{n,j}^{(i)}) \rho_{n,j}^{(i)}$$

- The number of nodes without a file:

$$X_{n+1}^{0} = B_n + \sum_{i=1}^{X_n^0} (1 - \xi_{n,0}^{(i)} \zeta_{n,0}^{(i)}) \rho_{n,0}^{(i)} + \sum_{i=1}^{X_n^K} (1 - \xi_{n,K}^{(i)} \zeta_{n,K}^{(i)}) \rho_{n,K}^{(i)}$$

- The number of nodes with file of age 1:

$$X_{n+1}^{1} = \sum_{j=0}^{K-1} \sum_{i=1}^{X_n^j} \xi_{n,j}^{(i)} \zeta_{n,j}^{(i)} \rho_{n,j}^{(i)}$$

- **Payoff:** Weighted sum of number of copies
- **Cost:** Transmission energy, proportional to X_n^1.

- Linear cost:

$$Maximize \quad \sum_{j=1}^{K} \alpha_j E[X_n^j] - cE[X_n^1]$$

Gives rise to a threshold optimal policy.

- Quadratic cost:

$$Maximize \quad \left(\sum_{j=1}^{K} \alpha_j E[X_n^j] \right)^2 - cE[X_n^1]^2$$

Gives rise to linear optimal policy.

9 Random Environment

Random column vectors $X_n \in \mathbb{R}^M$, satisfying

$$X_{n+1} = A_n(X_n, Y_n) + B_n(Y_n)\,, \quad n \in \mathbb{Z}\,. \tag{28}$$

$Y = \{Y_n\}$ denotes a Markov chain, taking values on a finite state-space $\Theta = \{1, 2, \ldots, N\}$ whereas A_n and B_n denote random vector-valued functions with domain $\mathbb{R}^M \times \Theta$ and Θ, respectively. The functions $A_n(\cdot, y)$ are independent random variables for all $y \in \Theta$, $n \in \mathbb{Z}$ and further adhere to the following assumptions.

- For each $y \in \Theta$, $A_n(\cdot, y)$ has a divisibility property. Let $x = x^1 + x^2 + \ldots + x^k \in \mathbb{R}^M$, then $A_n(x, y)$ has the following representation,

$$A_n(x, y) = \sum_{j=1}^{k} \hat{A}_n^{(j)}(x^j, y)\,, \tag{29}$$

whereby $\hat{A}_n^{(j)}(\cdot, y)$, $j = 1, \ldots, k$, are identically distributed, but not necessarily independent, and have the same distribution as $A_n(\cdot, y)$.

- For each $y \in \Theta$, $A_n(\cdot, y)$ is linear in the mean,

$$\mathrm{E}[A_n(x,y)] = \mathcal{A}_y^{(n)} x \,, \qquad x \in \mathbb{R}^M, y \in \Theta \,, n \in \mathbb{Z} \,. \tag{30}$$

Here $\{\mathcal{A}_y^{(n)}, y \in \Theta, n \in \mathbb{Z}\}$ is a set of fixed $M \times M$ matrices. Further, for each $y \in \Theta$, the correlation matrix of $A_n(x,y)$ is linear in xx' and in x. For all $x = [x_1, \ldots, x_M] \in \mathbb{R}^M$, we have the following representation,

$$\mathrm{E}[A_n(x,y)A_n'(x,y)] = F_y^{(n)}(xx') + \sum_{j=1}^{M} x_j \Gamma_{y,j}^{(n)} \,, \tag{31}$$

$y \in \Theta, n \in \mathbb{Z}$. For each $y \in \Theta$ and $n \in \mathbb{Z}$, $F_y^{(n)}$ is a linear operator that maps $M \times M$ non-negative definite matrices on $M \times M$ non-negative definite matrices and satisfies $F_y^{(n)}(0) = 0$. Further, $\{\Gamma_{y,j}^{(n)}, y, j \in \Theta, n \in \mathbb{Z}\}$ is a set of fixed $M \times M$ matrices.

Before proceeding to our main results, we introduce some additional notation. Let $p_{ij}^{(n)} = \Pr[Y_{n+1} = j | Y_n = i]$ denote the transition probability of the Markov chain Y_k at time n $(i, j \in \Theta)$ and let $P^{(n)} = [p_{ij}^{(n)}]$ denote the corresponding transition matrix. The probability that the Markov chain is in state k at time n is denoted by $\pi_k^{(n)} = \Pr[Y_n = k]$. For the immigration process B_n, the following notation is introduced for the first and second order moments,

$$b_i^{(n)} = \mathrm{E}[B_n(i)]\,, \quad \mathcal{B}_{ij}^{(m,n)} = \mathrm{E}[B_m(i)B_n(j)]$$

Finally, the following block matrices and block vector are defined to simplify further notation,

$$\hat{\mathcal{A}}^{(n)} = \begin{bmatrix} \mathcal{A}_1^{(n)} p_{11}^{(n)} & \mathcal{A}_2^{(n)} p_{21}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{N1}^{(n)} \\ \mathcal{A}_1^{(n)} p_{12}^{(n)} & \mathcal{A}_2^{(n)} p_{22}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{N2}^{(n)} \\ \vdots & & \ddots & \vdots \\ \mathcal{A}_1^{(n)} p_{1N}^{(n)} & \mathcal{A}_2^{(n)} p_{2N}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{NN}^{(n)} \end{bmatrix},$$

$$\hat{b}^{(n)} = \sum_{k \in \Theta} \pi_k^{(n)} \begin{bmatrix} p_{k1}^{(n)} b_k^{(n)} \\ p_{k2}^{(n)} b_k^{(n)} \\ \vdots \\ p_{kN}^{(n)} b_k^{(n)} \end{bmatrix},$$

$$\hat{\mathcal{B}}^{(m,n)} = \sum_{k \in \Theta} \pi_k^{(m)} \mathcal{L}_k^{(m,n)},$$

$$\mathcal{L}_k^{(m,n)} = \begin{bmatrix} \mathcal{B}_{k1}^{(m,n)} p_{k1}^{(m)} & \mathcal{B}_{k2}^{(m,n)} p_{k1}^{(m)} & \dots & \mathcal{B}_{kN}^{(m,n)} p_{k1}^{(m)} \\ \mathcal{B}_{k1}^{(m,n)} p_{k2}^{(m)} & \mathcal{B}_{k2}^{(m,n)} p_{k2}^{(m)} & \dots & \mathcal{B}_{kN}^{(m,n)} p_{k2}^{(m)} \\ \vdots & & \ddots & \\ \mathcal{B}_{k1}^{(m,n)} p_{kN}^{(m)} & \mathcal{B}_{k2}^{(m,n)} p_{kN}^{(m)} & \dots & \mathcal{B}_{kN}^{(m,n)} p_{kN}^{(m)} \end{bmatrix}.$$

Stationary analysis

Assumptions: •(i) the process $\{B_n, n \in \mathbb{Z}\}$ is stationary ergodic; •(ii) the Markov chain Y_n is ergodic and •(iii) the processes A_n are independent and identically distributed.

Simplify notation as follows: $\mathcal{A}_i^{(n)} = \mathcal{A}_i$, $b_i^{(n)} = b_i$, $\hat{b}^{(n)} = \hat{b}$, $\mathcal{B}_{ij}^{(m,n)} = \mathcal{B}_{ij}^{(n-m)}$, $p_{ij}^{(n)} = p_{ij}$, $P^{(n)} = P$, $\pi_k^{(n)} = \pi_k$, $\hat{\mathcal{A}}^{(n)} = \hat{\mathcal{A}}$ and $\hat{\mathcal{B}}^{(m,n)} = \hat{\mathcal{B}}^{(n-m)}$.

For any $y \in \mathbb{R}^M$, let $\bigotimes_{i=n}^k A_i(x, Y_i) = x$ for $k < n$ whereas, for $k \geq n$, this operator is defined by the following recursion,

$$\bigotimes_{i=n}^{k} A_i(x, Y_i) = A_k \left(\bigotimes_{i=n}^{k-1} A_i(x, Y_i), Y_k \right) .$$

•**Theorem** Assume that (i) $b_i < \infty$ component-wise for all $i \in \Theta$; and (ii) that all the eigenvalues of the matrix $\hat{\mathcal{A}}$ are within the open unit disk. Then, there exist a unique stationary solution X_n^*, distributed like,

$$X_n^* =_d \sum_{j=0}^{\infty} \bigotimes_{i=n-j}^{n-1} \hat{A}_i^{(n-j)} (B_{n-j-1}(Y_{n-j-1}), Y_i) \,, \tag{32}$$

for $n \in \mathbb{Z}$. The sum on the right side of the former expression converges absolutely almost surely. Furthermore, one can construct a probability space such that $\lim_{n\to\infty} \|X_n - X_n^*\| = 0$, almost surely, for any initial value X_0.

Let $\hat{\mu}_n(x)$ denote the block column vector with elements,

$$\mathrm{E}\left[\bigotimes_{i=0}^{n-1} \hat{A}_i^{(0)}(x, Y_i)\, 1\{Y_n = l\}\right], \quad l \in \Theta\,, x \in \mathbb{R}^M.$$

Let μ be the block column vector with elements $\mu_i \triangleq \mathrm{E}[X_0^* 1\{Y_0 = i\}]$, $i \in \Theta$.

Let Ω, be the block column vector with elements $\Omega_i \triangleq \mathrm{E}[X_0^*(X_0^*)' 1\{Y_0 = i\}]$, $i \in \Theta$.

Theorem Assume that the stability conditions of the previous Theorem are satisfied. The conditional first moment vector is then given by,

$$\mu = (\mathcal{I} - \hat{\mathcal{A}})^{-1}\hat{b}\,. \tag{33}$$

Under the additional assumption that the second moments of $B_0(i)$ are finite, $i \in \Theta$, the conditional second moment matrices Ω_i of X_0^* are the unique solution of,

$$\Omega_l = \sum_{k\in\Theta}\Big(F_k(\Omega_k) + \sum_{j=1}^{M}\mu_k^{(j)}\Gamma_k^{(j)} + \mathcal{B}_{kk}^{(0)}\pi_k + \mathcal{A}_k\Lambda_k + \Lambda_k'\mathcal{A}_k'\Big)p_{kl}\,,$$

$l \in \Theta$, where Λ_k denotes the kth diagonal (block) element of $\sum_{j=0}^{\infty}\hat{\mathcal{A}}^j\hat{\mathcal{B}}^{(j+1)}$ and with $\mu_k^{(j)}$ the jth element of μ_k.

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