From Mean Field Interaction to Evolutionary Game Dynamics

Hamidou Tembine, Jean-Yves Le Boudec, Rachid ElAzouzi, Eitan Altman
LCA, Ecole Polytechnique Federale de Lausanne
LIA/CERI, University of Avignon
MAESTRO Group, INRIA

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Plan

Mean Field Interactions (MFI)

Random selection among finite number players (Fast Simulation)

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Evolutionary games with random number of players

- Large population. At each time, there are several local interactions among random number of players.
- The population profile evolves according to some evolutionary process, learning process, adaptive process, optimization process etc.

Aims

- Evolution of the population profile of ”type-action” $M^N(t)$
- Convergence to mean field when the population size grows
  - Study of the random process $M^N = \frac{1}{N}\sum_j \delta_{S_j^N}$
  - Asymptotics of $M_s^N(t) = \frac{1}{N}\sum_j \delta_{\{S_j^N(t)=s\}}$ when t goes to $+\infty$.
  - Asymptotics of $M^N(t)$ when N goes to $+\infty$.
- ODE of $m(t) := \lim_{N\longrightarrow\infty} M^N(t)$ (or accumulation point, $w-$limits etc)
- From mean field interactions to population dynamics
- Evolutionary stability and equilibria

Mean Field Interactions (description)

Let $S = \{(\theta, a), a \in A_\theta\}$ be the set of "type-actions". Assume S **finite.** After an encounter between k players in $\mathcal{B}^N(t)$ (random set), the variation of the population profile $M^N(t) \nrightarrow M^N(t + \Delta_N)$. The player $j \in \mathcal{B}^N(t)$ receives an instantaneous cost $\mathcal{C}^{N,\theta^j}(S_j^N(t), S_{\mathcal{B}^N \backslash j}^N(t))$.

New states

$S_j^N(t + \Delta_N)$ is drawn according to

$L_{\theta^j}^N \left(. | S_j^N(t), S_{\mathcal{B}^N \backslash j}^N(t), j \in \mathcal{B}^N \right)$.

Drift

$$f^N(m) := \mathbb{E}\left(M^N(t + \Delta_N) - M^N(t) \mid M^N(t) = m, \mathcal{B}^N(t + \Delta_N) \right)$$

Non-commutative diagram ?

$$
\begin{tikzcd}
M^N(t) \arrow[r, "t \longrightarrow +\infty"] \arrow[d, "N \longrightarrow +\infty"] & \varpi^N \arrow[d, "N \longrightarrow +\infty"] \\
m(t) \arrow[r, "t \longrightarrow +\infty"] & ?
\end{tikzcd}
$$

Let

$$J_{k_1,\ldots,k_\Theta}(m) := \mathbb{P}\left(\sharp B_\theta^N(t+\Delta_N) = k_\theta,\ \theta = 1,\ldots\ \Theta \mid M^N(t) = m\right)$$

Assumption 1 :

$$\forall m, \sum_k (k_1 + \ldots + k_\theta)^2 J_{k_1,\ldots,k_\theta}(m) < \infty$$

$$\mathcal{C}^N \longrightarrow \mathcal{C},\ L^N \longrightarrow L,\ \Delta_N \longrightarrow 0$$

Result

(i) $\frac{1}{\Delta_N} f^N \longrightarrow f$. *(ii) Under assumption 1, the random process* $M^N = \frac{1}{N}\sum_j \delta_{S_j^N}$ *converges weakly (in Skorokhod topology) to a deterministic measure.*

Sketch of Proof : Convergence of marginal measures

Extend $\tilde{M}^N$ to continuous time

$$\tilde{M}^N(t) = \tilde{M}^N\left(\frac{\lfloor Nt \rfloor}{N}\right).$$

Define the filtration $\mathcal{F}_k = \sigma(S_1^N(t), \ldots, S_N^N(t), t \leq k)$.
$\phi = [\phi_1, \ldots, \phi_d]$ a bounded measurable function.

$$w^N(t) = \tilde{M}^N(t)) - \tilde{M}^N(0) - \sum_{k=0}^{Nt-1} f^N(\tilde{M}^N(\frac{k}{N})).$$

Then, w^N is a martingale.

Sketch of Proof : Convergence of marginal measures

$$\tilde{M}_x^N(t)) - \tilde{m}_x(t) = \tilde{M}_x^N(0)) - \tilde{m}_x(0) + \sum_{k=0}^{Nt-1} f_x^N(\tilde{M}^N(\frac{k}{N})) - \int_0^t f_x(\tilde{m}(\tau))d\tau$$

By the convergence of Darboux approximation of the Riemann integral term, $\frac{1}{N}\sum_{k=0}^{Nt-1} f_x(u, \tilde{M}^N(\frac{k}{N})) - \int_0^t f_x(\tilde{m}(\tau))d\tau$ is bounded by $C'\frac{1}{N}$ for some C'. By, Lipschitz continuity :

$$\| \tilde{M}^N(t) - \tilde{m}(t) \| \leq \| \tilde{M}^N(0) - \tilde{m}(0) + w^N(t) \| + K \int_0^t \| \tilde{M}^N(\tau) - \tilde{m}(\tau)$$

Sketch of Proof : Convergence of marginal measures

By Gronwall's inequality,

$$\| \tilde{M}^N(t) - \tilde{m}(t) \| \leq [\| \tilde{M}^N(0) - \tilde{m}(0) + w^N(t) \| + \frac{Kt}{N}] e^{Kt}$$

$$\sup_{0\leq t\leq T} \| \tilde{M}^N(t) - \tilde{m}(t) \| \leq [\| \tilde{M}^N(0) - \tilde{m}(0) \| + \sup_{0\leq t\leq T} \| w^N(t) \| + \frac{KT}{N}$$

By Doob's inequality one has

$$\mathbb{E}\left[\left(\sup_{0\leq t\leq T} \| w^N(t) \|\right)^2\right] \leq 4\mathbb{E}([w^N]_T)$$

$[w^N]_T$: total variation of the martingale w^N

$$[w^N]_T = \sum_{t=0}^{Nt-1} \| \tilde{M}^N(\frac{k+1}{N}) - \tilde{M}^N(\frac{k}{N}) - f^N(\tilde{M}^N(\frac{k}{N})) \| \leq \frac{Ct}{N}.$$

$$D^{T,N}[m_0] := \sup_{t\in[0,T]} \max_{\theta,a} |\tilde{M}^N_{\theta,a}(t) - \tilde{m}^N[m_0]_{\theta,a}(t)|$$

the maximal deviation in any population profile, from the flow induced by $\vec{f}^N(\tilde{m})$ through m_0, during $[0,T]$ where $\tilde{m}^N[m_0]$ is the solution of the ODE

$$\begin{cases} \frac{d}{dt}\tilde{m}^N(t) = f^N(\tilde{m}^N(t)) \\ \tilde{m}^N(0) = m_0 \end{cases}$$

(existence and uniqueness of $\tilde{m}^N[m0]$ follows from Picard-Lindelöf).
$\bar{D}^{T,N}[m_0] := \sup_{t\in[0,T]} \max_{\theta,a} |\tilde{M}^N_{\theta,a}(t) - m[m_0]_{\theta,a}(t)|$ maximal deviation from the flow induced by $\vec{f}(m)$ through m_0, during $[0,T]$.

Sketch of Proof : Convergence of marginal measures

$M^N(t) := \frac{1}{N}\sum_{j=1}^{N} \delta_{S_j^N(t)}$

Martingale+Legendre's transformation+Gronwall's inequality

Convergence to deterministic distribution

For every $\tau > 0$ there exists a constant C such that for every $\epsilon > 0$ and N large enough one has

$$\mathbb{P}\left(\sup_{0\leq\tau\leq T} ||M^N(\tau) - m(\tau)|| > \epsilon |\ M^N(0) = m_0\right) \leq 2de^{-\epsilon^2 CN}$$

for all $m_0 \in \Delta_d$,

ODE

Convergence of random measure in càdlàg function spaces

The random measure $\frac{1}{N}\sum_{j=1}^{N}\delta_{S_j^N}$ with marginal $\frac{1}{N}\sum_{j=1}^{N}\delta_{S_j^N(t)}$ converges (when $N \longrightarrow \infty$ a deterministic measure (solution of ODE) under mild assumptions on the expected number of interacting players that changes action at the same time and asymptotic indistinguishability [a].

a. This condition is weaker than **anonymity**

See also Tanabe (2006), Graham (2007), Le Boudec & Benaim (2008).

Sketch of Proof

$M^N := \frac{1}{N}\sum_{j=1}^{N} \delta_{S_j^N}$

Continuous, bounded functions ϕ_l

$$\lim_N \mathbb{E}\left(\prod_l \phi_l(S_l^N)\right) =?$$

Snitzman's theorem, Pair of type-state

$$\lim_{N\longrightarrow\infty} E[\phi(S_j^N)\phi(S_i^N)] = \phi(m_{\theta_i})\phi(m_{\theta_j}),\ \frac{1}{N}\sum_{j=1}^{N} E[\phi(S_j^N)] \longrightarrow \phi(m).$$

Derivation + Holder's inequality

$$\lim_{N\longrightarrow\infty} E\left[\phi(M^N) - \phi(m)\right]^2 = 0.$$

The mean field interaction is asymptotically equivalent to an evolutionary game

When N goes to infinity, the mean field interaction model with random set $\mathcal{B}^N(t)$ of players is equivalent to an evolutionary game [a] in which a local interaction at time t is described by

- each player is facing a population profile $m(t)$,
- the instantaneous expected cost of a player with the type θ and action a is

$$C_a^{\theta}(m(t)) := \lim_{N \longrightarrow \infty} C_a^{N,\theta}\left(M^N(t) | S_j^N(t) = (\theta, a), M^N(t) = m(t)\right)$$

a. Notice that players are not necessarily using the same strategies.

A class of evolutionary dynamics (homogenous population)

- revision of strategies : L,

$$f(m) = \sum_{k\geq 1} J_k(m) \sum_{a'_1,\ldots,a'_k} \sum_{a_1,\ldots,a_k} \left(\prod_{l=1}^{k} m_{a_l}\right) \times$$

$$L_{a;a'}(m,k)\left(\sum_{l=1}^{k}(\vec{e}_{a'_l} - \vec{e}_{a_l})\right)$$

- evolution of system's state, ODE : $\frac{d}{dt}m(t) = f(m(t))$.

For $\mathcal{B}^N(t) \nrightarrow \delta_1$ we obtain

$$\frac{d}{dt}m_a(t) = \sum_{a'\in\mathcal{A}} L_{a'a}(m(t))x_a(t) - m_a(t)\sum_{a'\in\mathcal{A}} L_{aa'}(m(t))$$

Evolutionary game dynamics (I)

Setting

- BNN : Brown and von Neumann (1950), Nash (1951)
- Replicator : Taylor & Jonker (1978)
- Smith dynamics : Smith (1984)

Differential equation

- $\dot{m}_a^\theta = g_a^\theta(m) - m_a^\theta \sum_{a' \in \mathcal{A}_\theta} g_{a'}^\theta(m)$, $g_a(m) = \max(0, -\mathcal{C}_a^\theta(m) + \sum_{a' \in \mathcal{A}_\theta} m_{a'}^\theta \mathcal{C}_{a'}^\theta(m))$
- $\dot{m}_a^\theta = m_a^\theta[-\mathcal{C}_a^\theta(m) + \sum_{a' \in \mathcal{A}_\theta} m_{a'}^\theta \mathcal{C}_{a'}^\theta(m)]$
- $\dot{m}_a^\theta = \sum_{a' \in \mathcal{A}_\theta} m_{a'}^\theta \max(0, -\mathcal{C}_a^\theta(m) + \mathcal{C}_{a'}^\theta(m)) - m_a^\theta \sum_{a' \in \mathcal{A}_\theta} \max(0, \mathcal{C}_{a'}^\theta(m) - \mathcal{C}_a^\theta(m))$

Evolutionary game dynamics (II)

Origin

- Best response : Gilboa & Matsui (1991), Fudenberg & Tirole (1991)
- Fictitious play : Brown (1951), Gilboa & Matsui (1991),
- Logit : Fudenberg & Levine (1998)

Dynamics

- $$\dot{m}^{\theta}(t) \in BR^{\theta}(m(t)) - m^{\theta}(t)$$
- $\dot{y}(t) \in \frac{1}{t} BR(y(t)) - y(t)$, $y(t) = \left(\frac{1}{t}\int_0^t m_1(\tau)\, d\tau,\ \frac{1}{t}\int_0^t m_2(\tau)\, d\tau\right)$
- $m_a^{\theta}(t) = \dfrac{e^{\frac{-c_a^{\theta}(m(t))}{\epsilon}}}{\sum_{a' \in \mathcal{A}_{\theta}} e^{\frac{-c_{a'}^{\theta}(m(t))}{\epsilon}}} - m_a^{\theta}(t)$

A population profile m is an **equilibrium state** if

$$\langle m - x, \mathcal{C}(m) \rangle \leq 0, \ \forall x$$

This variational inequality is equivalent to :

$$\forall \theta, \ \forall a \in \mathcal{A}_\theta, \ \left(m_a^\theta > 0 \Longrightarrow C_a^\theta(m) = \min_{a' \in \mathcal{A}_\theta} C_{a'}^\theta(m) \right)$$

Sketch of proof

$\Longleftarrow$: $(\min \leq any)$. $\Longrightarrow$: convex combination.

The last property is sometimes called **Wardrop first principle** of optimality.

Evolutionary stability [1]

Denote $m_\epsilon = \epsilon x + (1-\epsilon)m$. A population profile m is an **neutrally stable state** if $\forall x \neq m$ there exists $\epsilon_x > 0$ such that

$$\langle m - x, \mathcal{C}(m_\epsilon) \rangle \leq 0, \ \forall \epsilon \in (0, \epsilon_x)$$

A population profile m is an **evolutionarily stable state** if $\forall x \neq m$ there exists $\epsilon_x > 0$ such that

$$\langle m - x, \mathcal{C}(m_\epsilon) \rangle < 0, \ \forall \epsilon \in (0, \epsilon_x)$$

A population profile m is an **unbeatable state** if $\forall x \neq m$ one has

$$\langle m - x, \mathcal{C}(m_\epsilon) \rangle < 0, \ \forall \epsilon \in (0, 1)$$

1. Hamilton 1967, Smith'72,82, Weibull'95, Hofbauer& Sigmund'98, Gintis 2000, Cressman'03, Samuelson'03, Vincent'05, Sandholm'09

Immediate consequences

Relation between the solution concepts

$$ES \Longleftarrow NSS \Longleftarrow ESS \Longleftarrow unbeatable\ state$$

Price of Evolutionary Stability

$$PoA_{ESS} = \frac{\max_{m^*\ ESS} \langle m^*, C(m^*) \rangle}{SO}$$

$$1 \leq PoS_{ES} \leq PoS_{NSS} \leq PoS_{ESS} \leq PoS_{unbeat.state} \leq PoA_{unbeat.state} \leq PoA_{ESS} \leq PoA_{NSS} \leq PoA_{ES} \leq +\infty$$

Existence of equilibria in evolving games with random number of players

Let $d = \sharp S$.

Result

For any distribution of $\mathcal{B}^N(t)$ and any continuous function r on the non-empty, convex and compact subset $\prod_\theta \Delta(\mathcal{A}_\theta)$ of the Euclidean space $\mathbb{R}^d$, the evolving game has a least one "static" equilibrium state.

Sketch of proof

Connection target projection dynamics and best reply

A sufficient condition (PC)

Result

Suppose that the drift limit $\vec{f}$ *satisfies*
$\vec{f}(m) \neq 0 \Longrightarrow \langle \vec{f}(m), \mathcal{C}(m) \rangle = \sum_{\theta,a} \mathcal{C}_a^{\theta}(m) f_{\theta,a}(m) > 0$ *where*
$f_y(m) = \sum_{k \geq 1} J_k(m) f_y^k(m)$,

$$f_y^k(m) = \sum_{a_1,\ldots,a_k} \left(\prod_{l=1}^{k} m_{a_l} \right) \left(\sum_{j=1}^{k} \eta_{a;y}^{j}(m,k) \right)$$

$$-m_y \left(\sum_{j=1}^{k} \sum_{a_{-j}} \Big(\prod_{l=1, l \neq j}^{k} m_{a_l} \Big) \eta_{y,a_{-j}}^{t} \right),$$

Then any "stationary" equilibrium state is a rest point of ODE.

A sufficiency condition for stationarity (NS)

Result

Suppose that the polymatrix of transition L satisfies

$$L_{\theta,a,a_{-j};\theta,b}(m) > 0 \iff a, b \in \mathcal{A}_\theta,\ \mathcal{C}_a^\theta(m) > \mathcal{C}_b^\theta(m)$$

for each j, a_{-j} and m. Then,

- *The mean field dynamics is positively correlated.*
- *Any rest point of the ODE is a stationary equilibrium state.*

Particular class of games

Potential multi-type games

There exists a $\mathcal{C}^1-$function W

$$\frac{\partial}{\partial m_a^\theta} W(m) = \mathcal{C}_a^\theta(m)$$

Multi-type games with monotone expected cost

$$\forall x,\ \langle m - x, \mathcal{C}(m) - \mathcal{C}(x) \rangle \geq 0$$

Smith-stability : $\forall\ x \in BR(m) \backslash \{x\},\ \langle m - x, \mathcal{C}(x) \rangle < 0$ is equivalent to Evolutionary Stability [a]. Moreover, the set of equilibria is convex set ; $ES\ set \Longleftrightarrow NSS\ set.$

a. Notice that the cost function is non-linear.

Particular games (cont'd)

Games with cooperative dynamics

$$\frac{\partial}{\partial\, m_{a'}^{\theta'}} f_{\theta,a}(m) \geq 0$$

Uniqueness of equilibrium state, strict monotonicity

$$\forall m \neq x,\ \langle m^{\theta} - x^{\theta}, \mathcal{C}^{\theta}(m) - \mathcal{C}^{\theta}(x) \rangle > 0$$

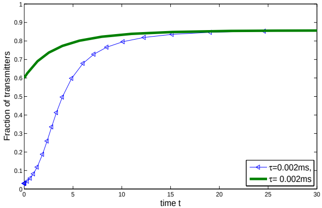
Evolving Games with Delayed Expected Cost

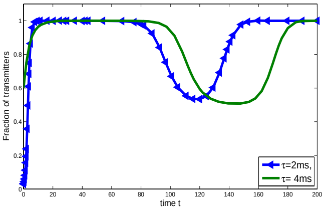
Delayed evolutionary game dynamics

$$\frac{d}{dt}m(t) = f\left(m(t),\ \{m(t-\tau_a^{\theta})\}_{\theta,a}\right)$$

Two important results

- Unbeatable state, EES, SES, ESS, NSS can be **unstable**. Evolutionary stable set can be **unstable set** under time delayed game dynamics.
- Possible **survival** of **dominated** strategies





Mean Field Asymptotics of Markov Decision Evolutionary Games

Mean Field Asymptotics of Markov Decision Evolutionary Games

Let $S = \{(\theta, s), s \in S_\theta\}$ be the set of "type-state" . $A_{\theta,s}$ set of action of type θ in state s . Assume S **finite.** $u_\theta(.|s) \in \Delta(A_{\theta,s})$. After an encounter between k players in $\mathcal{B}^N(t)$, the variation of the population profile $M^N(t) \nrightarrow M^N(t+\Delta_N)$. The player $j \in \mathcal{B}^N(t)$ receives an instantaneous cost $\nabla^N(X_j^N(t), X^N_{\mathcal{B}^N \setminus j}(t))$.

New states

$X_j^N(t+\Delta_N)$ is drawn according to $L^N_{\theta j}\left(.|X_j^N(t), X^N_{\mathcal{B}^N\setminus j}(t), j \in \mathcal{B}^N, \vec{u}\right)$.

Drift

$$f^N(m) := \mathbb{E}\left(M^N(t+\Delta_N) - M^N(t) \mid M^N(t) = m, \mathcal{B}^N(t+\Delta_N), \vec{u}\right)$$

Non-commutative diagram ?

Fix a Markov strategy profile u. Let $m[u, m_0](t)$ solution of $\dot{m} = f(u, m)$

$$\begin{array}{ccc} M^N[u,m_0](t) & \xrightarrow{t \longrightarrow +\infty} & ?\varpi^N[u,m_0] \\ \Big\downarrow N \longrightarrow +\infty & & \Big\downarrow N \longrightarrow +\infty \\ ?m[u,m_0](t) & \xrightarrow{t \longrightarrow +\infty} & ?? \end{array}$$

- Convergence/nonconvergence of $m[u, m_0](t)$ as t goes to infinity ?
- Convergence/nonconvergence of $\varpi^N[u, m_0]$ as N goes to ∞ ?

Non-commutative diagram ?

Fix a policy u. Let $m[u, m_0](t)$ solution of the ODE $\dot{m} = f(u, m)$ starting from m_0.

$$\begin{tikzcd} M^N[u,m_0](t) \arrow[r, "t \longrightarrow +\infty"] \arrow[d, "N \longrightarrow +\infty"] & \varpi^N[u,m_0] \arrow[d, "N \longrightarrow +\infty"] \\ m[u,m_0](t) \arrow[r, "t \longrightarrow +\infty"] & ? \end{tikzcd}$$

- Under which conditions, the two limits coincide (if they exist) ?
- If the dynamics do not converge, is there link between the time average of orbits of the ODE $\dot{m} = f(u, m)$ starting from $m(0) = m_0$, and the ω limit of $\varpi^N[u, m_0]$?.

Extension

Mean Field Games (continuum of players)

$$dm(t) = f(u, m)dt + \sigma(t)dW_t$$

Mixing atomic and non-atomic players

a single **"big player"** has a non-negligible influence in all the population. $\frac{1}{N}\sum_{j=1}\gamma_j\delta_{X_j^N}$.

Mean field limit under more general class of strategies

Mean field dynamics with migration

extend to the case type can change (inner and outer game). $\Longrightarrow$ **evolutionary game dynamics with migration** [a]

[a] Tembine H., Altman E., ElAzouzi R., Sandholm W. H., Evolutionary game dynamics with migration for hybrid power control in wireless communications, 47th IEEE CDC'2008

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