Visualizing geometrical statements with GeoView

by Yves BERTOT,
Frédérique GUILHOT
and Loïc POTTIER

Lemme Project – INRIA Sophia-Antipolis, France.

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Plan

1. Motivations
2. Pcoq and the geometry library
3. GeoplanJ
4. GeoView
5. Examples and demo
6. Conclusion

Motivations

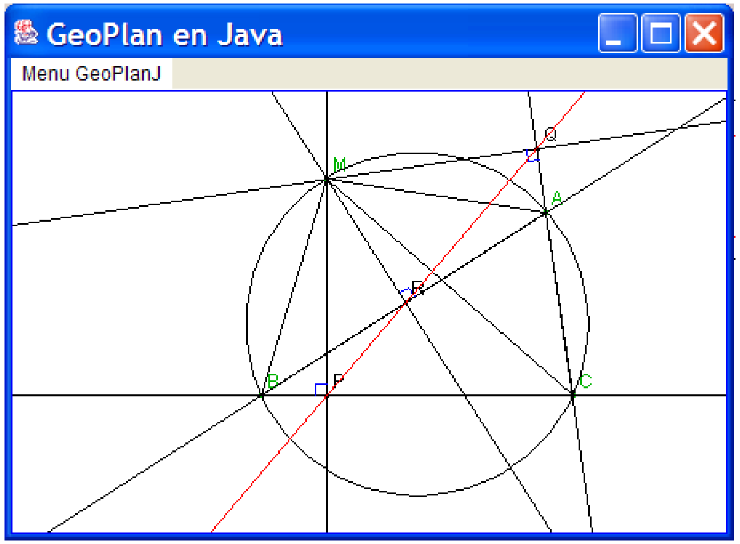
When mathematicians prove geometrical theorems,
they need to draw figures.

When proving geometrical theorems with a theorem prover,
we need also figures.

Geoview is a tool able to draw a dynamic and interactive figure,
representing a plane geometry theorem,
from its formal statement.

Simson's line theorem:

Theorem **Simson_line**:
Let A, B, C, M, P, Q and R be points.
If **ABC is a triangle,**
BCM is a triangle,
ABM is a triangle,
ACM is a triangle,
P is the orthogonal projection of M on line (BC),
Q is the orthogonal projection of M on line (AC),
and R is the orthogonal projection of M on line (AB)
then **M is on the circumcircle of triangle ABC if and only if P, Q and R are collinear.**



The Pcoq proof environment

– interface to the Coq proof assistant
– formulas and commands are structured data
– bi-dimensional display of mathematical formulas
– proof-by-pointing
– user defined notations

Lemma **isocele_mediane_bissectrice :**
Let A, B, C and I be points.
If **A ≠ I,**
B ≠ C,
I is the midpoint of [B C],
and A B C is an isosceles triangle in A
then $(\overrightarrow{AB}, \overrightarrow{AI}) = (\overrightarrow{AI}, \overrightarrow{AC})$.

A geometry library in Coq

Geometry theorems of courses taught in French high-schools:

- vectors, alignment, barycenter, midpoint, centroid, parallelism,
- orthogonality, orthocenter, orthogonal projection,
- euclidean distance, isosceles triangle, perpendicular bisector, circle,
- homothety, translation, reflection, rotation, direct similarity and composition of transformations.

Classical theorems in plane geometry, proved in the same way as they are proved in the high-school courses: Thales, Desargues, Pythagoras, Simson's line, Miquel, Euler's line, nine-point circle.

Trigonometry, complex numbers and three-dimensional geometry not yet connected to a drawing tool.

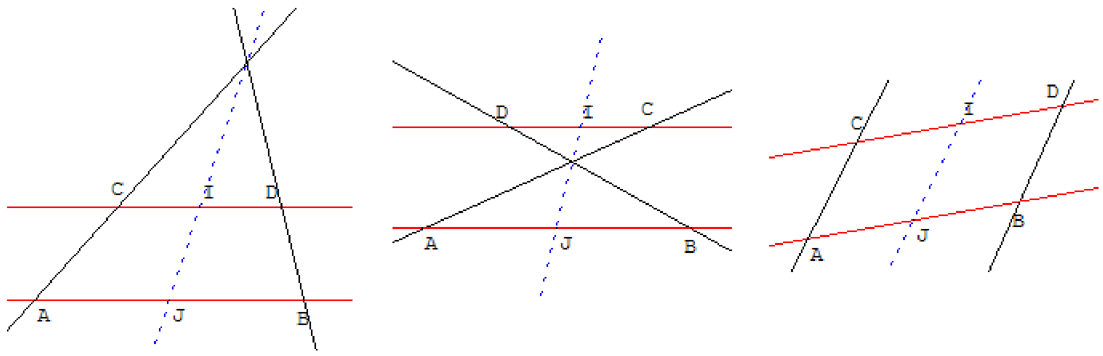
GeoplanJ

Written by F.Kotecki (CNAM, France).
GeoplanJ (available with a GNU GPL license) is a Java port of the software GeoplanW used in french high schools.

1. visualization of plane geometrical objects (points, lines, circles) from a text description of the figure (definition of objects and constraints between them: intersection, distance, etc).

2. interaction with objects (by mouse and keyboard): move, create, delete, modify. Geometrical constraints are kept: if A is the intersection of lines D and D', after moving D, A remains the intersection of D and D'.

Three figures of the same configuration:



GeoView

Input: theorem statement,
Output: figure drawn with GeoplanJ
From the theorem statement, we get points and geometrical constraints binding points.

Geometrical constraints

- notation $(C\, P_1 \ldots P_n)$: C is the constraint and $P_1 \ldots P_n$ are the points.
- for some choice of $n-1$ points among $P_1 \ldots P_n$, we can generally build the remaining point (called the linked point) by simple geometric constructions from the others.
- type $T(C)$ describes the degree of freedom of the linked point.
  $T(C) = 2$: the two coordinates of the linked point are determined
  $T(C) = 1$: the linked point can vary on a curve (circle or straight line)

From constraints to constructions

how to build $P_1, \ldots, P_m$ satifying constraints
$(C_1\, Q_{1\,1} \ldots Q_{1\,r_1}), \ldots, (C_n\, Q_{n\,1} \ldots Q_{n\,r_n})$ where $\forall i\, j\,, Q_{i\,j} \in \{P_1, \ldots, P_m\}$.

Our goal is to find a matrix M matrix with n lines and m columns verifying:

1. $\forall i\, j\,, M_{i\,j} \in \{-1, 0, 1, 2\}$,
2. $\forall i\, j\,, M_{i\,j} = -1 \Leftrightarrow P_j$ does not appear in the constraint C_i,
3. $\forall i\, j\,, M_{i\,j} > 0 \Rightarrow P_j$ appears in the constraint C_i of type $M_{i\,j}$,
4. each line of M has exactly one strictly positive entry,
5. $\forall j\,, \sum_i \sup(0, M_{i\,j}) \leqslant 2$,
6. the relation $\prec$ on points $P_1, \ldots, P_m$ defined by

   $$a \prec b \Leftrightarrow \exists i,\ M_{i\,a} = 0 \text{ and } M_{i\,b} > 0$$

   is such that its transitive closure $\lll$ is antisymetric.

Algorithm to find such a matrix

i. we first build a matrix M verifying conditions 1, 2, 3 and 4, where for each line the positive entry is left-most, which is easy.

ii. we enumerate all matrix verifying conditions 1, 2, 3 and 4, simply by lexicographically shifting the positive entry of each line to the right. We stop as soon as the matrix verifies conditions 5 and 6, which are easy to check. Otherwise, we fail.

! This method is not complete: it fails in some cases where a figure exists. (for a complete method, we should use real quantifier elimination!)

Example

C_1: (collinear $A\,B\,H$)
C_2: (ortho $H\,C\,A\,B$), $i.e.$ (HC) and $(A\,B)$ are perpendicular.
C_3: (circle $A\,B\,C$), $i.e.$ C is on the circle with diameter $[A\,B]$.
$T(\text{collinear}) = 1$
$T(\text{ortho}) = 1$
$T(\text{circle}) = 1$
Let $P_1, \ldots, P_n = A, B, C, H$.

The initial matrix, verifying conditions 1, 2, 3 and 4, is:

M	A	B	C	H
collinear	1	0	-1	0
ortho	1	0	0	0
circle	1	0	0	-1

It does not verify condition 5: the column A is linked by three constraints.

In lexicographic order, the next matrix verifying conditions 1, 2, 3 and 4 is:

M	A	B	C	H
collinear	1	0	-1	0
ortho	1	0	0	0
circle	0	1	0	-1

It verifies condition 5, but not condition 6: we have $B \prec A \prec B$

Continuing, we get the first next matrix that verifies conditions 5 and 6:

M	A	B	C	H
collinear	1	0	-1	0
ortho	0	0	1	0
circle	0	0	1	-1

Construction

We have then $B \prec A$, $H \prec A$, $A \prec C$.

1. Take B and H two free points in the plane.
2. Take A as a free point on the line (BH).
3. Take C as the intersection between the line orthogonal to (AB) and containing H and the circle with diameter $[AB]$.

Examples and demo

- nine-point circle theorem
- Simson's line theorem: equivalence of two propositions in the conclusion
- trapezoid exercice: conjunctions and disjunctions in the conclusion
- representation of real numbers by points on the real line: homothety, rotation etc.
- existential quantification in the conclusion
- an example where GeoView fails

Conclusion and future

- GeoView can be disconnected to its bases:
  axiomatization of plane geometry can be changed
  the proof tool can be changed
  the drawing tool also can be changed
- reversibility: figure as input, statement theorem as output ?
- evolution of the figure during a the proof.
- make the proof on the figure.
- 3D-geometry: problem of representation of planes on the 2D screen ?