Modeling of elastic Helmholtz equations by hybridizable discontinuous Galerkin method (HDG) for geophysical applications

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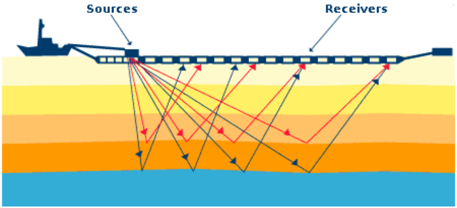
Motivation

Examples of seismic applications



Motivation

Imaging method : the full wave inversion

- Iterative procedure
- Inverse problem requiring to solve many direct problems

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Seismic imaging : time-domain or harmonic-domain ?

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Forward problem of the inversion process

- Elastic wave propagation in harmonic domain : Helmholtz equation
- Reduction of the size of the linear system

Motivation

Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities

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Use of unstructured meshes with FE methods

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DG method

- Flexible choice of interpolation orders ($p - adaptativity$)
- Highly parallelizable method
- Increased computational cost as compared to classical FEM

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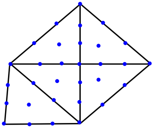
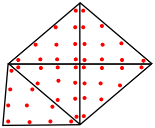


DOF of classical FEM



DOF of DGM

Motivation

Objective of this work

- Development of an hybridizable DG (HDG) method
- Comparison with a reference method : a standard nodal DG method

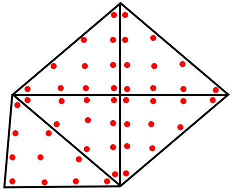


FIGURE: Degrees of freedom of DGM

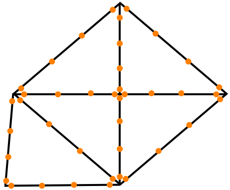


FIGURE: Degrees of freedom of HDGM

HDG methods

HDG methods

- B. Cockburn, J. Gopalakrishnan, R. Lazarov *Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, Vol. 47 (2009)
- S. Lanteri, L. Li, R. Perrussel, *Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations*, COMPEL, Vol. 32 (2013) (time-harmonic domain)
- N.C. Nguyen, J. Peraire, B. Cockburn, *High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics*, J. of Comput. Physics, Vol. 230 (2011) (time domain for seismic applications)

Contents

3D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla\cdot \underline{\underline{\sigma}}(\mathbf{x}) + f_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- Free surface condition : $\underline{\underline{\sigma}}\mathbf{n} = 0$ on Γ_l
- Absorbing boundary condition : $\underline{\underline{\sigma}}\mathbf{n} = PAP^T\mathbf{v}$ on Γ_a

- $\mathbf{v}$: velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

3D Helmholtz elastic equations

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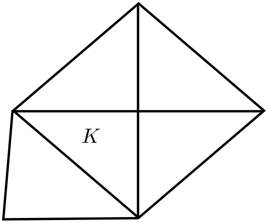
- ρ : mass density
- $\underline{\underline{C}}$: tensor of elasticity coefficients
- f_s : source term, $f_s \in L^2(\Omega)$

Contents

Notations and definitions

Notations

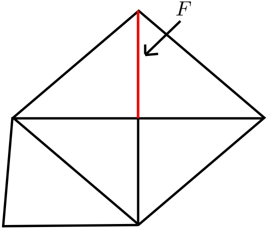
- $\mathcal{T}_h$ mesh of Ω composed of tetrahedrons K



Notations and definitions

Notations

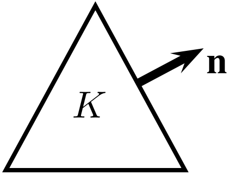
- $\mathcal{T}_h$ mesh of Ω composed of tetrahedrons K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$



Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of Ω composed of tetrahedrons K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$
- $\mathbf{n}$ the normal outward vector of an element K



Notations and definitions

Approximations spaces

- $P_p(K)$ set of polynomials of degree at most p on K

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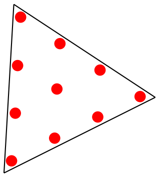

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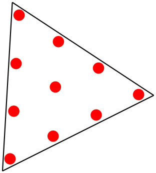

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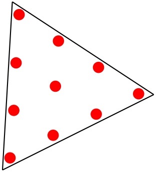
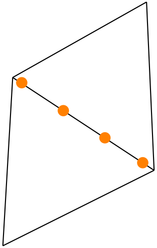
- $\mathbf{M}_h = \{\eta \in \left(L^2(\mathcal{F}_h)\right)^3 : \quad \eta|_F \in \left(P_p(F)\right)^3, \forall F \in \mathcal{F}_h\}$





Notations and definitions

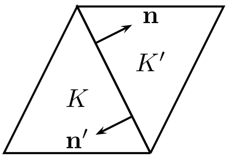
Definitions

- Jump $[\![\cdot]\!]$ of a vector $\mathbf{v}$ through F :

$$[\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^- = \mathbf{v}^+ \cdot \mathbf{n}^+ - \mathbf{v}^- \cdot \mathbf{n}^+$$

- Jump of a tensor $\underline{\underline{\sigma}}$ through F :

$$[\![\underline{\underline{\sigma}}]\!] = \underline{\underline{\sigma}}^+ \mathbf{n}^+ + \underline{\underline{\sigma}}^- \mathbf{n}^- = \underline{\underline{\sigma}}^+ \mathbf{n}^+ - \underline{\underline{\sigma}}^- \mathbf{n}^+$$



Contents

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{rcl} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} & = & 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\,\underline{\underline{\varepsilon}}(\mathbf{v}) & = & 0 \end{array} \right.$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}} \right) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and $\mathbf{v}^K$ respectively on ∂K

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Local HDG formulation

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We define :

$$\begin{aligned} \widehat{\mathbf{v}}^{\partial K} &= \widehat{\mathbf{v}}^F, \\ \widehat{\mathbf{v}}^F &= \lambda^F, && \forall F \in \mathcal{F}_h, \\ \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} &= \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F \right), && \text{on } \partial K \end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)

HDG formulation of the equations

Local HDG formulation

We replace $\widehat{\mathbf{v}}^K$ and $\left(\widehat{\underline{\underline{\sigma}}}^K \cdot \mathbf{n}\right)$ by their definitions into the local equations

$$\begin{cases} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla\mathbf{w} - \int_{\partial K} \underline{\underline{\sigma}}^K \cdot \mathbf{n} \cdot \mathbf{w} \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

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We define :

$$\underline{W}^K = \left(\underline{V_x}^K,\ \underline{V_y}^K,\ \underline{V_z}^K,\ \underline{\sigma_{xx}}^K,\ \underline{\sigma_{yy}}^K,\ \underline{\sigma_{zz}}^K,\ \underline{\sigma_{xy}}^K,\ \underline{\sigma_{xz}}^K,\ \underline{\sigma_{yz}}^K\right)^T$$

$$\underline{\Lambda} = \left(\underline{\Lambda}^{F_1},\ \underline{\Lambda}^{F_2}, ...,\ \underline{\Lambda}^{F_{n_f}}\right)^T, \text{ where } n_f = \mathrm{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\mathbb{A}^K \underline{W}^K + \sum_{F \in \partial K} \mathbb{C}^{K,F} \underline{\Lambda} = 0$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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$$\mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^K, the continuity of the normal component of $\underline{\underline{\widehat{\sigma}}}^K$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}]\!] \cdot \eta = 0$$

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Replacing $\left(\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}\right)$ and summing over all faces, the transmission condition becomes :

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \eta = 0$$

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Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

HDG formulation of the equations

Global HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}_K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \eta = 0 \end{cases}$$

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Global HDG discretization

$$\begin{cases} \mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}\right] = 0 \end{cases}$$

HDG formulation of the equations

Global HDG formulation

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Global HDG discretization

$$\begin{cases} \underline{W}^K = -(\mathbb{A}^K)^{-1} \mathbb{C}^K \underline{\Lambda} \\ \displaystyle\sum_{K\in\mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}\right] = 0 \end{cases}$$

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Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} \left[-\mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K + \mathbb{L}^K\right] \underline{\Lambda} = 0$$

Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $\mathbb{M}\underline{\Lambda}$

with $\mathbb{M} = \sum_{K \in \mathcal{T}_h} \left[-\mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K + \mathbb{L}^K \right]$

for $K = 1$ to Nb_{tri} **do**

Compute matrices $\mathbb{B}^K, (\mathbb{A}^K)^{-1}, \mathbb{C}^K$ and $\mathbb{L}^K$

Construction of the corresponding section of $\mathbb{M}$

end for

Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $\mathbb{M}\underline{\Lambda}$
2. Construction of the right hand side $\mathbb{S}$

Main idea of the algorithm using the HDG formulation

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2. Construction of the right hand side $\mathbb{S}$
3. Resolution $\mathbb{M}\underline{\Lambda} = \mathbb{S}$

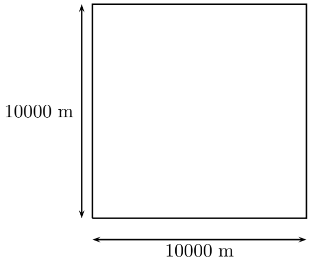
Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $\mathbb{M}\underline{\Lambda}$
2. Construction of the right hand side $\mathbb{S}$
3. Resolution $\mathbb{M}\underline{\Lambda} = \mathbb{S}$
4. Computation of the solutions of the initial problem

for $K = 1$ to Nb_{tri} **do**
  Compute $\underline{W}^K = -(\mathbb{A}^K)^{-1}\mathbb{C}^K\underline{\Lambda}$
end for

Contents

Plane wave



Computational domain Ω setting

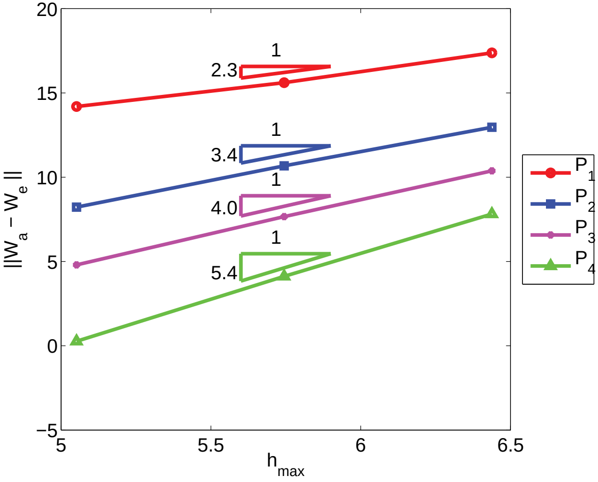
- Physical parameters :
  - $\rho = 2000 kg.m^{-3}$
  - $\lambda = 16 GPa$
  - $\mu = 8 GPa$
- Plane wave :

$$u = \nabla e^{i(k \cos\theta x + k \sin\theta y)}$$

where $k = \dfrac{\omega}{v_p}$

- $\theta = 0$
- Three meshes :
  - 3000 elements
  - 10000 elements
  - 45000 elements

Plane wave



Convergence order of the HDG scheme

Plane wave

Memory used

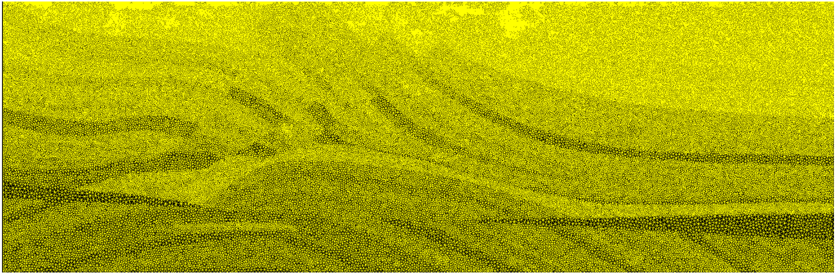
# Elements	Degree	Memory (MB) HDG	Memory (MB) UDG	Memory (MB) IPDG
3100	1	44	288	58
10300	1	161	1076	221
45000	1	797	5492	1156
3100	2	97	804	215
10300	2	355	3097	852
45000	2	1746	15965	4454
3100	3	170	1656	598
10300	3	624	6600	2394
45000	3	3080	34597	12362
3100	4	254	2749	1324
10300	4	947	10098	5251
45000	4	4653	50297	27314

Plane wave

Memory used

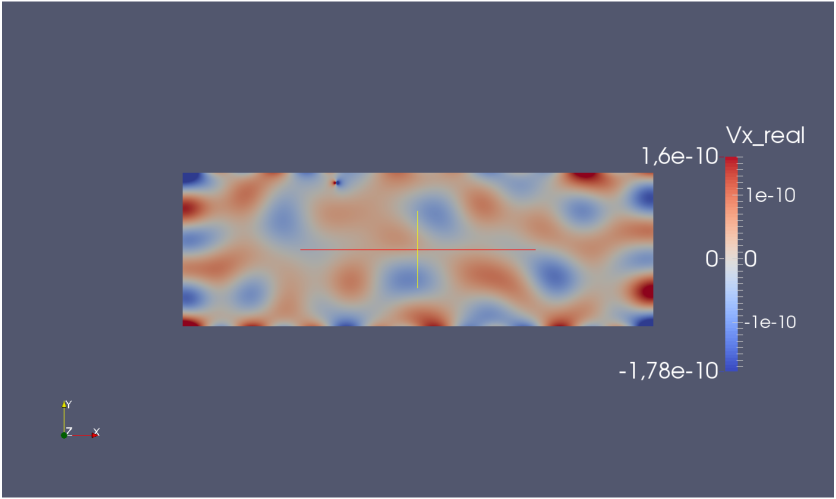
# Elements	Degree	Memory ratio HDG	Memory ratio UDG	Memory ratio IPDG
3100	1	1	6.55	1.32
10300	1	1	6.68	1.37
45000	1	1	6.89	1.45
3100	2	1	8.29	2.22
10300	2	1	8.72	2.4
45000	2	1	9.14	2.55
3100	3	1	9.74	3.52
10300	3	1	10.58	3.84
45000	3	1	11.23	4.01
3100	4	1	10.82	5.21
10300	4	1	10.66	5.54
45000	4	1	10.81	5.87

Marmousi test-case



Computational domain Ω composed of 235000 triangles

Parallel results for the Marmousi test-case with the HDG-P3 scheme, $f = 2\text{Hz}$



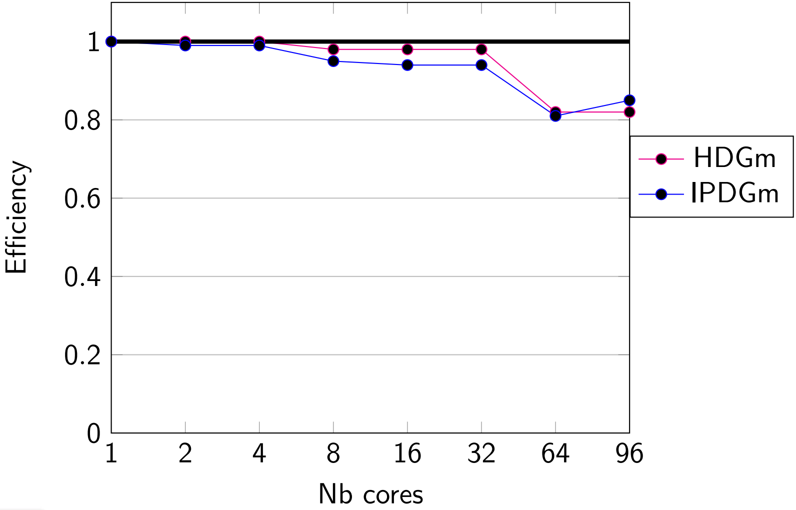
Caracteristics of the computing processors used

- **Plafrim** platform
- Hardware specification : 16 nodes, 12 cores by nodes
- Caracteristics of computing nodes :
  - 2 Hexa-core Westmere Intel® Xeon® X5670
  - Frequency : 2,93 GHz
  - Cache L3 : 12 Mo
  - RAM : 96 Go
  - Infiniband DDR : 20Gb/s
  - Ethernet : 1Gb/s

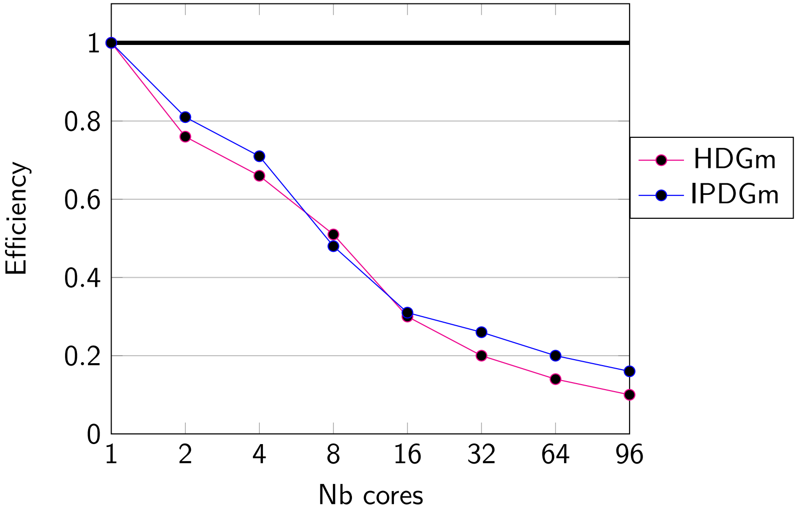
Efficiency of the parallelism of the global matrix construction

$$\text{Efficiency} = \frac{\text{sequential computational time}}{\text{nb\_cores} \times \text{computational time with one core}}$$

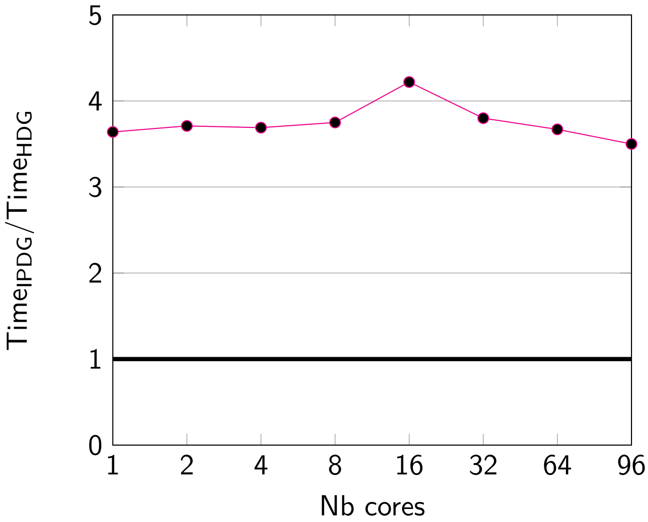
Efficiency of the parallelism of the global matrix construction



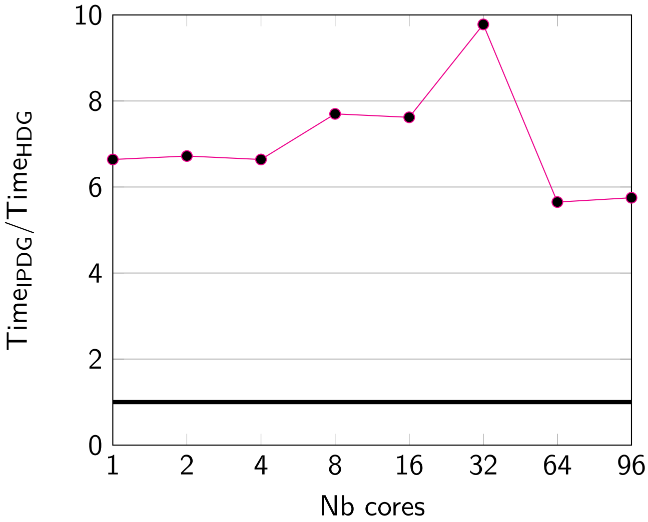
Efficiency of the parallelism of the resolution part



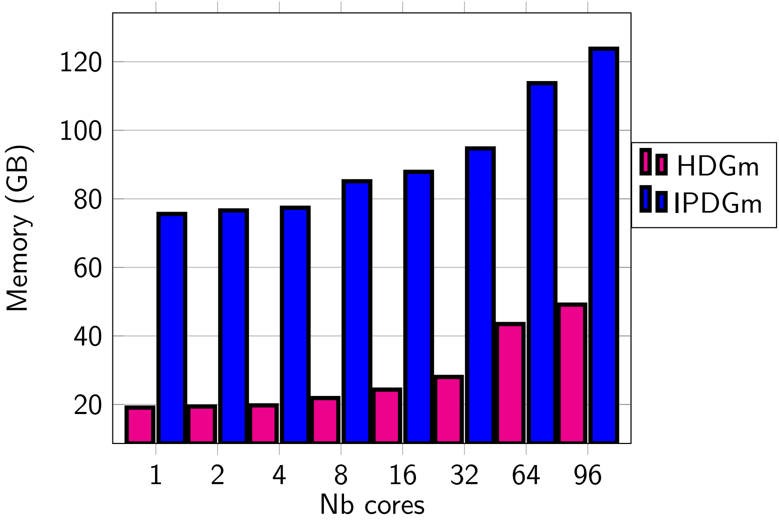
Speed up for the global matrix construction



Speed up (Total simulation time)



Memory required (GB) for the simulation



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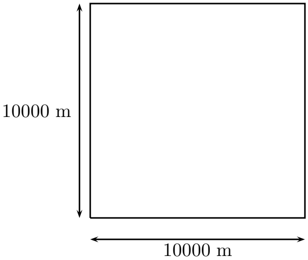
Plane wave



Computational domain Ω setting

- Physical parameters :
  - $\rho = 2000 kg.m^{-3}$
  - $\lambda = 16 GPa$
  - $\mu = 8 GPa$
- Plane wave :

$$u = \nabla e^{i(k \cos\theta x + k \sin\theta y)}$$

where $k = \frac{\omega}{v_p}$

- $\theta = 0$
- Two meshes :
  - 2600 elements
  - 6400 elements

3D preliminary numerical results

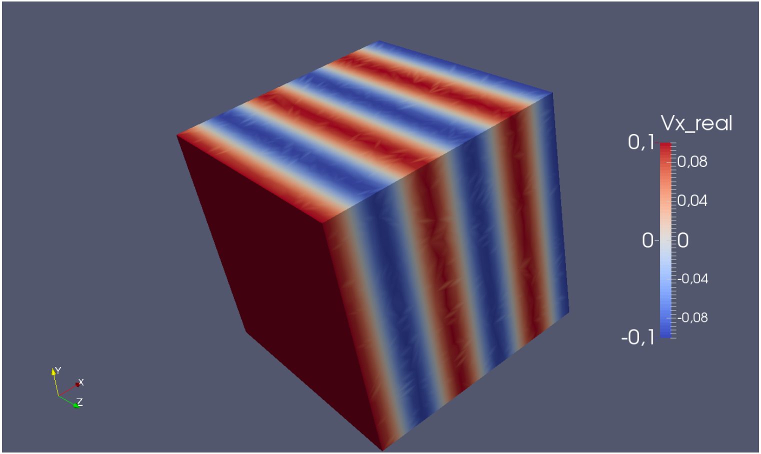


FIGURE: Exact solution

3D preliminary numerical results

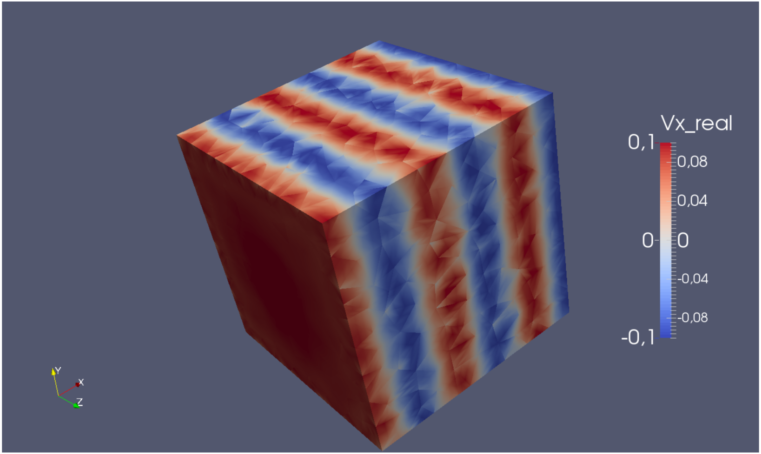


FIGURE: Numerical solution computed with P_3-HDG scheme on mesh composed of 6400 tetrahedrons

Elements	Order	CPU Time (s)		Memory (MB)	
		HDG	IPDG	HDG	IPDG
2600	1	11		572	
6400	1	59		1981	
2600	2	91		2254	
6400	2	498		7894	
2600	3	508		6242	
6400	3	2386		21899	

Elements	Order	CPU Time (s)		Memory (MB)	
		HDG	IPDG	HDG	IPDG
2600	1	11	12	572	461
6400	1	59	54	1981	1630
2600	2	91	123	2254	2687
6400	2	498	910	7894	10666
2600	3	508	1409	6242	13359
6400	3	2386	10370	21899	47922

Elements	Order	CPU Time (s)		Memory (MB)	
		HDG	IPDG	HDG	IPDG
2600	1	1		1	
6400	1	1		1	
2600	2	1		1	
6400	2	1		1	
2600	3	1		1	
6400	3	1		1	

Elements	Order	CPU Time (s)		Memory (MB)	
		HDG	IPDG	HDG	IPDG
2600	1	1	1.09	1	0.8
6400	1	1	0.9	1	0.8
2600	2	1	1.35	1	1.19
6400	2	1	1.83	1	1.35
2600	3	1	2.77	1	2.14
6400	3	1	4.35	1	2.19

Contents

Conclusions-Perspectives

Conclusions

On a same mesh, with the 2D HDG method :

- Memory gain
- Computational time gain

Conclusions-Perspectives

Conclusions

On a same mesh, with the 2D HDG method :

- Memory gain
- Computational time gain

Perspectives

- Study of the 3D HDG algorithm for Helmholtz equations
- Solution strategy for the HDG linear system

Thank you !