Multiscale Hybrid-Mixed Method for the Maxwell Equations in Time Domain

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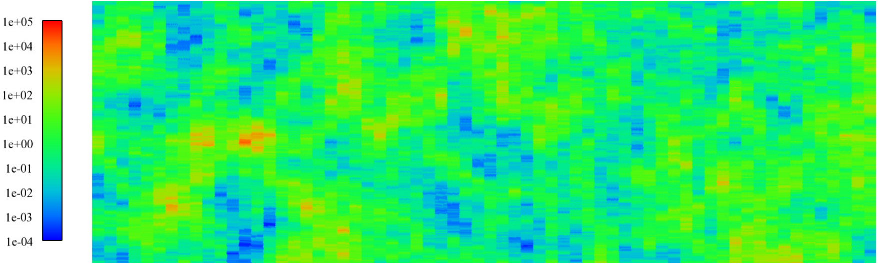
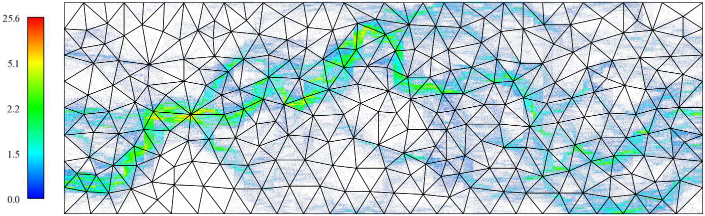
[1] Joint work with D. Paredes, S. Lanteri and C. Scheid

Motivation and Model

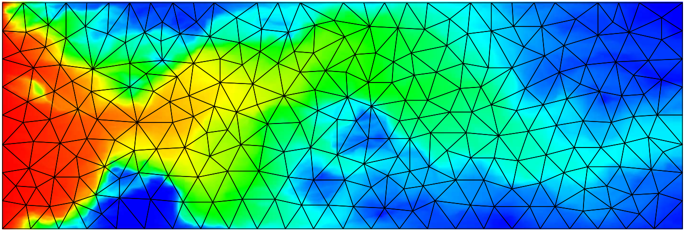
Heterogenous Domain



Flow Field



Saturation



The General MHM Idea

Hybrid Formulation (Raviart-Thomas '77)

Classical Weak Form: *Find $u \in U$ such that*

$$a(u, v)_\Omega = (f, v)_\Omega \quad \forall v \in U$$

Take $\mathcal{T}_h$ a (coarse) partition of Ω and set

$$V := \oplus \sum_{K \in \mathcal{T}_h} U(K)$$

Hybrid Form : *Find $(u,\, \lambda) \in V \times \Lambda$ such that*

$$\begin{aligned} a(u, v)_{\mathcal{T}_h} + (\lambda, v)_{\partial \mathcal{T}_h} &= (f,\, v)_{\mathcal{T}_h} \quad \forall v \in V \\ (\mu, u)_{\partial \mathcal{T}_h} &= 0 \quad \forall \mu \in \Lambda \end{aligned}$$

Solution Decomposition

Hybrid Form : *Find* $(u, \lambda) \in V \times \Lambda$ *such that*

$$a(u,v)_{\mathcal{T}_h} + (\lambda, v)_{\partial \mathcal{T}_h} = (f, v)_{\mathcal{T}_h} \quad \forall v \in V$$
$$(\mu, u)_{\partial \mathcal{T}_h} = 0 \quad \forall \mu \in \Lambda$$

$(v\,|_K, 0)$: First equation $\Leftrightarrow$ Collection of Local Problems[2]

$$a(u,v)_K = (f, v)_K - (\lambda, v)_{\partial K} \Rightarrow \quad u\,|_K = T\,\lambda + \hat{T}\,f$$

$(0, \mu)$: Second Equation $\Leftrightarrow$ Global Problem on Faces

$$(\mu, u)_{\partial \mathcal{T}_h} = 0 \quad \Leftrightarrow \quad (\mu, T\,\lambda)_{\partial \mathcal{T}_h} = -(\mu, \hat{T}\,f)_{\partial \mathcal{T}_h}$$

[2] Hyp: T, $\hat{T}$ are well-defined. Else see Harder-Paredes-Valentin, JCP '13

The MHM Method
(One Level)

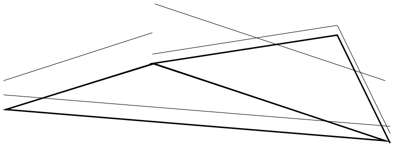
Finite Dimensional Space

We only need to select

$$\Lambda_H \subset \Lambda$$

Let $\lambda_H \in \Lambda_H$ be

$$\lambda_H = \sum_i c_i \, \psi_i$$



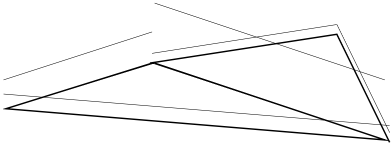
ψ_i

The MHM Method

Find $\lambda_H \in \Lambda_H$ *such that*

$$(\mu_H, T\,\lambda_H)_{\partial\mathcal{T}_H} = -(\mu_H, \hat{T}\,f)_{\partial\mathcal{T}_H} \quad \forall \mu_H \in \Lambda_H$$

where $T\,\lambda_H = \sum_i c_i \underbrace{T\,\psi_i}_{\eta_i}$ and $\hat{T}\,f$



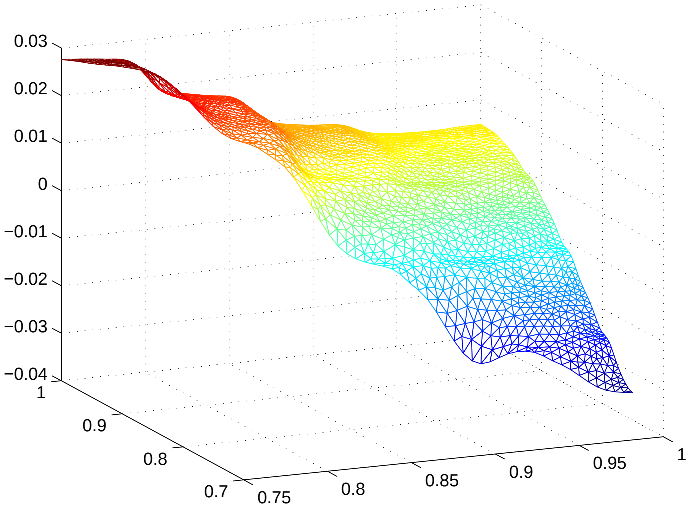
$$a(\eta_i, v)_K = -(\psi_i, v)_{\partial K}$$
$$a(\hat{T}\,f, v)_K = (f, v)_K$$

$$\lambda_H = \sum_i c_i\,\psi_i$$

Set

$$u_H = T\,\lambda_H + \hat{T}\,f$$

A Typical Heterogenous Base Function



The MHM Method
(Two Level)

How Does the Second Level Impact the MHM Method ?

Find $\lambda_H \in \Lambda_H$ *such that*

$$(\mu_H, T_h\, \lambda_H)_{\partial \mathcal{T}_H} = -(\mu_H, \hat{T}_h\, f)_{\partial \mathcal{T}_H} \quad \forall \mu_H \in \Lambda_H$$

$$T_h \approx T \quad \text{and} \quad \hat{T}_h \approx \hat{T}$$

Now

$$u \approx u_{H,h} := T_h\, \lambda_H + \hat{T}_h\, f$$

One Level

$$\Lambda_H \subset \Lambda \quad \text{and} \quad V(K)$$

$$\begin{cases} T\,\lambda_H = \sum_i c_i\,\eta_i \\ a(\eta_i,\, v)_K = -(\psi_i, v)_{\partial K} \end{cases}$$

$$\begin{cases} \hat{T}\,f \ \text{solves} \\ a(\hat{T}\,f,\, v)_K = (f, v)_K \end{cases}$$

Two Level

$$\Lambda_H \subset \Lambda \quad \text{and} \quad V_h(K) \subset V(K)$$

$$\begin{cases} T_h\,\lambda_H = \sum_i c_i\,\eta_i^h \\ a(\eta_i^h,\, v_h)_K = -(\psi_i, v_h)_{\partial K} \end{cases}$$

$$\begin{cases} \hat{T}_h\,f \ \text{solves} \\ a(\hat{T}_h\,f,\, v_h)_K = (f, v_h)_K \end{cases}$$

The Maxwell Case

The Model

Find $(\boldsymbol{e}, \boldsymbol{h}) : \Omega \times (0,T) \to \mathbb{R}^3$

$$\begin{cases} \varepsilon\, \partial_t \boldsymbol{e} - \nabla \times \boldsymbol{h} = \boldsymbol{f} \quad \text{in } \Omega \times (0,T) \\ \mu\, \partial_t \boldsymbol{h} + \nabla \times \boldsymbol{e} = \boldsymbol{0} \quad \text{in } \Omega \times (0,T) \\ \boldsymbol{e} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0,T) \\ \boldsymbol{e} = \boldsymbol{e}_0\,, \quad \boldsymbol{h} = \boldsymbol{h}_0 \quad \text{at } t = 0 \text{ on } \Omega \end{cases}$$

- μ and ε are the 3×3 symmetric tensor magnetic permeability and electric permittivity
- Initial field $\boldsymbol{e}_0$ and $\boldsymbol{h}_0$ and source $\boldsymbol{f}$ satisfy

$$\nabla \cdot \boldsymbol{f} = \nabla \cdot (\varepsilon\, \boldsymbol{e}_0) = \nabla \cdot (\mu\, \boldsymbol{h}_0) = 0 \quad \text{in } \Omega$$

To extend MHM strategy to handle

First Order PDE System

$+$

Time Dependent Model

Hybrid: Weak Continuity of $\boldsymbol{e} \times \boldsymbol{n}$

Find $(\boldsymbol{e}(t), \boldsymbol{h}(t), \boldsymbol{\lambda}(t)) \in \mathbf{V} \times \mathbf{V} \times \mathbf{\Lambda}$

$$\begin{cases} (\varepsilon\,\partial_t\,\boldsymbol{e}, \boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{h}, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{\lambda}, \boldsymbol{v})_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} \quad \forall \boldsymbol{v} \in \mathbf{V} \\ (\mu\,\partial_t\,\boldsymbol{h}, \boldsymbol{w})_{\mathcal{T}_h} + (\nabla \times \boldsymbol{e}, \boldsymbol{w})_{\mathcal{T}_h} = 0 \quad \forall \boldsymbol{w} \in \mathbf{V} \\ (\boldsymbol{\mu}, \boldsymbol{e})_{\partial\mathcal{T}_h} = 0 \quad \forall \boldsymbol{\mu} \in \mathbf{\Lambda} \end{cases}$$

- $(\boldsymbol{e}(t), \boldsymbol{h}(t))$ solves the standard weak form and

$$\boldsymbol{\lambda} = -\boldsymbol{h} \times \boldsymbol{n}^K \quad \text{on } \partial K \times (0, T)$$

- $(\boldsymbol{e}(t), \boldsymbol{h}(t)) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$

Semi-Discrete MHM method: Global Problem

Select

$$\mathbf{\Lambda}_H \subset \mathbf{\Lambda} \quad \text{and} \quad \mathbf{V}_h(K) \subset \mathbf{V}(K)$$

$$(\boldsymbol{e}(t), \boldsymbol{h}(t)) \approx (\boldsymbol{e}_h^{\boldsymbol{\lambda}_H}(t) + \boldsymbol{e}_h^{\boldsymbol{f}}(t), \boldsymbol{h}_h^{\boldsymbol{\lambda}_H}(t) + \boldsymbol{h}_h^{\boldsymbol{f}}(t))$$

Global Problem: Find $\boldsymbol{\lambda}_H(t) \in \mathbf{\Lambda}_H$

$$(\boldsymbol{\mu}_H, \boldsymbol{e}_h^{\boldsymbol{\lambda}_H})_{\partial\mathcal{T}_h} = -(\boldsymbol{\mu}_H, \boldsymbol{e}_h^{\boldsymbol{f}})_{\partial\mathcal{T}_h} \quad \text{for all } \boldsymbol{\mu}_H \in \mathbf{\Lambda}_H$$

Semi-Discrete MHM method: Local Maxwell Problems

$$\big(\varepsilon\,\partial_t \boldsymbol{e}_h^{\boldsymbol{\lambda}_H}, \boldsymbol{v}_h\big)_K - \big(\boldsymbol{h}_h^{\boldsymbol{\lambda}_H}, \nabla\times\boldsymbol{v}_h\big)_K = -(\boldsymbol{\lambda}_H, \boldsymbol{v}_h)_{\partial K} \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h(K)$$

$$\big(\mu\,\partial_t \boldsymbol{h}_h^{\boldsymbol{\lambda}_H}, \boldsymbol{w}_h\big)_K + \big(\nabla\times\boldsymbol{e}_h^{\boldsymbol{\lambda}_H}, \boldsymbol{w}_h\big)_K = 0 \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h(K)$$

with $\boldsymbol{e}_h^{\boldsymbol{\lambda}_H}(0) = \mathbf{0}$ and $\boldsymbol{h}_h^{\boldsymbol{\lambda}_H}(0) = \mathbf{0}$

$$\big(\varepsilon\,\partial_t \boldsymbol{e}_h^{\boldsymbol{f}}, \boldsymbol{v}_h\big)_K - \big(\boldsymbol{h}_h^{\boldsymbol{f}}, \nabla\times\boldsymbol{v}_h\big)_K = (\boldsymbol{f}, \boldsymbol{v}_h)_K \quad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h(K)$$

$$\big(\mu\,\partial_t \boldsymbol{h}_h^{\boldsymbol{f}}, \boldsymbol{w}_h\big)_K + \big(\nabla\times\boldsymbol{e}_h^{\boldsymbol{f}}, \boldsymbol{w}_h\big)_K = 0 \quad \text{for all } \boldsymbol{w}_h \in \mathbf{V}_h(K)$$

with $\boldsymbol{e}_h^{\boldsymbol{f}}(0) = \boldsymbol{e}_0$ and $\boldsymbol{h}_h^{\boldsymbol{f}}(0) = \boldsymbol{h}_0$

Semi-Discrete Approximation

We recall

$$(\boldsymbol{e}(t), \boldsymbol{h}(t)) \approx (\boldsymbol{e}_{H,h}(t), \boldsymbol{h}_{H,h}(t))$$

$$:= (\boldsymbol{e}_h^{\boldsymbol{\lambda}_H}(t) + \boldsymbol{e}_h^{\boldsymbol{f}}(t), \boldsymbol{h}_h^{\boldsymbol{\lambda}_H}(t) + \boldsymbol{h}_h^{\boldsymbol{f}}(t))$$

Fully-Discrete MHM method: Leap-Frog Case

Second-order Leap-Frog scheme $(n = 1, ..., N)$

$$(\boldsymbol{e}^{n+\frac{1}{2}}, \boldsymbol{h}^n) \approx (\boldsymbol{e}_{H,h}^{n+\frac{1}{2}}, \boldsymbol{h}_{H,h}^n)$$

Find $\boldsymbol{\lambda}_H^n \in \boldsymbol{\Lambda}_H$

$$(\boldsymbol{\mu}_H, \boldsymbol{e}_{H,h}^{n+\frac{1}{2}})_{\partial \mathcal{T}_h} = 0$$

$$\begin{cases} \left(\varepsilon \, \dfrac{\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} - \boldsymbol{e}_{H,h}^{n-\frac{1}{2}}}{\Delta t}, \boldsymbol{v}_h\right)_K - \left(\boldsymbol{h}_{H,h}^n, \nabla \times \boldsymbol{v}_h\right)_K = (\boldsymbol{f}, \boldsymbol{v}_h)_K - (\boldsymbol{\lambda}_H^n, \boldsymbol{v}_h)_{\partial K} \\ \left(\mu \, \dfrac{\boldsymbol{h}_{H,h}^n - \boldsymbol{h}_{H,h}^{n-1}}{\Delta t}, \boldsymbol{w}_h\right)_K + \left(\nabla \times \boldsymbol{e}_{H,h}^{n-\frac{1}{2}}, \boldsymbol{w}_h\right)_K = 0 \end{cases}$$

Numerical Algorithm and Validation

INVITATION
TO
DIEGO PAREDES' TALK

Conclusion

- New Multiscale Method for the Maxwell Equations
- Include Local Upscaling and Crossing Interfaces
- Capture Multiscale Features on Coarse Meshes
- Highly Adapted to Parallel Computation

Working in Progress (with Nachos Team / INRIA)

- Numerical analysis of the MHM Method (with C. Scheid and D. Paredes)
- Paralelization of the MHM method (with S. Lanteri, D. Paredes and R. Leger)
- Extension to the Elasto-dynamic model (with S. Lanteri and M.H. Lallemand)