

# Visualizing geometrical statements with GeoView

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# Plan

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# Motivations

When **mathematicians** prove geometrical theorems,  
they need to **draw figures**.

When **proving** geometrical theorems with a theorem prover,  
we need also **figures**.

Geoview is a tool able to draw a **dynamic and interactive figure**,  
representing a plane geometry theorem,  
**from its formal statement**.

# Simson's line theorem:

Theorem **Simson\_line**:

Let  $A, B, C, M, P, Q$  and  $R$  be points.

If  $ABC$  is a triangle,

$BCM$  is a triangle,

$ABM$  is a triangle,

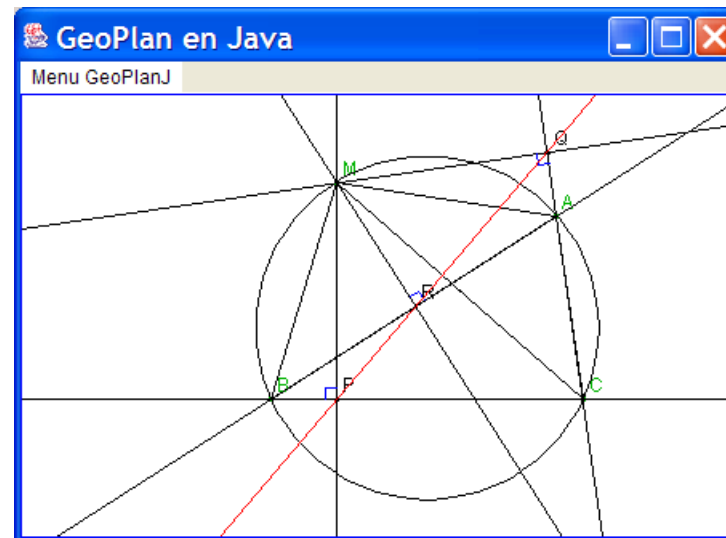
$ACM$  is a triangle,

$P$  is the orthogonal projection of  $M$  on line  $(BC)$ ,

$Q$  is the orthogonal projection of  $M$  on line  $(AC)$ ,

and  $R$  is the orthogonal projection of  $M$  on line  $(AB)$

then  $M$  is on the circumcircle of triangle  $ABC$  if and only if  $P, Q$  and  $R$  are collinear.



# The Pcoq proof environment

- interface to the Coq proof assistant
- formulas and commands are structured data
- bi-dimensional display of mathematical formulas
- proof-by-pointing
- user defined notations

Lemma **isocèle\_mediane\_bissectrice** :  
Let  $A, B, C$  and  $I$  be points.  
If  $A \neq I$ ,  
 $B \neq C$ ,  
 $I$  is the midpoint of  $[BC]$ ,  
and  $ABC$  is an isosceles triangle in  $A$   
then  $\overrightarrow{AB}, \overrightarrow{AI} = \overrightarrow{AI}, \overrightarrow{AC}$ .

## A geometry library in Coq

Geometry theorems of courses taught in French high-schools:

- vectors, alignment, barycenter, midpoint, centroid, parallelism,
- orthogonality, orthocenter, orthogonal projection,
- euclidean distance, isosceles triangle, perpendicular bisector, circle,
- homothety, translation, reflection, rotation, direct similarity and composition of transformations.

Classical theorems in plane geometry, proved in the same way as they are proved in the high-school courses: Thales, Desargues, Pythagoras, Simson's line, Miquel, Euler's line, nine-point circle.

Trigonometry, complex numbers and three-dimensional geometry not yet connected to a drawing tool.

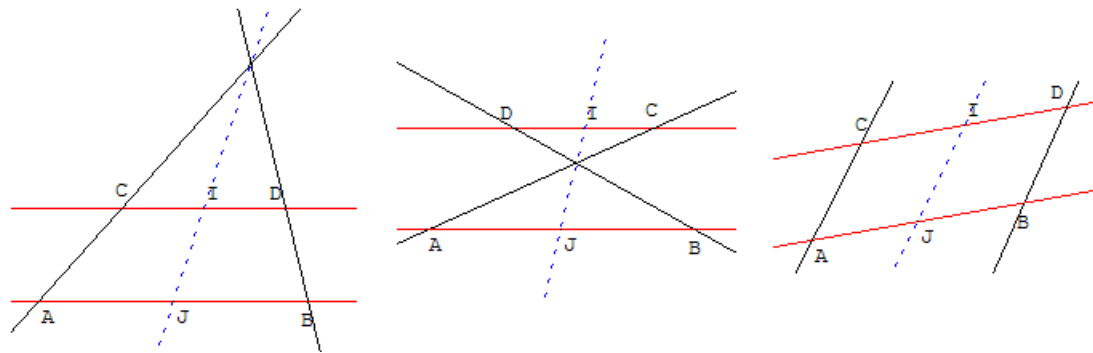
# GeoplanJ

Written by F.Kotecki (CNAM, France).

GeoplanJ (available with a GNU GPL license) is a Java port of the software GeoplanW used in french high schools.

1. visualization of plane geometrical objects (points, lines, circles) from a text description of the figure (definition of objects and constraints between them: intersection, distance, etc).
2. interaction with objects (by mouse and keyboard): move, create, delete, modify. Geometrical constraints are kept: if A is the intersection of lines D and D', after moving D, A remains the intersection of D and D'.

# Three figures of the same configuration:



# GeoView

Input: theorem statement,

Output: figure drawn with GeoplanJ

From the theorem statement, we get points and geometrical constraints binding points.

## Geometrical constraints

- notation  $(C P_1 \dots P_n)$ :  $C$  is the constraint and  $P_1 \dots P_n$  are the points.
- for some choice of  $n - 1$  points among  $P_1 \dots P_n$ , we can generally build the remaining point (called the linked point) by simple geometric constructions from the others.
- type  $T(C)$  describes the degree of freedom of the linked point.
  - $T(C) = 2$ : the two coordinates of the linked point are determined
  - $T(C) = 1$ : the linked point can vary on a curve (circle or straight line)

## From constraints to constructions

how to build  $P_1, \dots, P_m$  satisfying constraints

$(C_1 Q_{11} \dots Q_{1r_1}), \dots, (C_n Q_{n1} \dots Q_{nr_n})$  where  $\forall i j, Q_{ij} \in \{P_1, \dots, P_m\}$ .

Our goal is to find a matrix  $M$  matrix with  $n$  lines and  $m$  columns verifying:

1.  $\forall i j, M_{ij} \in \{-1, 0, 1, 2\}$ ,
2.  $\forall i j, M_{ij} = -1 \Leftrightarrow P_j$  does not appear in the constraint  $C_i$ ,
3.  $\forall i j, M_{ij} > 0 \Rightarrow P_j$  appears in the constraint  $C_i$  of type  $M_{ij}$ ,
4. each line of  $M$  has exactly one strictly positive entry,
5.  $\forall j, \sum_i \sup(0, M_{ij}) \leq 2$ ,
6. the relation  $\prec$  on points  $P_1, \dots, P_m$  defined by

$$a \prec b \Leftrightarrow \exists i, M_{ia} = 0 \text{ and } M_{ib} > 0$$

is such that its transitive closure  $\preccurlyeq$  is antisymmetric.

## Algorithm to find such a matrix

- i. we first build a matrix  $M$  verifying conditions 1, 2, 3 and 4, where for each line the positive entry is left-most, which is easy.
- ii. we enumerate all matrix verifying conditions 1, 2, 3 and 4, simply by lexicographically shifting the positive entry of each line to the right. We stop as soon as the matrix verifies conditions 5 and 6, which are easy to check. Otherwise, we fail.

! This method is not complete: it fails in some cases where a figure exists.  
(for a complete method, we should use real quantifier elimination!)

## Example

$C_1$ : (collinear  $A B H$ )

$C_2$ : (ortho  $H C A B$ ), *i.e.*  $(HC)$  and  $(AB)$  are perpendicular.

$C_3$ : (circle  $A B C$ ), *i.e.*  $C$  is on the circle with diameter  $[AB]$ .

$T(\text{collinear}) = 1$

$T(\text{ortho}) = 1$

$T(\text{circle}) = 1$

Let  $P_1, \dots, P_n = A, B, C, H$ .

The initial matrix, verifying conditions 1, 2, 3 and 4, is:

$M$	$A$	$B$	$C$	$H$
collinear	1	0	-1	0
ortho	1	0	0	0
circle	1	0	0	-1

It does not verify condition 5: the column  $A$  is linked by three constraints.

In lexicographic order, the next matrix verifying conditions 1, 2, 3 and 4 is:

$M$	$A$	$B$	$C$	$H$
collinear	1	0	-1	0
ortho	1	0	0	0
circle	0	1	0	-1

It verifies condition 5, but not condition 6: we have  $B \prec A \prec B$

Continuing, we get the first next matrix that verifies conditions 5 and 6:

$M$	$A$	$B$	$C$	$H$
collinear	1	0	-1	0
ortho	0	0	1	0
circle	0	0	1	-1

## Construction

We have then  $B \prec A, H \prec A, A \prec C$ .

1. Take  $B$  and  $H$  two free points in the plane.
2. Take  $A$  as a free point on the line  $(BH)$ .
3. Take  $C$  as the intersection between the line orthogonal to  $(AB)$  and containing  $H$  and the circle with diameter  $[AB]$ .

## Examples and demo

- nine-point circle theorem
- Simson's line theorem: equivalence of two propositions in the conclusion
- trapezoid exercise: conjunctions and disjunctions in the conclusion
- representation of real numbers by points on the real line: homothety, rotation etc.
- existential quantification in the conclusion
- an example where GeoView fails

## Conclusion and future

- GeoView can be disconnected to its bases:  
axiomatization of plane geometry can be changed  
the proof tool can be changed  
the drawing tool also can be changed
- reversibility: figure as input, statement theorem as output ?
- evolution of the figure during a the proof.
- make the proof on the figure.
- 3D-geometry: problem of representation of planes on the 2D screen ?