On Global Induction Mechanisms in a μ-Calculus with Explicit Approximations

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Motivation

	Local Induction	Global Induction
derivations	trees	graphs
soundness	internal, local induction rules	external, discharge condition
tradition	“classical“ logics	modal/temporal logics
μ-calculi	Park’s induction [Kozen]	sequent calculus [Dam/Gurov]
	fixed point tagging [Winskel]	model checking [Sti/Wal,Bra]

Programme: relate local and global inductive reasoning (proof translations)

This work: compare global discharge conditions in context of μ-calculi

Overview

1. Gentzen-style proof system for first-order μ-calculus

   (a) μ-Calculus with explicit approximations

   (b) Local proof rules and graph-shaped derivations (pre-proofs)

2. Induction discharge conditions:

   (A) semantical: runs [Dam/Gurov]

   (B) syntactical: traces

   (C) automata-theoretic

Theorem For a given pre-proof (A), (B) and (C) are equivalent.

μ-Calculus with Explicit Approximations (1)

Syntax first-order logic + (approximated) fixed points

$$\begin{array}{lcll} \phi & ::= & \text{FOL formula} \mid \Phi_X(\overline{t}) & \text{formulas} \\ \Phi_X & ::= & X \mid \mu X(\overline{x}).\phi \mid \mu^\kappa X(\overline{x}).\phi & \text{abstractions} \end{array}$$

Remarks

- ☞ individual, predicate and ordinal variables
- ☞ both X and $\overline{x}$ are bound in $\mu X(\overline{x}).\phi$ and $\mu^\kappa X(\overline{x}).\phi$
- ☞ usual syntactic monotonicity condition restricts fixed point formation

μ-Calculus with Explicit Approximations (2)

Models $\mathcal{M} = (\mathcal{A}, \rho)$ $\mathcal{A}$ a first-order structure, ρ a valuation

Semantics interpretation in lattice of predicates with point-wise ordering

$$\|\mu X(\overline{x}).\phi\|^{\mathcal{A}}_{\rho} = \mu\Psi \qquad \|\mu^{\kappa} X(\overline{x}).\phi\|^{\mathcal{A}}_{\rho} = \mu^{\rho(\kappa)}\Psi$$

where $\Psi = \lambda P.\, \lambda\overline{a}.\, \|\phi\|^{\mathcal{A}}_{\rho[P/X,\, \overline{a}/\overline{x}]}$ monotone predicate transformer

Proposition

1. $\mu\Psi = \bigvee_{\alpha} \mu^{\alpha}\Psi$
2. $\mu^{\alpha}\Psi = \bigvee_{\beta<\alpha} \Psi(\mu^{\beta}\Psi)$

Sequents and Validity

Sequents are of the form

$$\Gamma \vdash_{\mathcal{O}} \Delta$$

where $\mathcal{O} = (|\mathcal{O}|, \leq_{\mathcal{O}})$ is a finite partial order on ordinal variables recording ordinal constraints

Validity sequent $\Gamma \vdash_{\mathcal{O}} \Delta$ is valid if

$$\bigwedge \Gamma \to \bigvee \Delta$$

true in all models $(\mathcal{A}, \rho)$ where ρ respects $\mathcal{O}$, i.e. $\rho(\kappa) \leq \rho(\kappa')$ whenever $\kappa \leq_{\mathcal{O}} \kappa'$

Local Proof Rules for Fixed Points

$$(\mu\text{-L}) \qquad \frac{\Gamma, (\mu X(\overline{x}).\phi)(\overline{t}) \vdash_{\mathcal{O}} \Delta}{\Gamma, (\mu^{\kappa} X(\overline{x}).\phi)(\overline{t}) \vdash_{\mathcal{O}'} \Delta} \qquad \mathcal{O}' = \mathcal{O} \cup \{\kappa\}$$

$$(\mu\text{-R}) \qquad \frac{\Gamma \vdash_{\mathcal{O}} (\mu X(\overline{x}).\phi)(\overline{t}), \Delta}{\Gamma \vdash_{\mathcal{O}} \phi[\mu X(\overline{x}).\phi/X, \overline{t}/\overline{x}], \Delta}$$

$$(\mu^{\kappa}\text{-L}) \qquad \frac{\Gamma, (\mu^{\kappa} X(\overline{x}).\phi)(\overline{t}) \vdash_{\mathcal{O}} \Delta}{\Gamma, \phi[\mu^{\kappa'} X(\overline{x}).\phi/X, \overline{t}/\overline{x}] \vdash_{\mathcal{O}'} \Delta} \qquad \mathcal{O}' = \mathcal{O} \cup \{(\kappa', \kappa)\}$$

$$(\mu^{\kappa}\text{-R}) \qquad \frac{\Gamma \vdash_{\mathcal{O}} (\mu^{\kappa} X(\overline{x}).\phi)(\overline{t}), \Delta}{\Gamma \vdash_{\mathcal{O}} \phi[\mu^{\kappa'} X(\overline{x}).\phi/X, \overline{t}/\overline{x}], \Delta} \qquad \kappa' <_{\mathcal{O}} \kappa$$

Derivation Trees and Pre-Proofs

Derivation tree $\mathcal{D} = (\mathcal{N}, \mathcal{E}, \mathcal{L})$ sequent-labeled, consistent with proof rules

Repeat $R = (M, N, \sigma)$ leaf $N(\Gamma' \vdash_{\mathcal{O}'} \Delta')$, σ-instance of $M(\Gamma \vdash_{\mathcal{O}} \Delta)$

- ☞ more precisely: $\Gamma\sigma \subseteq \Gamma'$, $\Delta\sigma \subseteq \Delta'$ and $\mathcal{O}\sigma \subseteq \mathcal{O}'$
- ☞ N called repeat node and M its companion

Pre-proof $\mathcal{P} = (\mathcal{D}, \mathcal{R})$ pair of derivation tree $\mathcal{D}$ and set of repeats $\mathcal{R}$

- ☞ every non-axiom leaf appears in exactly one repeat of $\mathcal{R}$
- ☞ pre-proof graph: $\mathcal{G}(\mathcal{P}) = \mathcal{D} +$ repeat edges

Runs – Semantic Discharge (1)

Run of Pre-Proof $\mathcal{P}$ (rooted) path of $\mathcal{G}(\mathcal{P})$, labeled by valuations:

$$\Pi = (N_0, \rho_0) \cdots (N_i, \rho_i) \cdots$$

labels: ρ_i respects $\mathcal{O}_i$, and

tree edge: $(N_i, N_{i+1}) \in \mathcal{E}$ implies ρ_{i+1} agrees with ρ_i on all free variable common to N_{i+1} and N_i, and

repeat: $(N_{i+1}, N_i, \sigma) \in \mathcal{R}$ implies $\rho_{i+1} = \rho_i \circ \sigma$

Proofs – Semantic Discharge (2)

Proof pre-proof $\mathcal{P}$ such that all runs of $\mathcal{P}$ are finite

- ☞ proof = pre-proof + well-foundedness
- ☞ reference discharge condition to which we compare others

Theorem (Soundness) If there is a proof for $\Gamma \vdash_{\mathcal{O}} \Delta$ then $\Gamma \vdash_{\mathcal{O}} \Delta$ is valid.

Traces – Syntactic Discharge (1)

Trace path of $\mathcal{G}(\mathcal{P})$ labeled by ordinal constraints:

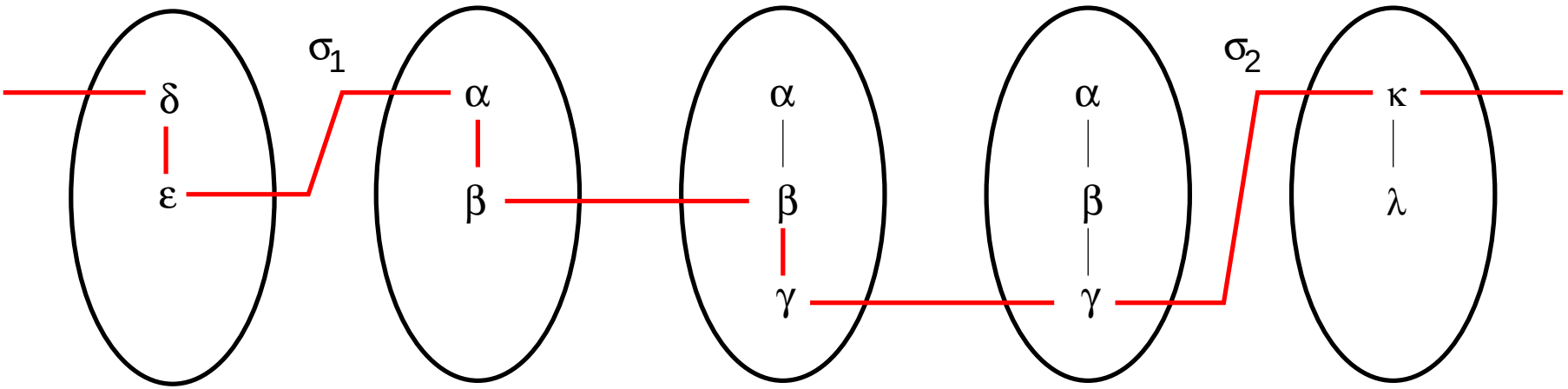
$$\tau = (N_0, (\kappa_0, \kappa_0')) \cdots (N_i, (\kappa_i, \kappa_i')) \cdots$$

labels: $\kappa_i' \leq_{\mathcal{O}_i} \kappa_i$ where $N_i(\Gamma_i \vdash_{\mathcal{O}_i} \Delta_i)$, and

tree edge: $(N_i, N_{i+1}) \in \mathcal{E}$ implies $\kappa_i' = \kappa_{i+1}$, and

repeat: $(N_{i+1}, N_i, \sigma) \in \mathcal{R}$ implies $\kappa_i' = \sigma(\kappa_{i+1})$

Example – Syntactic Discharge (2)



corresponding trace fragment:

$$(N_0, (\delta, \varepsilon)) \quad (N_1, (\alpha, \beta)) \quad (N_2, (\beta, \gamma)) \quad (N_3, (\gamma, \gamma)) \quad (N_4, (\kappa, \kappa))$$

repeat companion repeat companion

Progress – Syntactic Discharge (3)

Progress

trace τ progresses at i if $\kappa_i' <_{\mathcal{O}_i} \kappa_i$ (strict decrease),

is progressive if it progresses at infinitely many positions

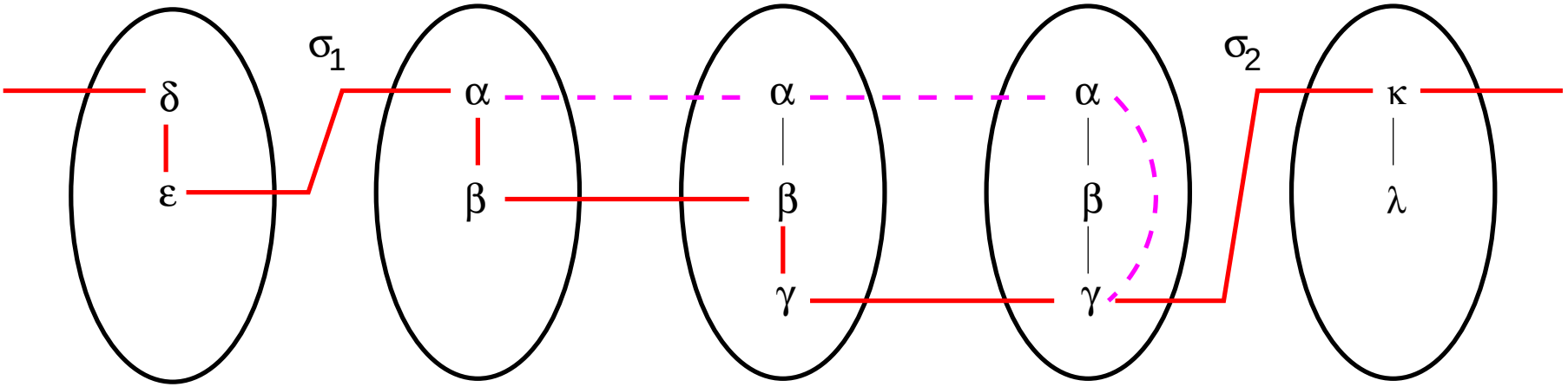
path π is progressive if there is a progressive trace along a suffix of π

Condition (T-DC): all infinite paths of $\mathcal{G}(\mathcal{P})$ are progressive

Theorem A pre-proof $\mathcal{P}$ satisfies (T-DC) iff it is a proof.

Normal Traces – Automata-Theoretic DC (1)

Observation any trace τ can be transformed into a normal trace $\hat{\tau}$ progressing at most at repeat nodes and with equivalent progress characteristics



$(N_0, (\delta, \varepsilon))$ $(N_1, (\alpha, \alpha))$ $(N_2, (\alpha, \alpha))$ $(N_3, (\alpha, \gamma))$ $(N_4, (\kappa, \kappa))$

repeat companion repeat companion

Automata-Theoretic Discharge (2)

Idea construct two Buechi automata, B_1 and B_2, over the alphabet $\mathcal{R}$ s.t.

- ☞ B_1 recognises sequences of repeats as traversed by paths of $\mathcal{G}(\mathcal{P})$,
- ☞ B_2 recognises a sequences of repeats potentially connected through a normal trace (provided the sequence is also accepted by B_1)
- ☞ $L(B_1) \subseteq L(B_2)$ characterises previous discharge conditions

Automata-Theoretic Discharge (3)

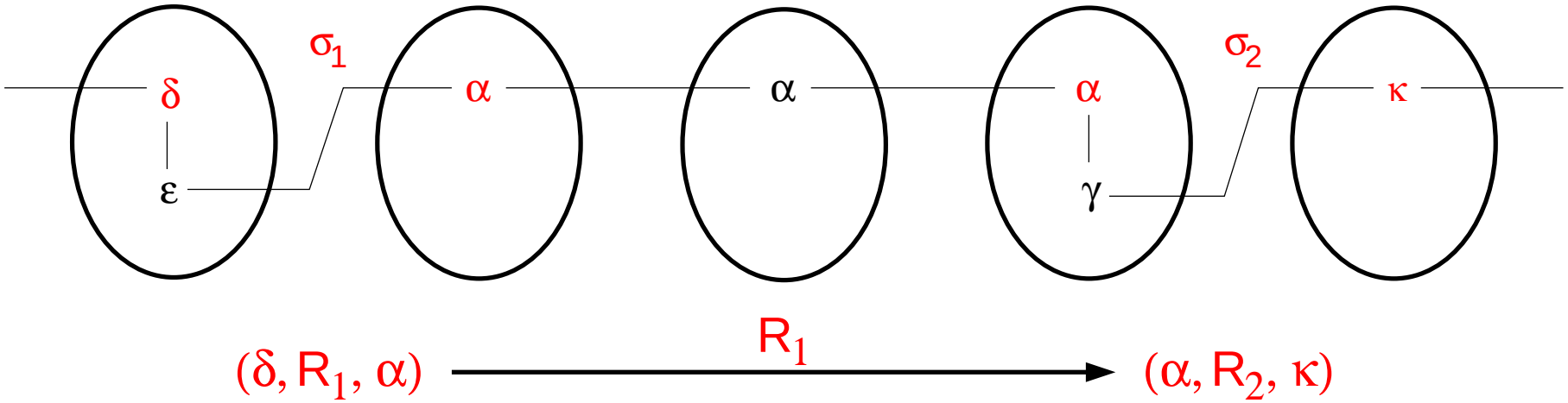
Automaton B_2 Details

States $\{(\kappa, R, \lambda) \mid R = (M, N, \sigma) \text{ and } \sigma(\lambda) \leq_{\mathcal{O}_N} \kappa\} \cup \{\spadesuit\}$

Accepting (κ, R, λ) with $\sigma(\lambda) <_{\mathcal{O}_N} \kappa$ (progress)

Transitions $(\kappa, R, \lambda) \xrightarrow{R} (\lambda, R', \iota)$

Example $\mathcal{B}_2$ Transition



Main Result and Summary

Theorem Let $\mathcal{P}$ be a pre-proof. Are equivalent:

(1) $\mathcal{P}$ is a proof (all runs of $\mathcal{P}$ are finite)

(2) $\mathcal{P}$ satisfies (T-DC) (all infinite paths of $\mathcal{G}(\mathcal{P})$ are progressive)

(3) $L(B_1) \subseteq L(B_2)$ (ditto, using normal traces)

The latter can be checked in time $2^{\mathcal{O}(n^3 \log n)}$, where $n = |\mathcal{N}|$.

Related and Future Work

Gentzen-style proof systems

- ☞ subsume Rabin-like syntactic conditions by [Sch/Sim] and [Dam et al.]:
  - obtained by restricting B_2 to states (κ, R, κ) (no renaming)
  - complexity drops to $2^{\mathcal{O}(n^2 \log n)}$ (time vs. space)
- ☞ but: do the new conditions provide more proof power?

Games (modal μ-calculus)

- ☞ generalisation of μ-/ν-traces [Walukiewicz]