Discontinuous Galerkin methods for solving Helmholtz isotropic wave equations for seismic applications

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Motivation

Examples of the seismic applications

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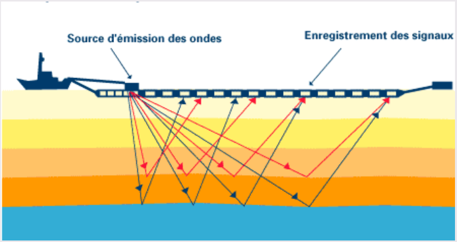
Examples of the seismic applications



Motivation

Imaging method : the full wave inversion

- Quantitative high resolution images of the subsurface physical parameters

Forward problem of the inversion process

- Elastic waves propagation in harmonic domain : Helmholtz equation

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Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities

DG method

- Use of triangular unstructured meshes
- Flexible choice of interpolation orders ($p - adaptativity$)

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Drawback of DG method

- Important computational cost

Main objective of the thesis

- Development of an hybridizable DG (HDG) method
- Development of a reference method, a classical DG method

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First order formulation of Helmoltz wave equations

$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla\cdot \underline{\underline{\sigma}}(\mathbf{x}) + f_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- Free surface condition : $\underline{\underline{\sigma}}\mathbf{n} = 0$ on Γ_l
- Absorbing boundary condition : $\underline{\underline{\sigma}}\mathbf{n} = v_p(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + v_s(\mathbf{v}\cdot\mathbf{t})\mathbf{t}$ on Γ_a

- $\mathbf{v}$: velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

2D Helmholtz elastic equations

First order formulation of Helmoltz wave equations

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- ρ : mass density
- $\underline{\underline{C}}$: tensor of elasticity coefficients
- f_s : source term, $f_s \in L^2(\Omega)$
- v_p : P-wave velocity
- v_s : S-wave velocity

2D Helmholtz isotropic elastic equations

First order formulation of Helmoltz isotropic wave equations

$$
\begin{cases}
i\omega v_x = \dfrac{1}{\rho}\left(\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xz}}{\partial z}\right) \\
i\omega v_z = \dfrac{1}{\rho}\left(\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{zz}}{\partial z}\right) \\
i\omega \sigma_{xx} = (\lambda + 2\mu)\dfrac{\partial v_x}{\partial x} + \lambda \dfrac{\partial v_z}{\partial z} \\
i\omega \sigma_{zz} = \lambda \dfrac{\partial v_x}{\partial x} + (\lambda + 2\mu)\dfrac{\partial v_z}{\partial z} \\
i\omega \sigma_{xz} = \mu\left(\dfrac{\partial v_x}{\partial z} + \dfrac{\partial v_z}{\partial x}\right)
\end{cases}
$$

λ and μ Lamé's constants and $v_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and $v_s = \sqrt{\dfrac{\mu}{\rho}}$

2D Helmholtz isotropic elastic equations

Vectorial form

$$i\omega \mathbf{Q} + \mathbf{A_x}\frac{\partial \mathbf{Q}}{\partial x} + \mathbf{A_z}\frac{\partial \mathbf{Q}}{\partial z} = 0$$

where $\mathbf{Q} = (v_x, v_z, \sigma_{xx}, \sigma_{zz}, \sigma_{xz})^{\mathrm{T}}$ and :

$$\mathbf{A_x} = -\begin{pmatrix} 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho} \\ \lambda+2\mu & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A_z} = -\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & \lambda+2\mu & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \end{pmatrix}$$

Contents

DG methods in time domain for seismic applications

DG methods in time domain for seismic applications

- M. Dumbser and M. Käser, *An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes - II; The three-dimensional isotropic case*, 2006 (upwind scheme)
- S. Delcourte, L.Fezoui and N. Glinsky-Olivier, *A high order discontinuous Galerkin method for the seismic wave propagation*, 2009 (centered scheme)

Notations and definitions

Notations

- Γ_l free surface boundary
- Γ_a the absorbing boundary
- $\mathcal{T}_h$ mesh of Ω composed of triangles K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$
- $\mathbf{n}$ the normal outward vector of an element K

Notations and definitions

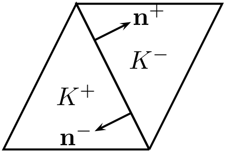
Definitions

- Jump $[\![\cdot]\!]$ of a vector $\mathbf{u}$ for F :

$$[\![\mathbf{u}]\!] = \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^- = \mathbf{u}^+ \cdot \mathbf{n}^+ - \mathbf{u}^- \cdot \mathbf{n}^+$$

- Jump of a tensor $\underline{\underline{\sigma}}$ for F :

$$[\![\underline{\underline{\sigma}}]\!] = \underline{\underline{\sigma}}^+ \mathbf{n}^+ + \underline{\underline{\sigma}}^- \mathbf{n}^- = \underline{\underline{\sigma}}^+ \mathbf{n}^+ - \underline{\underline{\sigma}}^- \mathbf{n}^+$$

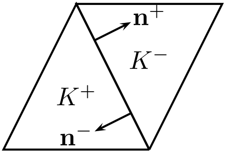


Notations and definitions

Definitions

- Average $\{\cdot\}$ of a variable u, for F :

$$\{u\} = \frac{u^+ + u^-}{2}$$



DG formulation of the original equation

Local DG formulation

$$i\omega \mathbf{Q} + \mathbf{A_x}\frac{\partial \mathbf{Q}}{\partial x} + \mathbf{A_z}\frac{\partial \mathbf{Q}}{\partial z} = 0$$

DG formulation of the original equation

Local DG formulation

$$i\omega \mathbf{Q}\varphi + \mathbf{A_x}\frac{\partial \mathbf{Q}}{\partial x}\varphi + \mathbf{A_z}\frac{\partial \mathbf{Q}}{\partial z}\varphi = 0$$

DG formulation of the original equation

Local DG formulation

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x{}^K \frac{\partial \mathbf{Q}^K}{\partial x}\varphi - \int_K \mathbf{A}_z{}^K \frac{\partial \mathbf{Q}^K}{\partial z}\varphi = 0$$

DG formulation of the original equation

Local DG formulation

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x{}^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A}_z{}^K \frac{\partial \varphi}{\partial z} + \sum_{F \in \mathcal{F}(K)} \int_F \mathbf{D_n} \mathbf{Q} \varphi = 0$$

$$\mathbf{D_n} = n_x \mathbf{A_x} + n_z \mathbf{A_z} = - \begin{pmatrix} 0 & 0 & \frac{n_x}{\rho} & 0 & \frac{n_z}{\rho} \\ 0 & 0 & 0 & \frac{n_z}{\rho} & \frac{n_x}{\rho} \\ n_x(\lambda + 2\mu) & n_z\lambda & 0 & 0 & 0 \\ n_x\lambda & n_z(\lambda + 2\mu) & 0 & 0 & 0 \\ n_z\mu & n_x\mu & 0 & 0 & 0 \end{pmatrix}$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K\in\mathcal{T}_h}\int_K i\omega\mathbf{Q}\varphi - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_x\mathbf{Q}\frac{\partial\varphi}{\partial x} - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_z\mathbf{Q}\frac{\partial\varphi}{\partial z} + \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{D_n}\mathbf{Q}\varphi]\!] = 0$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K\in\mathcal{T}_h}\int_K i\omega\mathbf{Q}\varphi - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_x\mathbf{Q}\frac{\partial\varphi}{\partial x} - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_z\mathbf{Q}\frac{\partial\varphi}{\partial z} + \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{D_n}\mathbf{Q}\varphi]\!] = 0$$

$$[\![\mathbf{D_n}\mathbf{Q}\varphi]\!] \simeq (\mathbf{D_n}\mathbf{Q})\,[\![\varphi]\!]$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K\in\mathcal{T}_h}\int_K i\omega\mathbf{Q}\varphi - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_x\mathbf{Q}\frac{\partial\varphi}{\partial x} - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{A}_z\mathbf{Q}\frac{\partial\varphi}{\partial z} + \sum_{F\in\mathcal{F}_h}\int_F (\mathbf{D}_\mathbf{n}\mathbf{Q})\,[\![\varphi]\!] = 0$$

Centered flux DG scheme

Centered flux on a face F

$$(\mathbf{D_n Q})|_F = \{\mathbf{D_n Q}\} = \frac{1}{2}\left(\mathbf{D_n}^K \mathbf{Q}^K + \mathbf{D_n}^{K'} \mathbf{Q}^{K'}\right)$$

Centered flux DG scheme

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x{}^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A}_z{}^K \mathbf{Q}^K \frac{\partial \varphi}{\partial z} + \sum_F \int_F \frac{1}{2}\left(\mathbf{D_n}^K \mathbf{Q}^K + \mathbf{D_n}^{K'} \mathbf{Q}^{K'}\right)\varphi = 0$$

Upwind flux DG scheme

Definition

$$\left\{\begin{array}{lcl} \mathbf{D_n}^{+} & = & \mathbf{R_n}\Gamma^{+}(\mathbf{R_n})^{-1} \\ \mathbf{D_n}^{-} & = & \mathbf{R_n}\Gamma^{-}(\mathbf{R_n})^{-1} \end{array}\right.$$

where

$$\Gamma^{-} = -\begin{pmatrix} v_p & 0 & 0 & 0 & 0 \\ 0 & v_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_s & 0 \\ 0 & 0 & 0 & 0 & v_p \end{pmatrix}$$

Upwind flux DG scheme

Definition

$$\left\{\begin{array}{lcl} \mathbf{D_n}^+ & = & \mathbf{R_n}\Gamma^+ (\mathbf{R_n})^{-1} \\ \mathbf{D_n}^- & = & \mathbf{R_n}\Gamma^- (\mathbf{R_n})^{-1} \end{array}\right.$$

and

$$\mathbf{R_n} = \begin{pmatrix} n_x v_p & -n_z v_s & 0 & n_z v_s & -n_x v_p \\ n_z v_p & n_x v_s & 0 & -n_x v_s & -n_z v_p \\ \lambda + 2n_x^2\mu & -2n_z n_x \mu & n_z^2 & -2n_z n_x \mu & \lambda + 2n_x^2\mu \\ \lambda + 2n_z^2\mu & 2n_z n_x \mu & n_x^2 & 2n_z n_x \mu & \lambda + 2n_z^2\mu \\ 2n_z n_x \mu & \mu(n_x^2 - n_z^2) & -n_x n_z & \mu(n_x^2 - n_z^2) & 2n_z n_x \mu \end{pmatrix}$$

Upwind flux DG scheme

Upwind flux on a face F

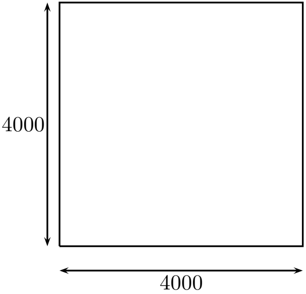
$$(\mathbf{D_n Q})\,|_F = \left(\mathbf{D_n}^K\right)^+ \mathbf{Q}^K + \left(\mathbf{D_n}^{K'}\right)^- \mathbf{Q}^{K'}$$

Upwind flux DG scheme

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A_x}^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A_z}^K \mathbf{Q}^K \frac{\partial \varphi}{\partial z} + \sum_F \int_F \left[\left(\mathbf{D_n}^K\right)^+ \mathbf{Q}^K + \left(\mathbf{D_n}^{K'}\right)^- \mathbf{Q}^{K'}\right] \varphi = 0$$

Contents

Plane wave



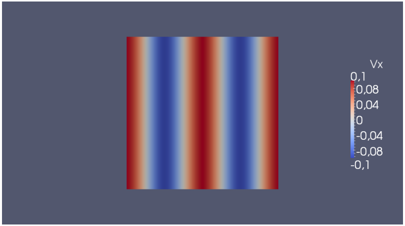
Computational domain Ω setting

- Physical parameters :
  - $\rho = 2.10^3 kg.m^{-3}$
  - $\lambda = 1,6.10^{1}0Pa$
  - $\mu = 8.10^9 Pa$
  - $v_p = 4.10^3 m.s^{-1}$
  - $v_s = 2.10^3 m.s^{-1}$
- Boundary :
  - boundary conditions on $\partial\Omega$ such as :

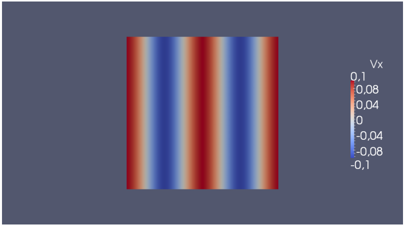
$$u = \nabla e^{i(k \cos\theta x + k \sin\theta y)}$$

$$\text{where } k = \frac{\omega}{v_p} \text{ or } k = \frac{\omega}{v_s}$$

Plane wave for a frequency $f = 2$ Hz, component V_x

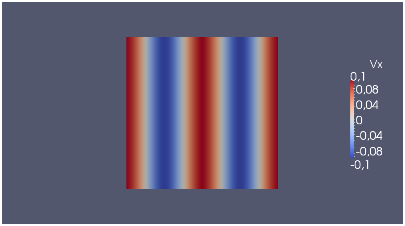


Exact solution

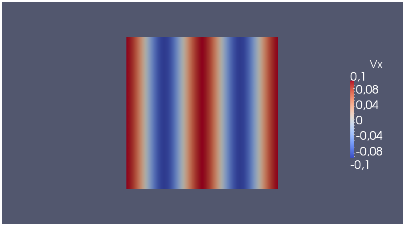


P_2 centered flux DG formulation

Plane wave for a frequency $f = 2$ Hz, component V_x

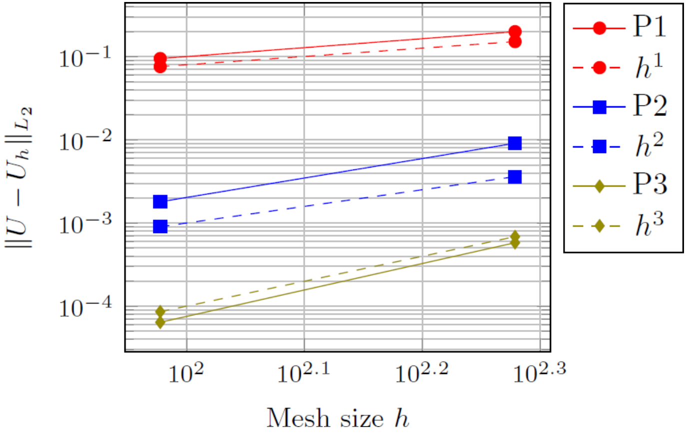


Exact solution



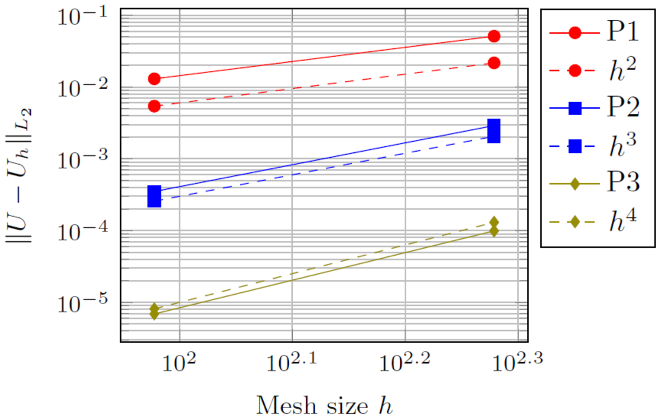
P_2 upwind flux DG formulation

Plane wave



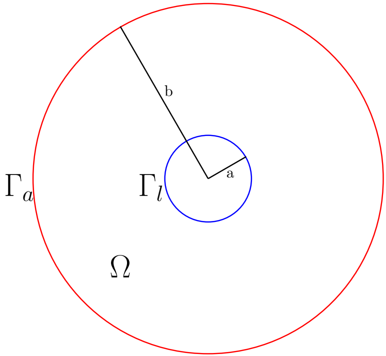
L_2 error for the centered flux

Plane wave



L_2-error for the upwind flux

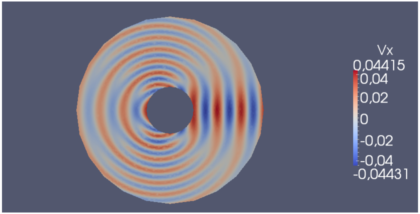
Circular diffraction



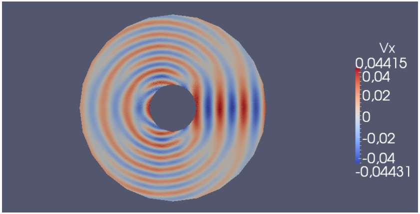
Configuration of the computational domain Ω

- Physical parameters :
  - $\rho = 2.10^3 kg.m^{-3}$
  - $\lambda = 1,6.10^10 Pa$
  - $\mu = 8.10^9 Pa$
  - $v_p = 4.10^3 m.s^{-1}$
  - $v_s = 2.10^3 m.s^{-1}$
- Boundary :
  - Γ_l is a free surface : $\underline{\underline{\sigma}}\mathbf{n} = 0$
  - Γ_a absorbing boundary : $\underline{\underline{\sigma}}\mathbf{n} = v_p(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + v_s(\mathbf{v}\cdot\mathbf{t})\mathbf{t}$
- $a = 2000m$
- $b = 8000m$
- $\mathcal{T}_h$ composed of 9653 elements

Results for various frequencies : $f = 2$ Hz

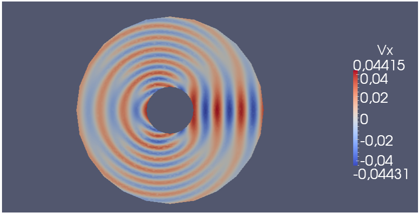


Exact solution

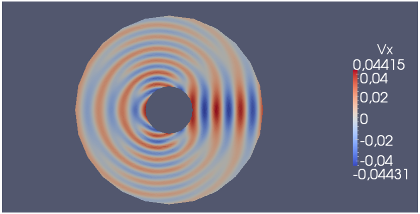


P_1 centered scheme

Results for various frequencies : $f = 2$ Hz



Exact solution



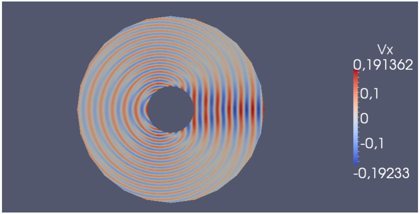
P_1 upwind scheme

Results for various frequencies : $f = 2$ Hz

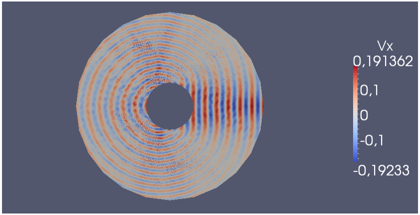
Nb dof		Centered	Upwind
144795	V_x L_2-error	3.44e-01	1.49e-01
(P_1)	Fact.-res. time (s)	13	16
	Memory	800	980
289590	V_x L_2-error	4.41e-02	4.67e-02
(P_2)	Fact.-res. time (s)	40	57
	Memory	1900	2800

Numerical statistics

Results for various frequencies : $f = 4$ Hz

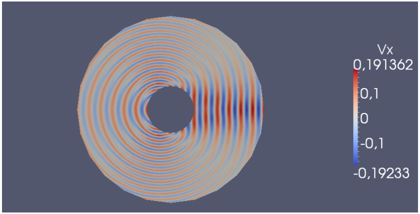


Exact solution

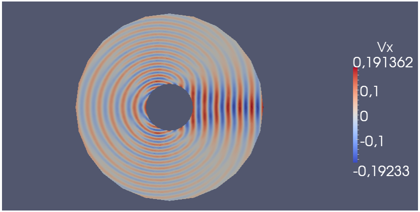


P_2 centered scheme

Results for various frequencies : $f = 4$ Hz



Exact solution



P_2 upwind scheme

Results for various frequencies : $f = 4$ Hz

Nb dof		Centered	Upwind
144795	V_x L_2-error	1.19	0.506
	Fact.-res. time (s)	13	16
	Memory	800	970
289590	V_x L_2-error	4.24e-01	1.66e-01
	Fact.-res. time (s)	40	57
	Memory	1900	2800

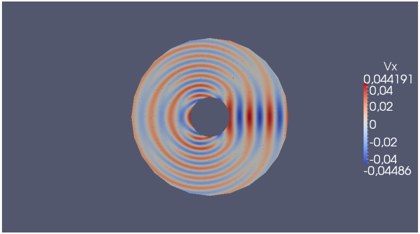
Numerical statistics

Results with $p - adaptativity$

Area of the triangle	Interpolation order	Number of triangles
]0 ;10000]	0	3
]10000 ;15000]	1	1745
]15000 ;20000]	2	3999
]20000 ;25000]	3	2658
]25000 ;30000]	4	1248

Distribution of the interpolation orders

Results with $p - adaptativity$: $f = 2$ Hz



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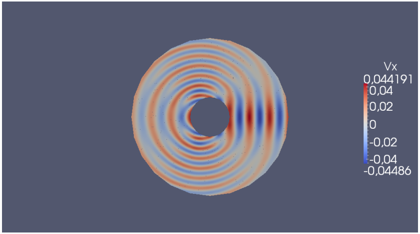
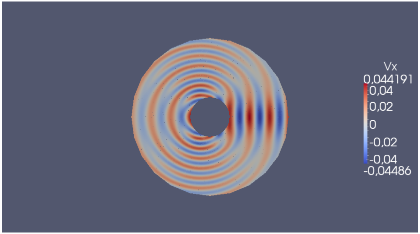


"p-local" centered scheme

Results with $p - adaptativity$: $f = 2$ Hz



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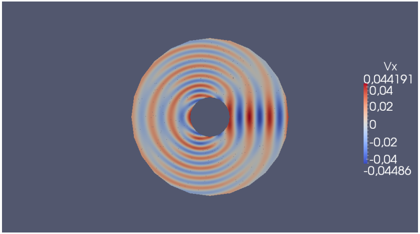
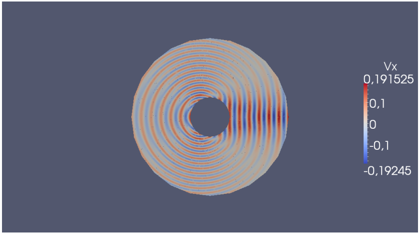
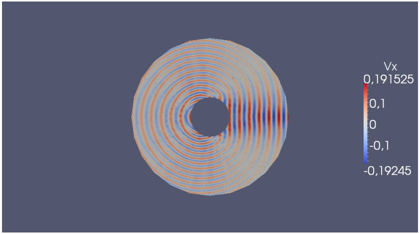


"p-local" upwind scheme

Results with $p - adaptativity$: $f = 4$ Hz

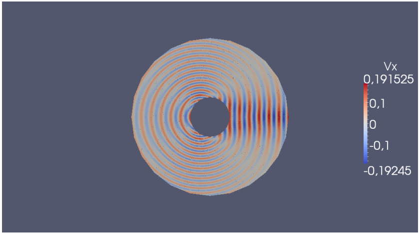




Exact solution

"p-local" centered scheme

Results with $p - adaptativity$: $f = 4$ Hz



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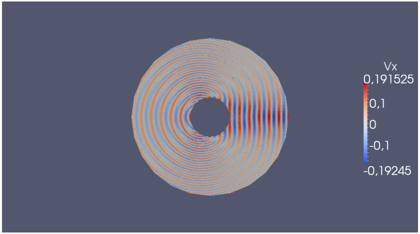

"p-local" upwind scheme

Results with p — *adaptativity*

	$f = 2$ Hz		$f = 4$ Hz	
	Centered	Upwind	Centered	Upwind
V_x L_2-error	5.02e-02	6.00e-02	2.83e-01	2.76e-01
Fact.-res. time (s)	57	79	57	80
Memory	3000	3800	3000	3800

Numerical statistics for both schemes as function of the frequency

Comparison between $p - adaptativity$ and $p - global$ for $f = 2Hz$

	$p - adaptativity$		$p - global$	
	Centered	Upwind	Centered	Upwind
V_x L_2-error	5.02e-02	6.00e-02	4.41e-02	4.41e-02
Fact.-res. time (s)	57	79	40	57
Memory	3000	3800	1900	2800

Comparison between $p - adaptativity$ and $p - global$ for $f = 4Hz$

	$p - adaptativity$		$p - global$	
	Centered	Upwind	Centered	Upwind
V_x L_2-error	2.83e-01	2.75e-01	4.24e-01	1.66e-01
Fact.-res. time (s)	57	80	40	57
Memory	3000	3800	1900	2800

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Conclusion-Perspectives

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