Multiscale Hybrid-Mixed Method for the Maxwell Equations in Time Domain

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Maxwell Equation - Transverse magnetic mode

Maxwell Equations

$$+$$

No dependence on z

$$\mathbf{h} = \begin{bmatrix} h_x \\ h_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h_z \end{bmatrix} \quad \mathbf{e} = \begin{bmatrix} e_x \\ e_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e_z \end{bmatrix}$$

$$e_x = e_y = h_z = 0$$

Maxwell Equation - Transverse magnetic mode

Find (h_x, h_y, e_z) *such that*

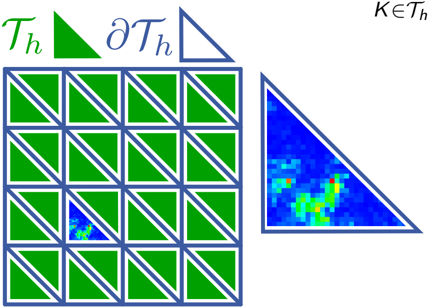
$$\left\{\begin{array}{rcll} \mu\,\partial_t h_x + \partial_y e_z &=& 0, & \text{in } \Omega \times (0,T) \\ \mu\,\partial_t h_y - \partial_x e_z &=& 0, & \text{in } \Omega \times (0,T) \\ \epsilon\,\partial_t e_z - \partial_x h_y + \partial_y h_x &=& j_z, & \text{in } \Omega \times (0,T) \\ e_z &=& 0, & \text{in } \Gamma \times (0,T) \\ (h_x, h_y, e_z) &=& (h_x^0, h_y^0, e_z^0), & \text{at } t=0, \text{ in } \Omega \end{array}\right.$$

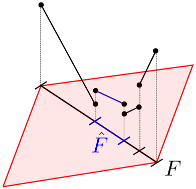
- $\epsilon(\mathbf{x})$, $\mu(\mathbf{x})$ and $j_z(\mathbf{x})$ may have *multi-scale* features

Objectives and Settings

- Captures multiscale features
- Parallelization in a natural way

$$V = \bigoplus_{K \in \mathcal{T}_h} V(K)$$





polynomial $\lambda_h \in \Lambda_h \subset \Lambda$

Variational Hybrid-Mixed Formulation (finite dimensional case)

Hybrid Formulation

Find $(h_x, h_y, e_z, \lambda) \in V \times V \times V \times \Lambda$

$$\begin{aligned} (\mu\, \partial_t h_x, w_x)_{\mathcal{T}_h} + (\partial_y e_z, w_x)_{\mathcal{T}_h} &= 0 \\ (\mu\, \partial_t h_y, w_y)_{\mathcal{T}_h} - (\partial_x e_z, w_y)_{\mathcal{T}_h} &= 0 \\ (\epsilon\, \partial_t e_z, w_z)_{\mathcal{T}_h} + (h_y, \partial_x w_z)_{\mathcal{T}_h} - (h_x, \partial_y w_z)_{\mathcal{T}_h} + (\lambda, w_z)_{\partial\mathcal{T}_h} &= (j_z, w_z)_{\mathcal{T}_h} \\ (e_z, \nu)_{\partial\mathcal{T}_h} &= 0 \end{aligned}$$

for all $(w_x, w_y, w_z, \nu) \in V \times V \times V \times \Lambda$

Global and local problems : $(h_x, h_y, e_z) = (h_x^\lambda, h_y^\lambda, e_z^\lambda) + (\bar{h}_x, \bar{h}_y, \bar{e}_z)$

Global problem

Find $\lambda \in \Lambda$

$$(e_z^\lambda, \nu)_{\partial\mathcal{T}_h} = -(\bar{e}_z, \nu)_{\partial\mathcal{T}_h}$$

for all $\nu \in \Lambda$

Global and local problems : $(h_x, h_y, e_z) = (h_x^\lambda, h_y^\lambda, e_z^\lambda) + (\bar{h}_x, \bar{h}_y, \bar{e}_z)$

Local problem for λ

Find $(h_x^\lambda, h_y^\lambda, e_z^\lambda) \in V(K) \times V(K) \times V(K)$

$$\begin{aligned}
(\mu\, \partial_t h_x^\lambda, w_x)_K + (\partial_y e_z^\lambda, w_x)_K &= 0 \\
(\mu\, \partial_t h_y^\lambda, w_y)_K - (\partial_x e_z^\lambda, w_y)_K &= 0 \\
(\epsilon\, \partial_t e_z^\lambda, w_z)_K + (h_y^\lambda, \partial_x w_z)_K - (h_x^\lambda, \partial_y w_z)_K &= -(\lambda, w_z)_{\partial K}
\end{aligned}$$

for all $(w_x, w_y, w_z) \in V(K) \times V(K) \times V(K)$

Global and local problems : $(h_x, h_y, e_z) = (h_x^\lambda, h_y^\lambda, e_z^\lambda) + (\bar{h}_x, \bar{h}_y, \bar{e}_z)$

Local problem (right-hand side)

Find $(\bar{h}_x, \bar{h}_y, \bar{e}_z) \in V(K) \times V(K) \times V(K)$

$$\begin{aligned}
(\mu\,\partial_t \bar{h}_x, w_x)_K + (\partial_y \bar{e}_z, w_x)_K &= 0 \\
(\mu\,\partial_t \bar{h}_y, w_y)_K - (\partial_x \bar{e}_z, w_y)_K &= 0 \\
(\epsilon\,\partial_t \bar{e}_z, w_z)_K + (\bar{h}_y, \partial_x w_z)_K - (\bar{h}_x, \partial_y w_z)_K &= (j_z, w_z)_K
\end{aligned}$$

for all $(w_x, w_y, w_z) \in V(K) \times V(K) \times V(K)$

Semi-discrete formulation $\Lambda_H \subset \Lambda$

$$\lambda_H = \sum_{i=1}^{N} c_i \, \psi_i$$

Global problem

Find $\lambda_H \in \Lambda_H$

$$(e_z^{\lambda_H}, \nu_H)_{\partial \mathcal{T}_h} \quad = \quad -(\bar{e}_z, \nu_H)_{\partial \mathcal{T}_h}$$

for all $\nu_H \in \Lambda_H$

$$\begin{bmatrix} h_x^H \\ h_y^H \\ e_z^H \end{bmatrix} = \begin{bmatrix} h_x^{\lambda_H} \\ h_y^{\lambda_H} \\ e_z^{\lambda_H} \end{bmatrix} + \begin{bmatrix} \bar{h}_x \\ \bar{h}_y \\ \bar{e}_z \end{bmatrix} = \sum_{i=1}^{N} c_i \begin{bmatrix} \eta_i^x \\ \eta_i^y \\ \eta_i^z \end{bmatrix} + \begin{bmatrix} \bar{h}_x \\ \bar{h}_y \\ \bar{e}_z \end{bmatrix}$$

Basis functions

Local basis functions, $i = 1, \ldots, N$

Find $(\eta_i^x, \eta_i^y, \eta_i^z) \in V(K) \times V(K) \times V(K)$

$$\begin{aligned} (\mu\, \partial_t \eta_i^x, w_x)_K + (\partial_y \eta_i^z, w_x)_K &= 0 \\ (\mu\, \partial_t \eta_i^y, w_y)_K - (\partial_x \eta_i^z, w_y)_K &= 0 \\ (\epsilon\, \partial_t \eta_i^z, w_z)_K + (\eta_i^y, \partial_x w_z)_K - (\eta_i^x, \partial_y w_z)_K &= -(\psi_i, w_z)_{\partial K} \end{aligned}$$

for all $(w_x, w_y, w_z) \in V(K) \times V(K) \times V(K)$

Temporal and spatial discretization choices

- Temporal discretization scheme
  - θ-scheme
  - Leap-frog scheme
- Second level (spatial) discretization
  - Classic Galerkin with $\mathbb{P}^k$ elements
  - Discontinuous Garlerkin

Algorithm

1. Calculate the local ($K \in \mathcal{T}_h$) basis functions $\eta_1, \ldots, \eta_{N_s}$
2. Mount the global matrix A
3. For $n = 1, \ldots, N_t$
    1. Calculate the functions $\bar{h}_x$, $\bar{h}_y$, $\bar{e}_z$
    2. Mount the right-hand side $\mathbf{b}$
    3. Solve the system $A\boldsymbol{\lambda}^n = \mathbf{b}$, where $\boldsymbol{\lambda}^n = [c_1^n, \ldots c_{N_s}^n]^T$
    4. Build the solution

$$\begin{bmatrix} h_x^{H,n} \\ h_y^{H,n} \\ e_z^{H,n} \end{bmatrix} = \begin{bmatrix} h_x^{\lambda_H^n} \\ h_y^{\lambda_H^n} \\ e_z^{\lambda_H^n} \end{bmatrix} + \begin{bmatrix} \bar{h}_x^n \\ \bar{h}_y^n \\ \bar{e}_z^n \end{bmatrix} = \sum_{i=1}^{N} c_i^n \begin{bmatrix} \eta_i^x \\ \eta_i^y \\ \eta_i^z \end{bmatrix} + \begin{bmatrix} \bar{h}_x^n \\ \bar{h}_y^n \\ \bar{e}_z^n \end{bmatrix}$$

Option 1

Classic Galerkin

$+$

θ-scheme

Semi-discrete formulation $\Lambda_H \subset \Lambda$

For $n = 0, \ldots, N$

$$\lambda_H^n = \sum_{i=1}^{N} c_i^n \, \psi_i$$

Global problem

Find $\lambda_H^{n+1} \in \Lambda_H$

$$(e_z^{\lambda_H^{n+1}}, \nu_H)_{\partial\mathcal{T}_h} = -(\bar{e}_z^{n+1}, \nu_H)_{\partial\mathcal{T}_h}$$

for all $\nu_H \in \Lambda_H$

$$\begin{bmatrix} h_x^{H,n} \\ h_y^{H,n} \\ e_z^{H,n} \end{bmatrix} = \begin{bmatrix} h_x^{\lambda_H^n} \\ h_y^{\lambda_H^n} \\ e_z^{\lambda_H^n} \end{bmatrix} + \begin{bmatrix} \bar{h}_x^n \\ \bar{h}_y^n \\ \bar{e}_z^n \end{bmatrix} = \sum_{i=1}^{N} c_i^n \begin{bmatrix} \eta_i^x \\ \eta_i^y \\ \eta_i^z \end{bmatrix} + \begin{bmatrix} \bar{h}_x^n \\ \bar{h}_y^n \\ \bar{e}_z^n \end{bmatrix}$$

Basis functions

Local basis functions

Find $(\eta_i^x, \eta_i^y, \eta_i^z) \in V(K) \times V(K) \times V(K)$

$$\begin{aligned}
(\mu\, \eta_i^x, w_x)_K + \theta\, \Delta t\, (\partial_y \eta_i^z, w_x)_K &= 0 \\
(\mu\, \eta_i^y, w_y)_K - \theta\, \Delta t\, (\partial_x \eta_i^z, w_y)_K &= 0 \\
(\epsilon\, \eta_i^z, w_z)_K + \theta\, \Delta t\, (\eta_i^y, \partial_x w_z)_K - \theta\, \Delta t\, (\eta_i^x, \partial_y w_z)_K &= -\theta\, \Delta t\, (\psi_i, w_z)_{\partial K}
\end{aligned}$$

for all $(w_x, w_y, w_z) \in V(K) \times V(K) \times V(K)$

No time-dependence

Right-hand side term

We assume $j_z = 0$

Local problem for $n = 0, \dots, N$

Find $(\bar{h}_x^{n+1}, \bar{h}_y^{n+1}, \bar{e}_z^{n+1}) \in V(K) \times V(K) \times V(K)$

$$\begin{aligned}
(\mu\, \bar{h}_x^{n+1}, w_x)_K + \theta\, \Delta t(\partial_y \bar{e}_z^{n+1}, w_x)_K &= (\mu\, h_x^n, w_x)_K \\
(\mu\, \bar{h}_y^{n+1}, w_y)_K - \theta\, \Delta t(\partial_x \bar{e}_z^{n+1}, w_y)_K &= (\mu\, h_y^n, w_y)_K \\
(\epsilon\, \bar{e}_z^{n+1}, w_z)_K + \theta\, \Delta t(\bar{h}_y^{n+1}, \partial_x w_z)_K - \theta\, \Delta t(\bar{h}_x^{n+1}, \partial_y w_z)_K &= (\epsilon\, e_z^n, w_z)_K
\end{aligned}$$

for all $(w_x, w_y, w_z) \in V(K) \times V(K) \times V(K)$

Option 2

Classic Galerkin

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Leap-Frog scheme

Semi-discrete formulation $\Lambda_H \subset \Lambda$

For $n = 0, \ldots, N$

$$\lambda_H^n = \sum_{i=1}^{N} c_i^n \, \psi_i$$

Global problem

Find $\lambda_H^{n+1} \in \Lambda_H$

$$(e_z^{\lambda_H^{n+1}}, \nu_H)_{\partial \mathcal{T}_h} = -(\bar{e}_z^{n+1}, \nu_H)_{\partial \mathcal{T}_h}$$

for all $\nu_H \in \Lambda_H$

$$e_z^{n,H} = e_z^{\lambda_H^n} + \bar{e}_z^n, \quad e_z^{\lambda_H^n} = \sum_{i=1}^{N} c_i^n \, \eta_i^z$$

Basis functions

Local basis functions

Find $\eta_i^z \in V(K)$

$$(\epsilon\, \eta_i^z, w_z)_K = \quad -\Delta t\, (\psi_i, w_z)_{\partial K}$$

for all $w_z \in V(K)$

Local problem for $n = 0, \dots, N$

Find $\bar{e}_z^{n+1} \in V(K)$

$$(\epsilon\, \bar{e}_z^{n+1}, w_z)_K = \quad -\Delta t(\bar{h}_y^{n+1}, \partial_x w_z)_K + \Delta t(\bar{h}_x^{n+1}, \partial_y w_z)_K + (\epsilon\, e_z^n, w_z)_K$$

for all $w_z \in V(K)$

Right-hand side term

Local problem for $n = 0, \ldots, N$

Find $h_x^{n+\frac{3}{2}} \in V(K)$

$$(\mu\, h_x^{n+\frac{3}{2}}, w_x)_K = \quad (\mu\, h_x^{n+\frac{1}{2}}, w_x)_K - \Delta t\, (\partial_y e_z^{n+1}, w_x)_K$$

for all $w_x \in V(K)$

Local problem for $n = 0, \ldots, N$

Find $h_y^{n+\frac{3}{2}} \in V(K)$

$$(\mu\, h_y^{n+\frac{3}{2}}, w_y)_K = \quad (\mu\, h_y^{n+\frac{1}{2}}, w_y)_K + \Delta t\, (\partial_x e_z^{n+1}, w_y)_K$$

for all $w_y \in V(K)$

Option 3 (The Goal)

Discontinous Galerkin

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Leap-Frog scheme

Semi-discrete formulation $\Lambda_H \subset \Lambda$

For $n = 0, \ldots, N$

$$\lambda_H^n = \sum_{i=1}^{N} c_i^n \, \psi_i$$

Global problem

Find $\lambda_H^{n+1} \in \Lambda_H$

$$(e_z^{\lambda_H^{n+1}}, \nu_H)_{\partial \mathcal{T}_h} = -(\bar{e}_z^{n+1}, \nu_H)_{\partial \mathcal{T}_h}$$

for all $\nu_H \in \Lambda_H$

$$e_z^{n,H} = e_z^{\lambda_H^n} + \bar{e}_z^n, \quad e_z^{\lambda_H^n} = \sum_{i=1}^{N} c_i^n \, \eta_i^z$$

Basis functions

$$V(K) = \bigoplus_{k \in \mathcal{T}_h(K)} V(k)$$

Local basis functions

Find $\eta_i^z \in V(K)$

$$(\epsilon\, \eta_i^z, w_z)_{\mathcal{T}_h(K)} = \quad -\Delta t \ \sum_{F \in \mathcal{E}_h^{ext}}, (\psi_i, w_z)_F$$

for all $w_z \in V(K)$

Local problem for $n = 0, \dots, N$

Find $\bar{e}_z^{n+1} \in V(K)$

$$\begin{aligned}(\epsilon\, \bar{e}_z^{n+1}, w_z)_{\mathcal{T}_h(K)} = \quad & -\Delta t(\bar{h}_y^{n+1}, \partial_x w_z)_{\mathcal{T}_h(K)} + \Delta t(\bar{h}_x^{n+1}, \partial_y w_z)_{\mathcal{T}_h(K)} \\ & -\Delta t \sum_{F \in \mathcal{E}_h^{int}} (n_y\, h_x - n_x\, h_y, w_z)_F + (\epsilon\, e_z^n, w_z)_{\mathcal{T}_h(K)}\end{aligned}$$

for all $w_z \in V(K)$

Right-hand side term

Local problem for $n = 0, \ldots, N$

Find $h_x^{n+\frac{3}{2}} \in V(K)$

$$(\mu\, h_x^{n+\frac{3}{2}}, w_x)_{\mathcal{T}_h(K)} = \begin{array}{l} (\mu\, h_x^{n+\frac{1}{2}}, w_x)_{\mathcal{T}_h(K)} + \Delta t\,(e_z^{n+1}, \partial_y w_x)_{\mathcal{T}_h(K)} \\ +n_y \sum_{F\in\mathcal{E}_h}(\{e_z\}, w_x)_F \end{array}$$

for all $w_x \in V(K)$

Local problem for $n = 0, \ldots, N$

Find $h_y^{n+\frac{3}{2}} \in V(K)$

$$(\mu\, h_y^{n+\frac{3}{2}}, w_y)_{\mathcal{T}_h(K)} = \begin{array}{l} (\mu\, h_y^{n+\frac{1}{2}}, w_y)_{\mathcal{T}_h(K)} - \Delta t\,(e_z^{n+1}, \partial_x w_y)_{\mathcal{T}_h(K)} \\ -n_x \sum_{F\in\mathcal{E}_h}(\{e_z\}, w_y)_F \end{array}$$

for all $w_y \in V(K)$

Exact Solution

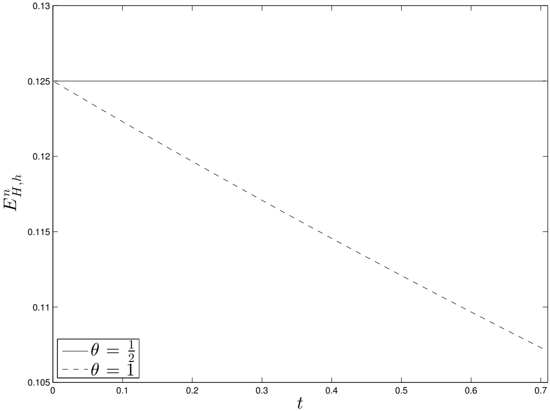
$$h_x(x, y, t) = \frac{\sqrt{2}}{2} \cos(2\sqrt{2}\pi t) \sin(2\pi x) \cos(2\pi y)$$

$$h_y(x, y, t) = -\frac{\sqrt{2}}{2} \cos(2\sqrt{2}\pi t) \cos(2\pi x) \sin(2\pi y)$$

$$e_z(x, y, t) = \sin(2\sqrt{2}\pi t) \sin(2\pi x) \cos(2\pi y)$$

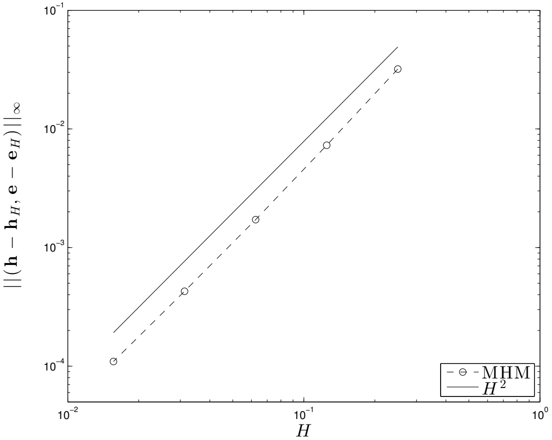
Continuous Galerkin + θ-scheme

$$E_h = \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K h_x^2 + h_y^2 + e_z^2$$



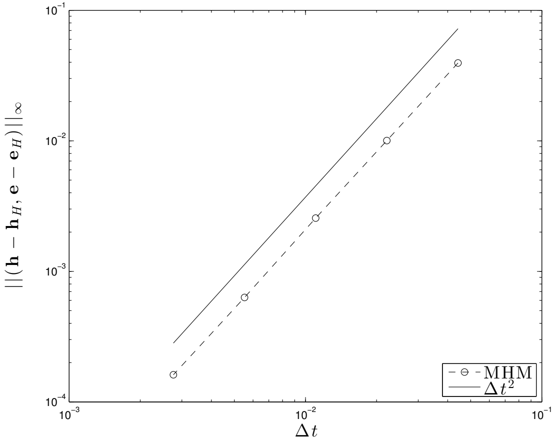
Continuous Galerkin + θ-scheme

$$l = 1,\ k = 2,\ \theta = \frac{1}{2}$$



Continuous Galerkin + θ-scheme

$$l = 1, \; k = 2, \; \theta = \frac{1}{2}$$



In progress...

1. Error analysis
2. Validate for multiscale experiments

Thank you!