The Multiscale Hybrid Mixed method for the Helmholtz equation

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The Helmholtz Equation

- Wave propagation at a given frequency
- Radar, Acoustics, Seismic (Inverse Problems)
- Heterogeneous media (Seismic)

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- Integral Equations (Homogeneous case)
- Finite Differences
- Finite Elements

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Difficulties

- Pollution effect (High frequency)
- Highly heterogeneous media

Model Problem

Find $u \in H^1(\Omega)$ such that

$$\begin{cases} -k^2 u - \Delta u = f & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - iku = 0 & \text{on } \partial\Omega, \end{cases}$$

where k is the wave number, and $f \in L^2(\Omega)$ is the source.

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Weak Form

Find $u \in H^1(\Omega)$ such that $a(u,v) = (f,v)_\Omega$ for all $v \in H^1(\Omega)$, with

$$a(u,v) = -k^2(u,v)_\Omega - ik(u,v)_{\partial\Omega} + (\nabla u, \nabla v)_\Omega.$$

The MHM method

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- Primal Hybrid Formulation.
- Multiscale Method.
- Adapted to highly heterogenous media.
- Interpretation as modified basis functions.
- Fine scales are localy captured by the basis functions.

Partition

- Let $\mathcal{T}_h$ be a partition of Ω.
- Let $\mathcal{E}_h$ be the set of edges in $\mathcal{T}_h$.
- Let $\mathcal{E}_h^{int}$ be the set of internal edges.
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$$V = H^1(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) \mid v \in H^1(K) \quad \forall K \in \mathcal{T}_h \right\},$$

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and

$$\Lambda = \left\{ \mu \in \prod_{K \in \mathcal{T}_h} H^{-1/2}(\partial K) \,\middle|\, \begin{array}{rl} \mu_+|_F + \mu_-|_F = 0 & \forall F \in \mathcal{E}_h^{int} \\ \mu|_F = 0 & \forall F \in \mathcal{E}_h^{ext} \end{array} \right\}.$$

Bilinear form

Define $a : V \times V \to \mathbb{C}$ and $b : V \times \Lambda \to \mathbb{C}$ by

$$\begin{aligned} a(u,v) &= -k^2(u,v)_{\mathcal{T}_h} - ik(u,v)_{\mathcal{E}_h^{ext}} + (\nabla u, \nabla v)_{\mathcal{T}_h}, \\ b(u,\mu) &= -\sum_{K \in \mathcal{T}_h} \langle \mu, u \rangle_{\partial K}. \end{aligned}$$

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Primal Hybrid Formulation

Find $(u,\lambda) \in V \times \Lambda$ such that

$$\left\{ \begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_{\mathcal{T}_h} && \forall v \in V \\ b(u,\mu) &= 0 && \forall \mu \in \Lambda. \end{aligned} \right.$$

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$$a(u, v) = (f, v)_{\mathcal{T}_h} - b(v, \lambda) \quad \forall v \in V.$$

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Now, pick $v_K \in V$ such that $\operatorname{supp} v_K \subset K$, then

$$\begin{aligned} a(u,v_K) &= -k^2(u,v_K)_K - ik(u,v_K)_{\partial K \cap \partial \Omega} + (\nabla u_K, \nabla v_K)_K \\ b(v_K,\lambda) &= -\langle \lambda, v_K \rangle_{\partial K \setminus \partial \Omega} \\ (f,v_K)_{\mathcal{T}_h} &= (f,v_K)_K. \end{aligned}$$

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So that $u_K = u|_K$ is solution of

$$\begin{cases} -k^2 u_K - \Delta u_K = f & \text{in } K \\ \nabla u_K \cdot \mathbf{n}_K = \lambda & \text{on } \partial K \setminus \partial\Omega \\ \nabla u_K \cdot \mathbf{n} - iku_K = 0 & \text{on } \partial K \cap \partial\Omega. \end{cases}$$

Well posedness of the local problems

We only need check for unicity

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Example

For an interior square $K = (0,h) \times (0,h) \subset \mathbb{R}^2$.

$$\lambda_0 = 0, \quad \lambda_1 = h^2\pi^2.$$

If $kh < \pi$, then $\lambda_0 < k^2 < \lambda_1$, and we have well posedness.

Local operators

Local operators

We define $T_K : H^{-1/2}(\partial K) \to H^1(K)$ for all $\mu \in H^{-1/2}(\partial K)$ with

$$a(T_K\mu, v_K) = -\langle \mu, v_K \rangle_{\partial K} \quad \forall v_K \in H^1(K).$$

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Local expression of u_K

$$u_K = T_K\lambda + \hat{T}_K f$$

Gobal expression of u

Grouping up the pieces we define $T : \Lambda \to V$ and $\hat{T} : L^2(\Omega) \to V$. Then

$$u = T\lambda + \hat{T} f.$$

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Global Problem

To obtain the global problem, we simply substitute $u = T\lambda + \hat{T}f$ in the the second equation $(b(u,\mu) = 0)$.

$$b(T\lambda, \mu) = -b(\hat{T}f, \mu) \quad \forall \mu \in \Lambda$$

The MHM method

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- First, localy define the operator T and $\hat{T}$ such that

$$
\begin{aligned}
a(T\mu, v) &= -b(\mu, v) \\
a(\hat{T}g, v) &= (g, v)_\Omega
\end{aligned}
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for all $\mu \in \Lambda$, $g \in L^2(\Omega)$ and $v \in V$.

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- Then $u = T\lambda + \hat{T}f$.

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$$-b(\hat{T}f, \mu) = a(T\mu, \hat{T}f) = a(\hat{T}f, T\mu) = (f, T\mu)_{\mathcal{T}_h}.$$

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- We can rewrite the global problem as

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- We only need to discretise $\Lambda_h \subset \Lambda$.
- The quality of the solution only depends upon Λ_h.
- We have to approximate $\lambda = \nabla u \cdot \mathbf{n}$.
- In 2d, we need basis functions living on one dimensional edges.

Some notations

- We note $(\varphi_j)_j$ a global basis of Λ_h.
- In each cell K, we note $(\psi_m^K)_m$ a local basis in each K.
- If we consider a φ_j, it lives on an edge $F = \partial K_+ \cap \partial K_-$.
- There are corresponding functions $\psi_{m_+}^{K_+}$ and $\psi_{m_-}^{K_-}$.

Local Problems

Local Problems

- Pick a cell $K \in \mathcal{T}_h$. For all ψ_m^K, we solve

$$a(\eta_m^K, v_K) = -\langle v_K, \psi_m^K \rangle_{\partial K} \quad \forall v_K \in H^1(K),$$

as well as

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- $\hat{f}$ will be used in the second member of the global problem.
- We only need to store their value on the edge of the mesh.

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- So that we can easily assemble the matrix and solve the linear system. Then we have

$$u_h = \sum_j \alpha_j \eta_j + \hat{f}.$$

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- It looks like we have solve the classical FEM formulation, with special basis functions η_j.
- The basis functions η_j being computed localy as solution of local subproblems.

A simple test

Let $\Omega = (0,1) \times (0,1)$ and $y = (0.5, 1.1)$. We consider the problem

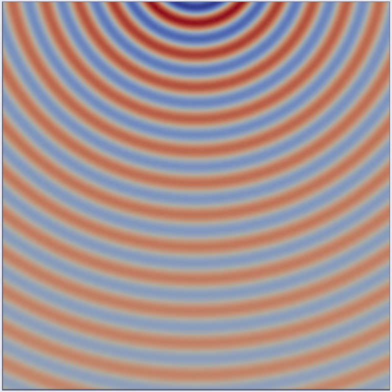
$$\left\{ \begin{array}{rcll} -k^2 u - \Delta u & = & 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - iku & = & \nabla \mathcal{H} \cdot \mathbf{n} - ik\mathcal{H} & \text{on } \partial\Omega, \end{array} \right.$$

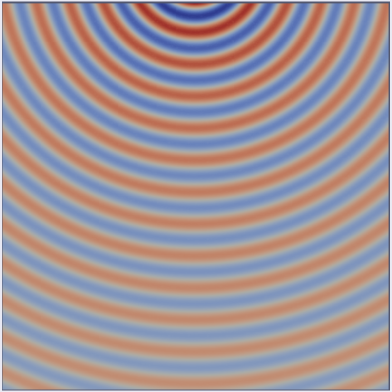
where

$$\mathcal{H}(x) = Y(k|x-y|) + iJ(k|x-y|),$$

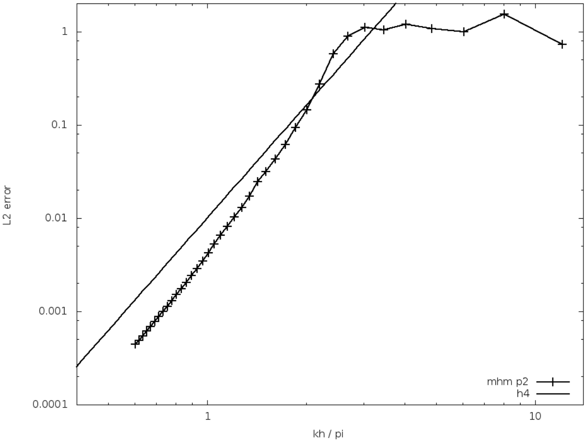
for all $x \in \Omega$. We set $k = 2\pi f$, with $f = 12.1$.

Real and Imaginary part of the solution

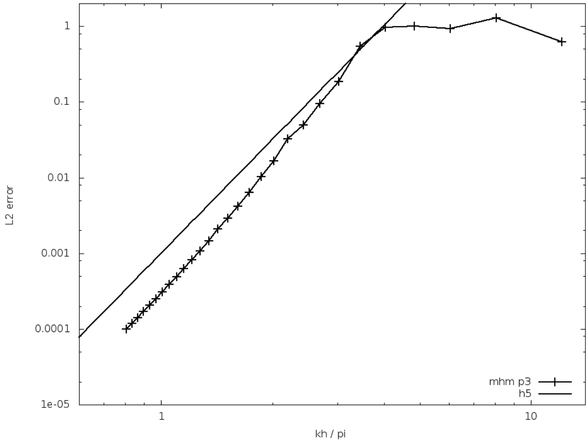




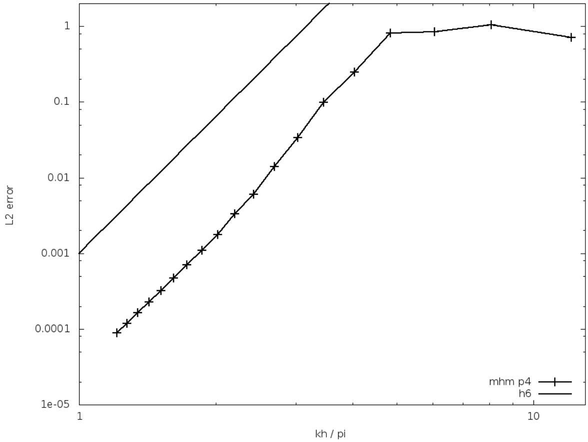
Convergence curve $l = 2$



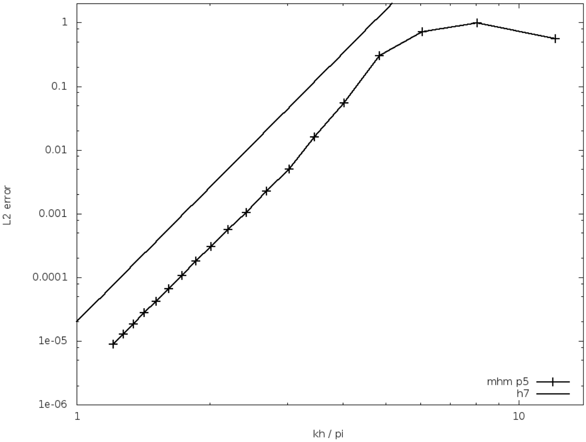
Convergence curve $l = 3$



Convergence curve $l = 4$



Convergence curve $l = 5$



Anisotropy study

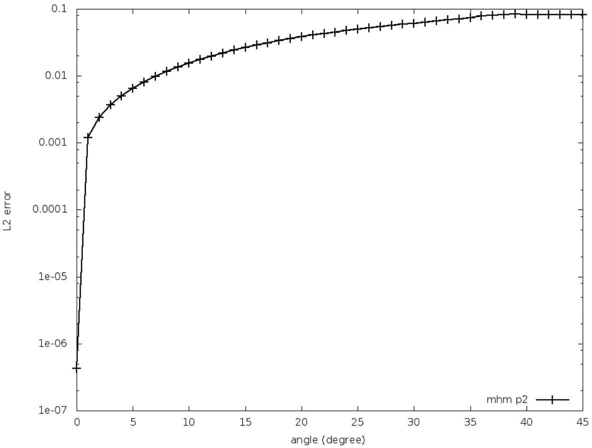
We still consider $\Omega = (0,1) \times (0,1)$ and $f = 12.1$. For every angle $\theta \in [0, \pi/4]$, we consider the plane wave

$$\mathbf{e}_\theta(x) = e^{ik\nu \cdot x},$$

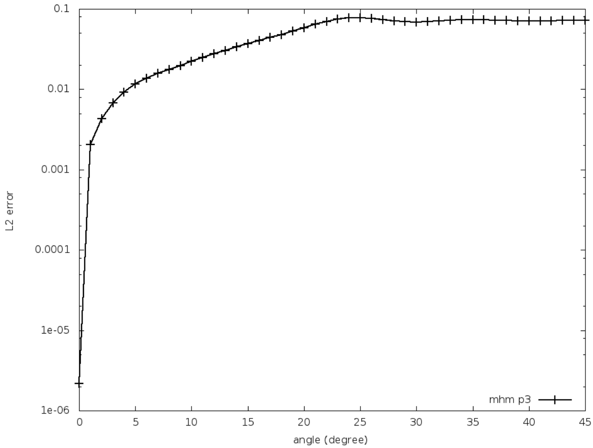
for $x \in \Omega$ with $\nu = (\cos\theta, \sin\theta)$. We solve the following problem

$$\begin{cases} -k^2 u - \Delta u &= 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - iku &= \nabla \mathbf{e}_\theta \cdot \mathbf{n} - ik\mathbf{e}_\theta & \text{on } \partial\Omega. \end{cases}$$

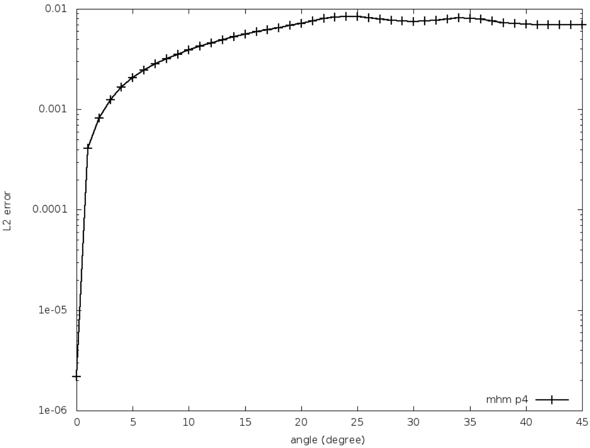
$n = 15, l = 2$


0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
L2 error
0
5
10
15
20
25
30
35
40
45
angle (degree)
mhm p2

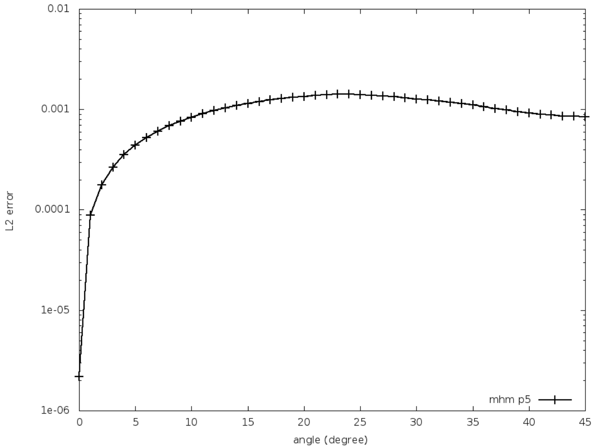
$n = 10, l = 3$


0.1
0.01
0.001
0.0001
1e-05
1e-06
L2 error
0
5
10
15
20
25
30
35
40
45
angle (degree)
mhm p3

$n = 10, l = 4$


0.01
0.001
0.0001
1e-05
1e-06
L2 error
0
5
10
15
20
25
30
35
40
45
angle (degree)
mhm p4

$n = 10, l = 5$


0.01
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0
5
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25
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angle (degree)
mhm p5

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- On a cartesian mesh, we have either $\nu = (1,0)$ or $(0,1)$ and

$$\nu \cdot \mathbf{v}_F = \pm\frac{\sqrt{2}}{2}.$$

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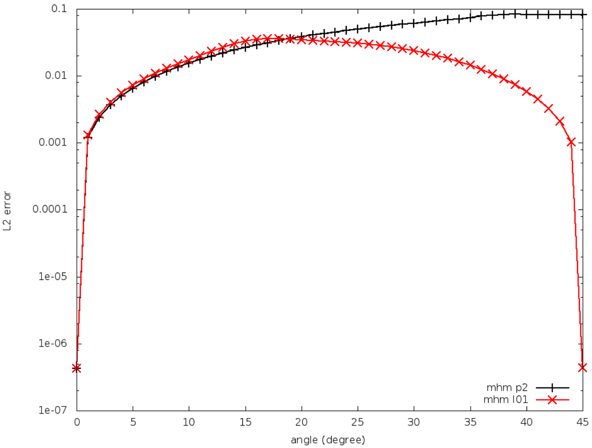
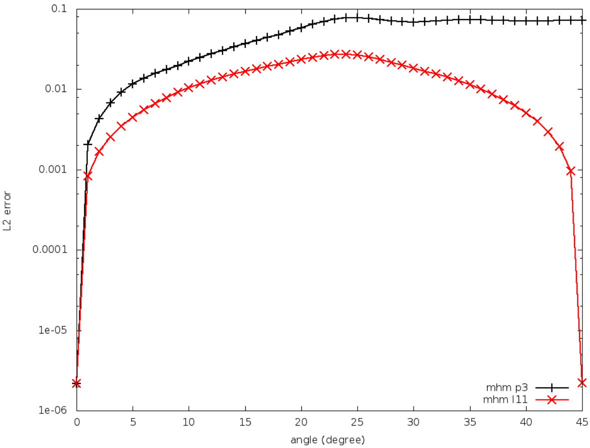
$$\phi_0(t) = 1, \quad \phi_+^m(t) = e^{ik\alpha_m t}, \quad \phi_-^m(t) = e^{ik\alpha_m t},$$

with $\alpha_m = \cos(m\pi/2n)$, and $m = 1, n-1$.

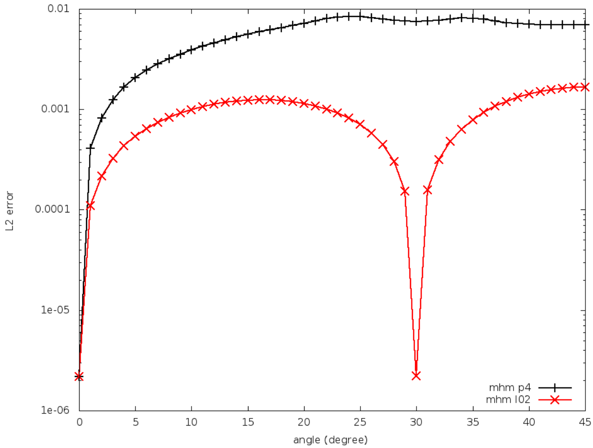
$l = 15$, Λ_2 vs Λ_0^1



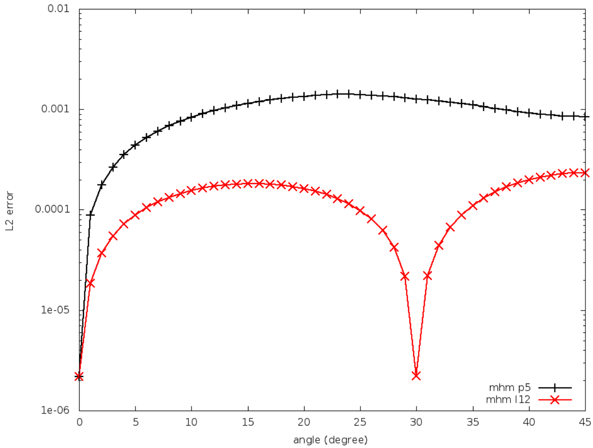
$l = 10$, Λ_3 vs Λ_1^1



$l = 10$, Λ_4 vs Λ_0^2



$l = 10$, Λ_5 vs Λ_1^2



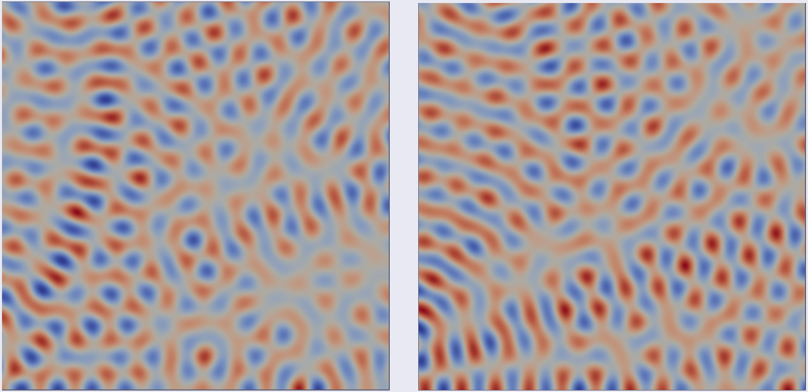
A more complex test

We solve

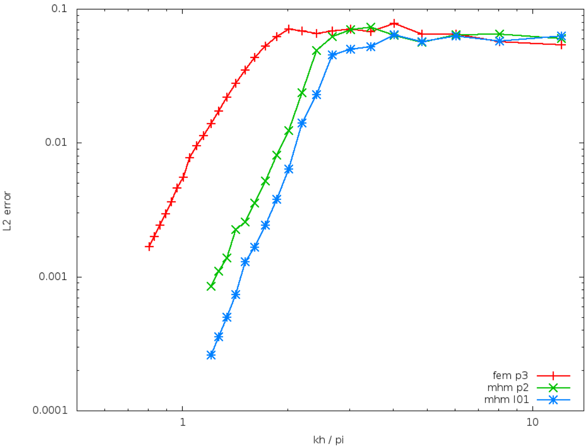
$$\left\{ \begin{array}{rcll} -k^2u - \Delta u & = & 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - iku & = & g & \text{on } \partial\Omega \setminus \Gamma \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{on } \Gamma, \end{array} \right.$$

where g is composed of 3 ponctual sources with different locations and amplitudes.

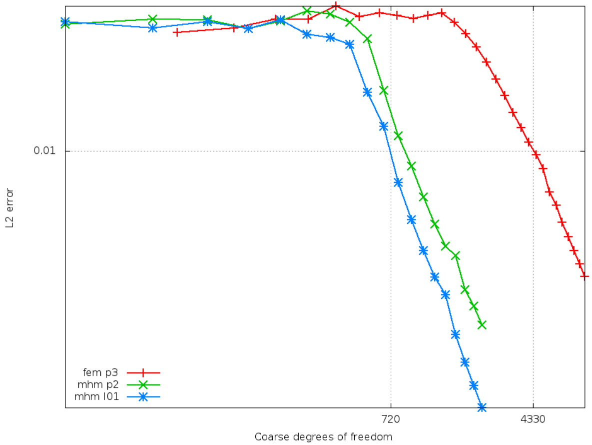
Real and Imaginary part of the solution



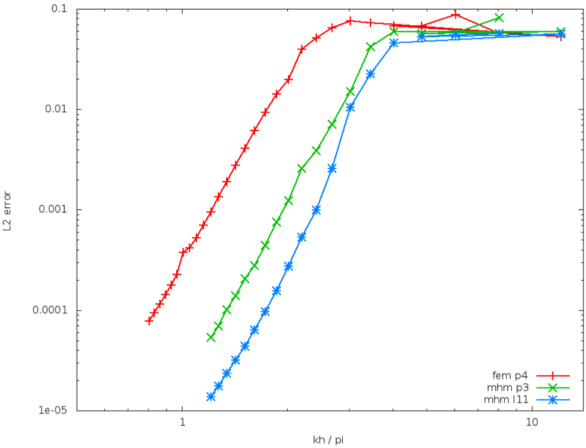
$\mathcal{P}_3$, Λ_2 vs Λ_0^1



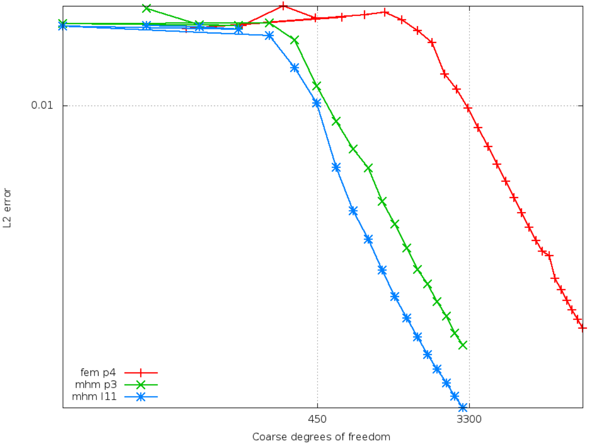
$\mathcal{P}_3$, Λ_2 vs Λ_0^1



$\mathcal{P}_4$, Λ_3 vs Λ_1^1



$\mathcal{P}_4$, Λ_3 vs Λ_1^1



Conclusion and perspectives

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- We want to tackle (highly) heterogeneous problems.
- Condition on the mesh to avoid resonance?
- Choice of basis functions?