A MHM method for time-domain linear elastodynamics

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Complex media

- heterogeneous and/or anisotropic materials (e.g. geosismics, composite materials)
- linear elastodynamic waves
- MHM method as a candidate (F. Valentin)

Potential approaches

1. mixed displacement/stress formulation (second order in time)
   - time discretization first, then MHM strategy
   - time/space MHM method
2. velocity/stress formulation (first order in time)
   $\longrightarrow$ our choice

Displacement formulation

Find $\mathbf{u} : (0,T)\times\Omega \to \mathbb{R}^d$ ($d = 2,3$), such that

$$\begin{cases} \rho\,\partial_{tt}\mathbf{u} - \mathbf{div}\,\mathcal{C}\,\mathcal{E}(\mathbf{u}) = \mathbf{f}\,, & \text{in } (0,T)\times\Omega\,,\\ \mathbf{u} = \mathbf{0}\,, & \text{on } (0,T)\times\partial\Omega\,,\\ \mathbf{u} = \mathbf{u}_0\,,\quad \partial_t\mathbf{u} = \mathbf{v}_0 & \text{for } t=0 \quad \text{in } \Omega\,, \end{cases} \tag{1}$$

- $\rho := \rho(\mathbf{x})\ ,\mathbf{x}\in\Omega$, mass density, uniformally bounded.
- linearized strain tensor $\mathcal{E}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^T\mathbf{u}\right)$
- $\mathcal{C}$ stiffness (or elastic) tensor (fourth-rank, symmetric), time independent, uniformally positive and bounded
- $\boldsymbol{\sigma} := \mathcal{C}\,\mathcal{E}(\mathbf{u})$ stress tensor (2nd-rank, symmetric)
- $\mathcal{A} := \mathcal{C}^{-1}$ fourth-rank compliance tensor (same properties as $\mathcal{C}$)

Displacement formulation - Unicity and regularity results

$$V_0 = [H_0^1(\Omega)]^d,\ H = [L^2(\Omega)]^d,\ V_0' = [H^{-1}(\Omega)]^d,$$

If $\mathbf{u}_0 \in V_0,\ \mathbf{v}_0 \in H,\ \mathbf{f}, \partial_t \mathbf{f} \in [L^2(]0,T[\times\Omega)]^3$ *then*

(1) *has a unique weak solution* $\mathbf{u}$ *with*

- $\mathbf{u} \in L^\infty(0,T;V_0)$,
- $\mathbf{u}' \in L^\infty(0,T;H)$,
- $\mathbf{u}'' \in L^\infty(0,T;V_0')$.

cf. Duvaut-Lions, *Inequalities in Mechanics and Physics*, Springer-verlag, 1976

The velocity/stress formulation

- $\mathbf{v} := \partial_t \mathbf{u}\,,\ \boldsymbol{\sigma} := \mathcal{C}\,\mathcal{E}(\mathbf{u})$
- Find $\mathbf{v} : (0,T) \times \Omega \to \mathbb{R}^d$, such that

$$\begin{cases} \rho\,\partial_t \mathbf{v} - \mathbf{div}\boldsymbol{\sigma} = \mathbf{f}\,, \quad \text{in } (0,T)\times\Omega\,, \\ \partial_t \boldsymbol{\sigma} - \mathcal{C}\,\mathcal{E}(\mathbf{v}) = 0\,, \text{in } (0,T)\times\Omega\,, \\ \mathbf{v} = \mathbf{0}\,, \quad \text{on } (0,T)\times\partial\Omega\,, \\ \mathbf{v} = \mathbf{v}_0\,, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 := \mathcal{C}\ \mathcal{E}(\mathbf{u}_0) \quad \text{at } t=0 \quad \text{in } \Omega\,, \end{cases} \tag{2}$$

2nd equation replaced by $\mathcal{A}\,\partial_t \boldsymbol{\sigma} - \mathcal{E}(\mathbf{v}) = 0$.

Existence and unicity of the solution couple $(\mathbf{v},\boldsymbol{\sigma})$

- $\mathbb{S}$:= set of symmetric second-rank tensors
- $S := H(\mathbf{div},\Omega) \cap \mathbb{S}$

If

- $\boldsymbol{\sigma}_0 \in S$,
- $\mathbf{v}_0 \in [H_0^1(\Omega)]^d$,
- $f_i \in W^{1,1}([0,T]), \forall i = 1,\dots,d$

then (2) *has a unique solution* $(\mathbf{v},\boldsymbol{\sigma})$ *and*

$$\boldsymbol{\sigma} \in C^0([0,T];S) \cap C^1([0,T];L^2(\Omega;\mathbb{S})) \; ,$$
$$\mathbf{v} \in C^0([0,T];[H_0^1(\Omega)]^d \cap C^1([0,T_0];[L^2(\Omega)]^d) \; .$$

(J. Lee, thèse (D. N. Arnold), *Mixed methods with weak symmetry for time dependent problems of elasticity and viscoelasticity*, 2012)

Weak formulation associated with (2)

- $\mathcal{T}_H$ an arbitrary given regular discretization of $\Omega = \bigcup_{K \in \mathcal{T}_H} K$;
- Functional space notations (fixed t, space) $\mathbb{M} \equiv \mathbb{R}^{d \times d}$:

$$
\begin{aligned}
V &:= \{\mathbf{v} \in [L^2(\Omega)]^d,\ \mathbf{v} \in [H^1(K)]^d,\ \forall K \in \mathcal{T}_H\} ,\\
H(\mathbf{div}, \mathcal{T}_H) &:= \{\boldsymbol{\tau} \in L^2(\Omega) \cap \mathbb{M} ,\ \boldsymbol{\tau} \in H(\mathbf{div}, K) ,\ \forall K \in \mathcal{T}_H\} ,\\
W &:= H(\mathbf{div}, \mathcal{T}_H) \cap \mathbb{S} ,\\
\boldsymbol{\Lambda} &:= \{\boldsymbol{\tau}\mathbf{n}^K|_{\partial K},\ \boldsymbol{\tau} \in W,\ K \in \mathcal{T}_H\} \equiv \bigcup_{K \in \mathcal{T}_H} H^{-1/2}(\partial K) ,\\
Q &:= \{\mathbf{q} \in [L^2(\Omega)]^d,\ \mathbf{q} \in [L^2(K)]^d,\ \forall K \in \mathcal{T}_H\} .
\end{aligned}
$$

The classical weak formulation associated with (2)

For any fixed t in $(0,T)$, find $(\mathbf{v},\boldsymbol{\sigma}) \in V \times W$, such that

$$\begin{cases} (\rho\,\partial_t \mathbf{v}\,,\,\mathbf{w})_{\mathcal{T}_H} + (\boldsymbol{\sigma}\,,\,\boldsymbol{\mathcal{E}}(\mathbf{w}))_{\mathcal{T}_H} - \displaystyle\sum_{K\in\mathcal{T}_H} \left(\boldsymbol{\sigma}\mathbf{n}^K\,,\,\mathbf{w}|_{\partial K}\right)_{\partial K} = (\mathbf{f}\,,\,\mathbf{w})_{\mathcal{T}_H}\;, \\ (\boldsymbol{\mathcal{A}}\,\partial_t\boldsymbol{\sigma}\,,\,\boldsymbol{\tau})_{\mathcal{T}_H} - (\boldsymbol{\mathcal{E}}(\mathbf{v})\,,\,\boldsymbol{\tau})_{\mathcal{T}_H} = 0\;, \end{cases} \tag{3}$$

$\forall(\mathbf{w},\boldsymbol{\tau}) \in V \times W$.

The classical hybrid weak formulation associated with (3)

For any fixed t in $(0,T)$, find $(\mathbf{v},\boldsymbol{\sigma},\lambda) \in V \times W \times \boldsymbol{\Lambda}$, such that

$$\begin{cases} (\rho\,\partial_t\mathbf{v}\,,\,\mathbf{w})_{\mathcal{T}_H} + (\boldsymbol{\sigma}\,,\,\boldsymbol{\mathcal{E}}(\mathbf{w}))_{\mathcal{T}_H} + \sum\limits_{K\in\mathcal{T}_H} (\lambda\,,\,\mathbf{w}|_{\partial K})_{\partial K} = (\mathbf{f}\,,\,\mathbf{w})_{\mathcal{T}_H}\,, \\ (\boldsymbol{\mathcal{A}}\,\partial_t\boldsymbol{\sigma}\,,\,\boldsymbol{\tau})_{\mathcal{T}_H} - (\boldsymbol{\mathcal{E}}(\mathbf{v})\,,\,\boldsymbol{\tau})_{\mathcal{T}_H} = 0\,, \\ \sum\limits_{K\in\mathcal{T}_H} (\mu\,,\,\mathbf{v}|_K)_{\partial K} = 0\,, \end{cases} \tag{4}$$

$\forall(\mathbf{w},\boldsymbol{\tau},\mu) \in V \times W \times \boldsymbol{\Lambda}$.

$$\Longrightarrow \lambda = -\boldsymbol{\sigma} n^K \text{ sur } \partial K\,,\ \forall K$$

Global/local formulation derived from (4) - 1

(4) is equivalent, for any fixed $t \in (0, T)$, to find $(\mathbf{v}, \boldsymbol{\sigma}, \lambda) \in V \times W \times \boldsymbol{\Lambda}$ satisfying

- the global problem

$$(\mu, \mathbf{v})_{\partial \mathcal{T}_H} := \sum_{K \in \mathcal{T}_H} (\mu, \mathbf{v})_{\partial K} = 0 \, , \quad \forall \mu \in \boldsymbol{\Lambda} \, , \tag{5}$$

- the local (independent) problems on each $K \in \mathcal{T}_H$

$$(\rho \, \partial_t \mathbf{v} \, , \mathbf{w})_K + (\boldsymbol{\sigma} \, , \mathcal{E}(\mathbf{w}))_K + (\lambda \, , \mathbf{w}|_{\partial K})_{\partial K} = (\mathbf{f} \, , \mathbf{w})_K \, , \; \forall \mathbf{w} \in V(K) \, , \tag{6}$$

$$(\mathcal{A} \partial_t \boldsymbol{\sigma} \, , \boldsymbol{\tau})_K - (\mathcal{E}(\mathbf{v}) \, , \boldsymbol{\tau})_K = 0 \, , \; \forall \boldsymbol{\tau} \in W(K) \, . \tag{7}$$

with $\mathbf{v}(0) = \mathbf{v}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$.

Remark

The local problem (6)-(7) has the following formulation (at least in the distribution sense)

$$\begin{cases} \rho\,\partial_t \mathbf{v} - \mathbf{div}\boldsymbol{\sigma} = \mathbf{f}\,, & \text{in } (0,T)\times K\,,\\ \mathcal{A}\partial_t\boldsymbol{\sigma} - \mathcal{E}(\mathbf{v}) = 0\,, & \text{in } (0,T)\times K\,,\\ \boldsymbol{\sigma}\mathbf{n}^K = -\lambda\,, & \text{on } (0,T)\times \partial K\,,\\ \mathbf{v} = \mathbf{v}_0\,, \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0\,, & \text{in } K \text{ at } t=0\,. \end{cases}$$

$\Longrightarrow$ If λ is known, that can be solved since $\mathbf{f}$, $\mathbf{v}_0$ and $\boldsymbol{\sigma}_0$ are given.

Starting point of MHM

Idea : split the solution couple $(\mathbf{v}, \boldsymbol{\sigma})$ as

- $\mathbf{v} := \mathbf{v}^{\lambda} + \mathbf{v}^{\mathbf{f}}$, $\boldsymbol{\sigma} := \boldsymbol{\sigma}^{\lambda} + \boldsymbol{\sigma}^{\mathbf{f}}$ and solve accordingly each elementwise local problem,
- write down consequently the corresponding global problem
- $(\mathbf{v}^{\lambda}, \boldsymbol{\sigma}^{\lambda})$ and $(\mathbf{v}^{\mathbf{f}}, \boldsymbol{\sigma}^{\mathbf{f}})$ respectively defined by the splitting operators P and $\hat{P}$:
  - $P : \mu \in \boldsymbol{\Lambda} \longmapsto P(\mu) = (P^{\mathbf{v}}(\mu), P^{\boldsymbol{\sigma}}(\mu)) = (\mathbf{v}^{\mu}, \boldsymbol{\sigma}^{\mu}) \in V \times W$
  - $\hat{P} : \mathbf{q} \in Q \longmapsto \hat{P}(\mathbf{q}) = (\hat{P}^{\mathbf{v}}(\mathbf{q}), \hat{P}^{\boldsymbol{\sigma}}(\mathbf{q})) = (\mathbf{v}^{\mathbf{q}}, \boldsymbol{\sigma}^{\mathbf{q}}) \in V \times W$

Definition of the splitting operators

- P and $\hat{P}$ are defined from (6) and (7)
- P and $\hat{P}$ are bounded linear operators and respectively defined locally on each $K \in \mathcal{T}_H$ by

$$\begin{cases} (\rho\,\partial_t P^{\mathbf{v}}\mu, \mathbf{w})_K + (P^{\boldsymbol{\sigma}}\mu, \boldsymbol{\mathcal{E}}(\mathbf{w}))_K = -(\mu, \mathbf{w})_{\partial K}, & \forall \mathbf{w} \in V(K), \\ (\boldsymbol{\mathcal{A}}\partial_t P^{\boldsymbol{\sigma}}\mu, \boldsymbol{\tau})_K - (\boldsymbol{\mathcal{E}}(P^{\mathbf{v}}\mu), \boldsymbol{\tau})_K = 0, & \forall \boldsymbol{\tau} \in W(K); \end{cases} \tag{8}$$

and

$$\begin{cases} \left(\rho\,\partial_t \hat{P}^{\mathbf{v}}\mathbf{q}, \mathbf{w}\right)_K + \left(\hat{P}^{\boldsymbol{\sigma}}\mathbf{q}, \boldsymbol{\mathcal{E}}(\mathbf{w})\right)_K = (\mathbf{q}, \mathbf{w})_K, & \forall \mathbf{w} \in V(K), \\ \left(\boldsymbol{\mathcal{A}}\partial_t \hat{P}^{\boldsymbol{\sigma}}\mathbf{q}, \boldsymbol{\tau}\right)_K - \left(\boldsymbol{\mathcal{E}}(\hat{P}^{\mathbf{v}}\mathbf{q}), \boldsymbol{\tau}\right)_K = 0, & \forall \boldsymbol{\tau} \in W(K); \end{cases} \tag{9}$$

for any given $\mu \in \boldsymbol{\Lambda}$ and any given $\mathbf{q} \in Q$ with initial conditions

- $P^{\mathbf{v}}\mu(0) = 0$, $P^{\boldsymbol{\sigma}}\mu(0) = 0$,
- $\hat{P}^{\mathbf{v}}\mathbf{q}(0) = \mathbf{v}_0$ and $\hat{P}^{\boldsymbol{\sigma}}\mathbf{q}(0) = \boldsymbol{\sigma}_0$.

"strong formulation" of (8) and (9) with $(\lambda, \mathbf{f})$

- From (8) :

$$\begin{cases} \rho\,\partial_t \mathbf{v}^\lambda - \mathbf{div}\boldsymbol{\sigma}^\lambda = \mathbf{0}\,, & \text{in } (0,T)\times K\,, \\ \mathcal{A}\partial_t \boldsymbol{\sigma}^\lambda - \mathcal{E}(\mathbf{v}^\lambda) = 0\,, & \text{in } (0,T)\times K\,, \\ \boldsymbol{\sigma}\mathbf{n}^K = -\lambda\,, & \text{on } (0,T)\times \partial K\,, \\ \mathbf{v}^\lambda = \mathbf{0}\,, & \text{and } \boldsymbol{\sigma}^\lambda = 0\,, \text{ in } K \text{ at } t=0\,. \end{cases} \tag{10}$$

- From (9)

$$\begin{cases} \rho\,\partial_t \mathbf{v}^\mathbf{f} - \mathbf{div}\boldsymbol{\sigma}^\mathbf{f} = \mathbf{f}\,, & \text{in } (0,T)\times K\,, \\ \mathcal{A}\partial_t \boldsymbol{\sigma}^\mathbf{f} - \mathcal{E}(\mathbf{v}^\mathbf{f}) = 0\,, & \text{in } (0,T)\times K\,, \\ \boldsymbol{\sigma}^\mathbf{f}\mathbf{n}^K = 0\,, & \text{on } (0,T)\times \partial K\,, \\ \mathbf{v}^\mathbf{f} = \mathbf{v}_0\,, & \text{and } \boldsymbol{\sigma}^\mathbf{f} = \boldsymbol{\sigma}_0\,, \text{ in } K \text{ at } t=0\,. \end{cases} \tag{11}$$

Summary before any discretization

Find $(\lambda, \mathbf{v} = \mathbf{v}^\lambda + \mathbf{v}^\mathbf{f}, \boldsymbol{\sigma} = \boldsymbol{\sigma}^\lambda + \boldsymbol{\sigma}^\mathbf{f}) \in \boldsymbol{\Lambda} \times V \times W$ such that

- Global problem

$$\sum_{K \in \mathcal{T}_H} \left(\mu, \mathbf{v}^\lambda\right)_{\partial K} = - \sum_{K \in \mathcal{T}_H} \left(\mu, \mathbf{v}^\mathbf{f}\right)_{\partial K} , \; \forall \mu \in \boldsymbol{\Lambda} ; \tag{12}$$

- Local splitted independent problems derived from (8) and (9) with $(\lambda, \mathbf{f})$

$$\begin{cases} \left(\rho \, \partial_t \mathbf{v}^\lambda , \mathbf{w}\right)_K + \left(\boldsymbol{\sigma}^\lambda , \mathcal{E}(\mathbf{w})\right)_K = -(\lambda, \mathbf{w})_{\partial K} , \quad \forall \mathbf{w} \in V(K) , \\ \left(\mathcal{A} \partial_t \boldsymbol{\sigma}^\lambda , \boldsymbol{\tau}\right)_K - \left(\mathcal{E}(\mathbf{v}^\lambda) , \boldsymbol{\tau}\right)_K = 0 , \quad \forall \boldsymbol{\tau} \in W(K) ; \end{cases}$$

and

$$\begin{cases} \left(\rho \, \partial_t \mathbf{v}^\mathbf{f} , \mathbf{w}\right)_K + \left(\boldsymbol{\sigma}^\mathbf{f} , \mathcal{E}(\mathbf{w})\right)_K = (\mathbf{f}, \mathbf{w})_K , \quad \forall \mathbf{w} \in V(K) , \\ \left(\mathcal{A} \partial_t \boldsymbol{\sigma}^\mathbf{f} , \boldsymbol{\tau}\right)_K - \left(\mathcal{E}(\mathbf{v}^\mathbf{f}) , \boldsymbol{\tau}\right)_K = 0 , \quad \forall \boldsymbol{\tau} \in W(K) . \end{cases}$$

Approximation of $\boldsymbol{\Lambda}$

λ uniquely determines $\mathbf{v}^{\lambda}, \boldsymbol{\sigma}^{\lambda}$ in $V \times W$. Let $\boldsymbol{\Lambda}_H$ be some FE space approximating $\boldsymbol{\Lambda}$. The coupled global/local problems to be solved are

- for any fixed t in $(0,T)$, find $\lambda_H(t) \in \boldsymbol{\Lambda}_H$, such that

$$\sum_{K \in \mathcal{T}_H} \left(\mu_H \, , \mathbf{v}^{\lambda_H} \right)_{\partial K} = - \sum_{K \in \mathcal{T}_H} \left(\mu_H \, , \mathbf{v}^{\mathbf{f}} \right)_{\partial K} \, , \tag{13}$$

  with $\mathbf{v}^{\lambda_H} := P^{\mathbf{v}} \lambda_H$, $\mathbf{v}^{\mathbf{f}} := \hat{P}^{\mathbf{v}} \mathbf{f}$.

- approximation of $(\mathbf{v}, \boldsymbol{\sigma})$ by $(\mathbf{v}_H, \boldsymbol{\sigma}_H)$, with

$$\mathbf{v}_H := \mathbf{v}^{\lambda_H} + \mathbf{v}^{\mathbf{f}} \, , \qquad \boldsymbol{\sigma}_H := \boldsymbol{\sigma}^{\lambda_H} + \boldsymbol{\sigma}^{\mathbf{f}} \, .$$

How getting $(\mathbf{v}_H, \boldsymbol{\sigma}_H)$?

$(\mathbf{v}^{\lambda_H}, \boldsymbol{\sigma}^{\lambda_H})$, $(\mathbf{v}^{\mathbf{f}}, \boldsymbol{\sigma}^{\mathbf{f}})$ are respectively solutions of

$$\begin{cases} \left(\rho\,\partial_t \mathbf{v}^{\lambda_H}, \mathbf{w}\right)_K + \left(\boldsymbol{\sigma}^{\lambda_H}, \mathcal{E}(\mathbf{w})\right)_K = -(\lambda_H, \mathbf{w})_{\partial K}, & \forall \mathbf{w} \in V(K), \\ \left(\mathcal{A}\partial_t \boldsymbol{\sigma}^{\lambda_H}, \boldsymbol{\tau}\right)_K - \left(\mathcal{E}(\mathbf{v}^{\lambda_H}), \boldsymbol{\tau}\right)_K = 0, & \forall \boldsymbol{\tau} \in W(K); \end{cases} \tag{14}$$

with $\mathbf{v}^{\lambda_H}(0) = \mathbf{0}$, $\boldsymbol{\sigma}^{\lambda_H}(0) = 0$, and

$$\begin{cases} \left(\rho\,\partial_t \mathbf{v}^{\mathbf{f}}, \mathbf{w}\right)_K + \left(\boldsymbol{\sigma}^{\mathbf{f}}, \mathcal{E}(\mathbf{w})\right)_K = (\mathbf{f}, \mathbf{w})_K, & \forall \mathbf{w} \in V(K), \\ \left(\mathcal{A}\partial_t \boldsymbol{\sigma}^{\mathbf{f}}, \boldsymbol{\tau}\right)_K - \left(\mathcal{E}(\mathbf{v}^{\mathbf{f}}), \boldsymbol{\tau}\right)_K = 0, & \forall \boldsymbol{\tau} \in W(K); \end{cases} \tag{15}$$

with $\mathbf{v}^{\mathbf{f}}(0) = \mathbf{v}_0$, $\boldsymbol{\sigma}^{\mathbf{f}}(0) = \boldsymbol{\sigma}_0$, in every K in $\mathcal{T}_H$

Approximations of $V(K)$ and $W(K)$

Let us assume we are given some stable FE pair $(V_h(K), W_h(K))$ with $\mathcal{T}_h(K)$. Suppose it is of continuous Galerkin type. We define

$$V_h := \bigoplus_{K \in \mathcal{T}_H} V_h(K) \subset V \,, \quad W_h := \bigoplus_{K \in \mathcal{T}_H} W_h(K) \subset W \,.$$

$(\mathbf{v}_H, \boldsymbol{\sigma}_H)$ are approximated by $(\mathbf{v}_{H,h}, \boldsymbol{\sigma}_{H,h})$, with

$$\mathbf{v}_{H,h} := \mathbf{v}_h^{\lambda_H} + \mathbf{v}_h^{\mathbf{f}} \,, \qquad \boldsymbol{\sigma}_{H,h} := \boldsymbol{\sigma}_h^{\lambda_H} + \boldsymbol{\sigma}_h^{\mathbf{f}} \,,$$

where each splitted component is defined by the discrete linear operators P_h and $\hat{P}_h$ and are solutions of the discrete equivalent systems associated with (14) and (15).

The global problem is written by : for any fixed t in $(0, T)$, find $\lambda_H \in \mathbf{\Lambda}_H$ such that

$$\sum_{K \in \mathcal{T}_H} \sum_{k \in \mathcal{T}_h(K)} \left(\mu_H \,, \mathbf{v}_h^{\lambda_H}\right)_{\partial k \cap \partial K} = - \sum_{K \in \mathcal{T}_H} \sum_{k \in \mathcal{T}_h(K)} \left(\mu_H \,, \mathbf{v}_h^{\mathbf{f}}\right)_{\partial k \cap \partial K} \,, \quad \forall \mu_H \in \mathbf{\Lambda}_H \,. \tag{16}$$

θ-scheme

$$t_{n_\theta} = (1-\theta)t_{n-1} + \theta t_n \ , \quad \theta \in]0,1].$$

- Global problem : find $\lambda_H^n \in \mathbf{\Lambda}_H$ such that

$$\sum_{K \in \mathcal{T}_H} \sum_{k \in \mathcal{T}_h(K)} \left(\mu_H \, , \mathbf{v}_h^{\lambda_H^n}\right)_{\partial k \cap \partial K} = - \sum_{K \in \mathcal{T}_H} \sum_{k \in \mathcal{T}_h(K)} \left(\mu_H \, , \mathbf{v}_h^{\mathbf{f},n}\right)_{\partial k \cap \partial K} \, , \quad \forall \mu_H \in \mathbf{\Lambda}_H \, . \tag{17}$$

- Local problem (λ_H^n dependent) for $(\mathbf{v}_h^{\lambda_H^n}, \boldsymbol{\sigma}_h^{\lambda_H^n})$

$$\left\{ \begin{array}{l} \frac{1}{\Delta t}\left(\rho \, \mathbf{v}_h^{\lambda_H^n} \, , \mathbf{w}_h\right)_K + \theta \left(\boldsymbol{\sigma}_h^{\lambda_H^n} \, , \mathcal{E}(\mathbf{w}_h)\right)_K = -\left(\lambda_H^n \, , \mathbf{w}_h\right)_{\partial K} \, , \ \forall \mathbf{w}_h \in V_h(K) \, , \\ \frac{1}{\Delta t}\left(\mathcal{A}\boldsymbol{\sigma}_h^{\lambda_H^n} \, , \boldsymbol{\tau}_h\right)_K - \theta \left(\mathcal{E}(\mathbf{v}_h^{\lambda_H^n}) \, , \boldsymbol{\tau}_h\right)_K = 0 \, , \ \forall \boldsymbol{\tau}_h \in W_h(K) \, , \end{array} \right. \tag{18}$$

with $\mathbf{v}_h^{\lambda_H^0} = \mathbf{0}$, $\boldsymbol{\sigma}_h^{\lambda_H^0} = 0$.

θ-scheme . . .

- Local problem ($\mathbf{f}$ dependent) for $(\mathbf{v}_h^{\mathbf{f},n}, \boldsymbol{\sigma}_h^{\mathbf{f},n})$

$$
\left\{
\begin{aligned}
\frac{1}{\Delta t}\left(\rho\,\mathbf{v}_h^{\mathbf{f},n}, \mathbf{w}_h\right)_K + \theta\left(\boldsymbol{\sigma}_h^{\mathbf{f},n}, \mathcal{E}(\mathbf{w}_h)\right)_K &= (\mathbf{f}, \mathbf{w}_h)_K + \frac{1}{\Delta t}\left(\rho\,\mathbf{v}_{H,h}^{n-1}, \mathbf{w}_h\right)_K \\
&\quad -(1-\theta)\left(\boldsymbol{\sigma}_{H,h}^{n-1}, \mathcal{E}(\mathbf{w}_h)\right)_K , \\
\frac{1}{\Delta t}\left(\mathcal{A}\boldsymbol{\sigma}_h^{\mathbf{f},n}, \boldsymbol{\tau}_h\right)_K - \theta\left(\mathcal{E}(\mathbf{v}_h^{\mathbf{f},n}), \boldsymbol{\tau}_h\right)_K &= (1-\theta)\left(\mathcal{E}(\mathbf{v}_{H,h}^{n-1}), \boldsymbol{\tau}_h\right)_K \\
&\quad + \frac{1}{\Delta t}\left(\mathcal{A}\boldsymbol{\sigma}_{H,h}^{n-1}, \boldsymbol{\tau}_h\right)_K ,
\end{aligned}
\right.
\tag{19}
$$

for all $(\mathbf{w}_h, \boldsymbol{\tau}_h) \in V_h(K) \times W_h(K)$, with $\mathbf{v}_h^{\mathbf{f},0} = \mathbf{v}_0$ and $\boldsymbol{\sigma}_h^{\mathbf{f},0} = \boldsymbol{\sigma}_0$.

How getting $\mathbf{v}_h^{\lambda_H^n}$, $\boldsymbol{\sigma}_h^{\lambda_H^n}$ and λ_H^n ?

Let $\{\boldsymbol{\psi}_j\}_{1\leq j\leq \dim \boldsymbol{\Lambda}_H}$ be a basis of $\boldsymbol{\Lambda}_H$, $\{\boldsymbol{\phi}_j\}_{1\leq j\leq \dim V_h}$ a basis of V_h and $\{\boldsymbol{\varphi}_j\}_{1\leq j\leq \dim W_h}$ be a basis of W_h. At each time step, we want to find $\{\beta_j^n\}_{1\leq j\leq \dim \lambda_H}$ such that

$$\lambda_H^n = \sum_j \beta_j^n \boldsymbol{\psi}_j \ .$$

Then it can be shown that

$$\mathbf{v}_h^{\lambda_H^n} = \sum_l \beta_l^n \boldsymbol{\eta}_l^H \ , \quad \boldsymbol{\eta}_l^H = T^{\mathbf{v}_{H,h}}(\boldsymbol{\psi}_l^H) \in V_h \ ,$$
$$\boldsymbol{\sigma}_h^{\lambda_H^n} = \sum_l \beta_l^n \boldsymbol{\chi}_l^H \ , \quad \boldsymbol{\chi}_l^H = T^{\boldsymbol{\sigma}_{H,h}}(\boldsymbol{\psi}_l^H) \in W_h \ ,$$

and

$$\boldsymbol{\eta}_l^H = \sum_{k\in\mathcal{T}_h(K)} \sum_j \left(\boldsymbol{\eta}_l^H, \boldsymbol{\phi}_j\right)_k \boldsymbol{\phi}_j \ , \quad \boldsymbol{\chi}_l^H = \sum_{k\in\mathcal{T}_h(K)} \sum_m \left(\boldsymbol{\chi}_l^H, \boldsymbol{\varphi}_m\right)_k \boldsymbol{\varphi}_l \ ,$$

for all $K \in \mathcal{T}_H$.

CGFE MHM algorithm with θ-scheme

1. **INITIALIZATION**
   Set $\mathbf{v}^0_{H,h} = \mathbf{v}_0$ and $\sigma^0_{H,h} = \sigma_0$ where $\mathbf{v}_0$, σ_0 are the initial conditions.
2. **BASIS FUNCTION COMPUTATION**
   Given the basis functions $\boldsymbol{\psi}_j$ of $\boldsymbol{\Lambda}_H(K)$, for each $K \in \mathcal{T}_H$, solve the following local problems to get $(\boldsymbol{\eta}_i, \boldsymbol{\chi}_i) \in V_h(K) \times W_h(K)$, for all $i = 1, \ldots, \dim \boldsymbol{\Lambda}_H(K)$

$$\begin{cases} \dfrac{1}{\Delta t}(\rho \boldsymbol{\eta}_i, \boldsymbol{\phi}_j)_K + \theta\left(\boldsymbol{\chi}_i, \mathcal{E}(\boldsymbol{\phi}_j)\right)_K & = -(\boldsymbol{\psi}_i, \boldsymbol{\phi}_j)_{\partial K}, \quad \forall \boldsymbol{\phi}_j \in V_h(K), \\ \dfrac{1}{\Delta t}(\mathcal{A}\boldsymbol{\chi}_i, \boldsymbol{\varphi}_j)_K - \theta\left(\mathcal{E}(\boldsymbol{\eta}_i), \boldsymbol{\varphi}_j\right)_K & = 0, \quad \forall \boldsymbol{\varphi}_j \in W_h(K). \end{cases}$$

CGFE MHM algorithm with θ-scheme . . .

3 TIME UPDATING

Do $n = 1$ to N

(3.1) COMPUTE $(\mathbf{v}_h^{\mathbf{f},n}, \boldsymbol{\sigma}_h^{\mathbf{f},n}) \in V_h(K) \times W_h(K)$, for each $K \in \mathcal{T}_H$, from

$$\begin{cases} \dfrac{1}{\Delta t}\left(\rho\, \mathbf{v}_h^{\mathbf{f},n}, \boldsymbol{\phi}_j^{k}\right)_K + \theta\left(\boldsymbol{\sigma}_h^{\mathbf{f},n}, \mathcal{E}(\boldsymbol{\phi}_j^{k})\right)_K & = \left(\mathbf{f}, \boldsymbol{\phi}_j^{k}\right)_K + \dfrac{1}{\Delta t}\left(\mathbf{v}_{H,h}^{n-1}, \boldsymbol{\phi}_j\right)_K \\ & \quad -(1-\theta)\left(\boldsymbol{\sigma}_{H,h}^{n-1}, \mathcal{E}(\boldsymbol{\phi}_j)\right)_K, \\ \dfrac{1}{\Delta t}\left(\mathcal{A}\boldsymbol{\sigma}_h^{\mathbf{f},n}, \boldsymbol{\varphi}_j\right)_K - \theta\left(\mathcal{E}(\mathbf{v}_h^{\mathbf{f},n}), \boldsymbol{\varphi}_j\right)_K & = \dfrac{1}{\Delta t}\left(\mathcal{A}\boldsymbol{\sigma}_{H,h}^{n-1}, \boldsymbol{\varphi}_j\right)_K \\ & \quad +(1-\theta)\left(\mathcal{E}(\mathbf{v}_{H,h}^{n-1}), \boldsymbol{\varphi}_j\right)_K, \end{cases}$$

for all $(\boldsymbol{\phi}_j, \boldsymbol{\varphi}_j) \in V_h(K) \times W_h(K)$.

CGFE MHM algorithm with θ-scheme ...

(3.2) GET THE DEGREES OF FREEDOM OF $\lambda_H^n \in \boldsymbol{\Lambda}_H$ by solving the following global problem

$$\sum_{j=1}^{\dim \boldsymbol{\Lambda}_H} \beta_j^n \left(\boldsymbol{\psi}_i\,,\boldsymbol{\eta}_j\right)_{\partial\mathcal{T}_H} = -\left(\boldsymbol{\psi}_i\,,\mathbf{v}_h^{\mathbf{f},n}\right)_{\partial\mathcal{T}_H}\,, \quad \forall i = 1,\ldots,\dim \boldsymbol{\Lambda}_H\,.$$

(3.3) UPDATE $(\mathbf{v}_{H,h}^n, \boldsymbol{\sigma}_{H,h}^n)$ from

$$\mathbf{v}_{H,h}^n = \sum_{j=1}^{\dim \boldsymbol{\Lambda}_H} \beta_j^n \boldsymbol{\eta}_j + \mathbf{v}_h^{\mathbf{f},n} \quad \text{and} \quad \boldsymbol{\sigma}_{H,h}^n = \sum_{j=1}^{\dim \boldsymbol{\Lambda}_H} \beta_j^n \boldsymbol{\chi}_j + \boldsymbol{\sigma}_h^{\mathbf{f},n}\,,$$

$$\lambda_H^n = \sum_{j=1}^{\dim \boldsymbol{\Lambda}_H} \beta_j^n \boldsymbol{\psi}_j\,.$$

End do

Conclusion and perspectives

- A general framework for MHM strategy to solve elastodynamic equations in heterogeneous anisotropic media.
- MHM is very attractive as being an additive multi-level method allowing both parallelism and adaption.
- Important questions concerning FE space pair for $\mathbf{v}$ and $\boldsymbol{\sigma}$, since symmetry of the stress tensor should be preserved and both compatibility and stability should be ensured.
- MHM restricted to heterogeneous but isotropic case with stress tensor vectorization formulation, DG solver and Leapfrog scheme.