Multiscale Hybrid-Mixed Method for Reactive-Advective Dominated Models

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[1] Joint work with C. Harder and D. Paredes

HOSCAR Inspiration

Switch from

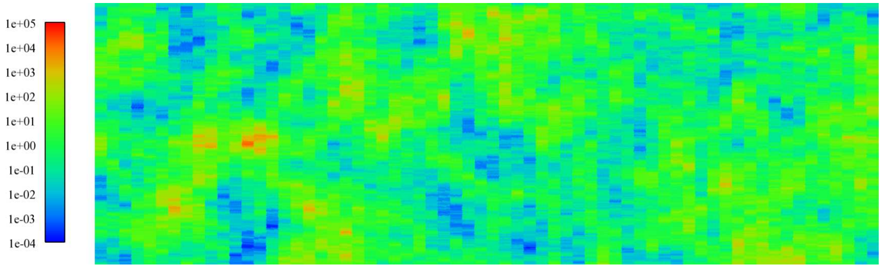
Making Existing Numerical Algorithm Parallel

to

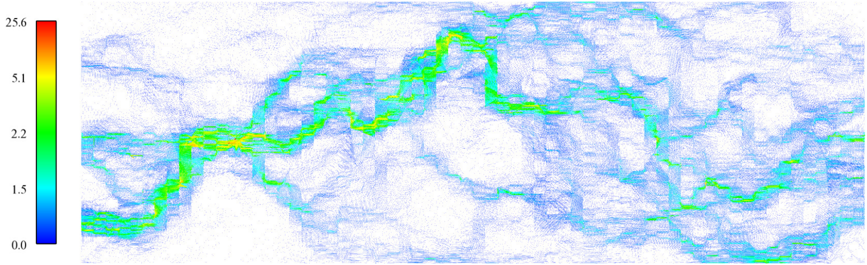
Making Parallel Numerical Algorithm

Motivation and Model

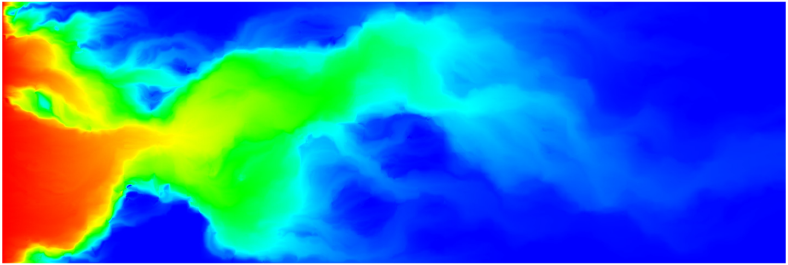
A Log-Normal Permeability Field



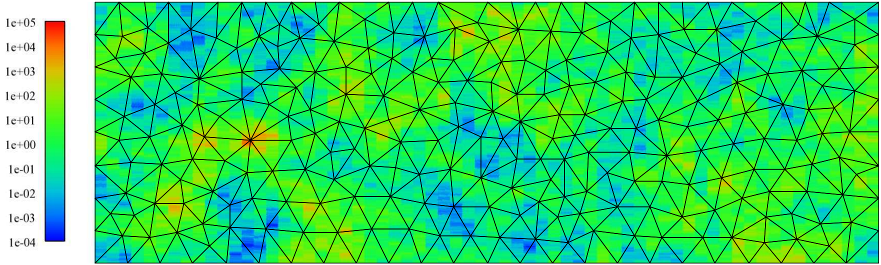
A Typical Velocity Field



A Typical Advective Dominated Transport



Upscalling



Reactive-Advective-Diffusive Model

Find u such that

$$-\nabla \cdot D\, u + \sigma\, u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where

$$D\, u := \varepsilon\, \nabla u - \boldsymbol{\alpha}\, u$$

- ε, $\sigma > 0$ and $\nabla \cdot \boldsymbol{\alpha} = 0$
- $\varepsilon \ll |\boldsymbol{\alpha}|\, h$ and/or $\varepsilon \ll \sigma\, h^2$
- ε and $\boldsymbol{\alpha}$ may have *multi-scale* features

Finite Element: u and $D\,u$

Galerkin method with $\mathbb{P}_k$ elements

Basis functions are *“ignorant” of multi-scale* structures

$\Downarrow$

Lack of approximation on coarse meshes

A (incomplete) State-of-the-Art: RAD

Boundary Layers

RFB	Brezzi et al. (1996)
VMS	Hughes et al. (2000)
MsFEM	Hou et al. (2004)
PGEM	Franca et al. (2005)
LDG-H	Cockburn et al. (2009)

Heterogenous Media

MsFEM	Hou et al. (1997)
UpFEM	Sangalli (2003)
RFB	Arbogast (2004)
MMMFEM	Yotov et al. (2011)
	Wheeler et al. (2011)

The General MHM Idea

Hybrid Formulation (Raviart-Thomas '77)

Classical Weak Form: *Find* $u \in U$ *such that*

$$a(u,v)_\Omega = (f,v)_\Omega \quad \forall v \in U$$

Take $\mathcal{T}_h$ a (coarse) partition of Ω and set

$$V := \oplus \sum_{K \in \mathcal{T}_h} U(K)$$

Hybrid Form : *Find* $(u,\, \lambda) \in V \times \Lambda$ *such that*

$$\begin{aligned} a(u,v)_{\mathcal{T}_h} + (\lambda, v)_{\partial \mathcal{T}_h} &= (f,\, v)_{\mathcal{T}_h} \quad \forall v \in V \\ (\mu, u)_{\partial \mathcal{T}_h} &= 0 \quad \forall \mu \in \Lambda \end{aligned}$$

Solution Decomposition

Hybrid Form : *Find* $(u, \lambda) \in V \times \Lambda$ *such that*

$$a(u,v)_{\mathcal{T}_h} + (\lambda, v)_{\partial\mathcal{T}_h} = (f,\, v)_{\mathcal{T}_h} \quad \forall v \in V$$
$$(\mu, u)_{\partial\mathcal{T}_h} = 0 \quad \forall \mu \in \Lambda$$

- First equation ⇔ Collection of Local Problems[2]

$$a(u,v)_K = (f,\, v)_K - (\lambda, v)_{\partial K} \Rightarrow \quad u\,|_K = T\,\lambda + \hat{T}\,f$$

- Second Equation ⇔ Global Coarse Problem on Faces

$$(\mu, u)_{\partial\mathcal{T}_h} = 0 \quad \Leftrightarrow \quad (\mu, T\,\lambda)_{\partial\mathcal{T}_h} = -(\mu, \hat{T}\,f)_{\partial\mathcal{T}_h}$$

[2] Hyp: T, $\hat{T}$ are well-defined. Else see Harder-Paredes-Valentin, JCP '13

The MHM Method

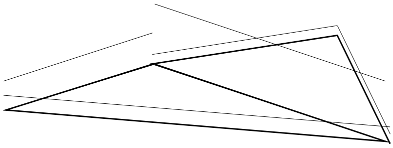
Finite Dimensional Space

We only need to select

$$\Lambda_h \subset \Lambda$$

Let $\lambda_h \in \Lambda_h$ be

$$\lambda_h = \sum_i c_i \, \psi_i$$



ψ_i

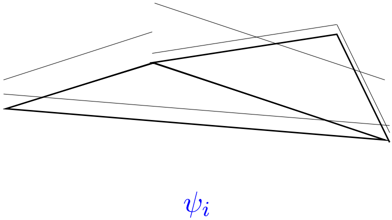
The MHM Method

Find $\lambda_h \in \Lambda_h$ *such that*

$$(\mu_h, T\,\lambda_h)_{\partial\mathcal{T}_h} = -(\mu_h, \hat{T}\,f)_{\partial\mathcal{T}_h} \quad \forall \mu_h \in \Lambda_h$$

where $T\,\lambda_h = \sum_i c_i\,\eta_i$ and $\hat{T}\,f$

$$a(\eta_i, v)_K = -(\psi_i, v)_{\partial K}$$
$$a(\hat{T}\,f, v)_K = (f, v)_K$$



Set

$$u_h = T\,\lambda_h + \hat{T}\,f$$

Reactive-Advective-Diffusive Case

Setting MHM for RAD

$$V := H^1(\mathcal{T}_h) \quad \text{and} \quad \Lambda := H^{-\frac{1}{2}}(\partial\mathcal{T}_h)$$

$$a(u,v) := (\varepsilon \nabla u, \nabla v)_{\mathcal{T}_h} + \frac{1}{2}(\boldsymbol{\alpha}\nabla u, v)_{\mathcal{T}_h} - \frac{1}{2}(u, \boldsymbol{\alpha}\nabla v)_{\mathcal{T}_h} + (\sigma u, v)_{\mathcal{T}_h}$$

$$a(\eta_i, v)_K = -(\psi_i, v)_{\partial K}$$

$$\Updownarrow$$

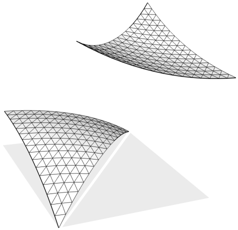
$$-\nabla\cdot(D\,\eta_i) + \sigma\,\eta_i = 0$$
$$\varepsilon\nabla_{\boldsymbol{n}}\eta_i - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{n}}{2}\eta_i = \psi_i$$

$$a(\hat{T}\,f, v)_K = (f, v)_{\partial K}$$

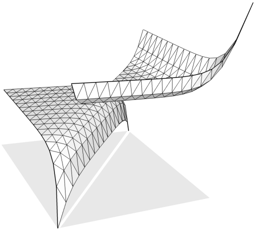
$$\Updownarrow$$

$$-\nabla\cdot(D\,\hat{T}f) + \sigma\,\hat{T}\,f = f$$
$$\varepsilon\nabla_{\boldsymbol{n}}\hat{T}\,f - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{n}}{2}\hat{T}\,f = 0$$

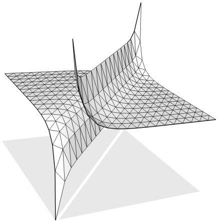
Diffusion Dominates



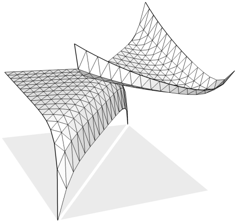
Advection Dominates



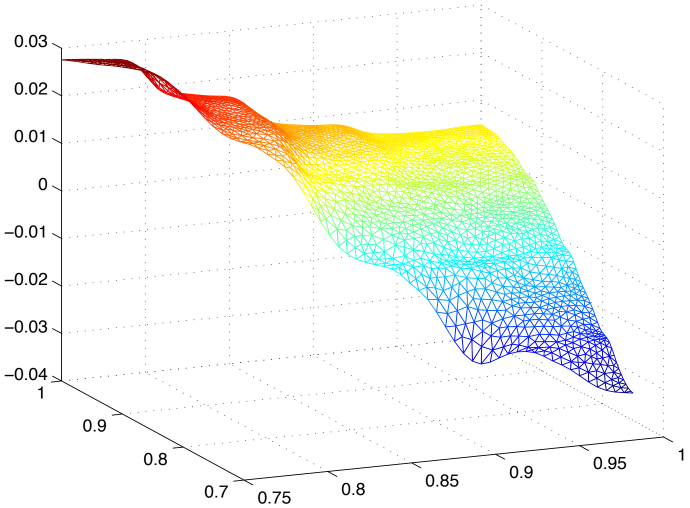
Reaction Dominates



Reaction-Advection Dominates



A Typical Heterogenous Base Function



Considerations on the Method

HMH Method is Well-Posed and Optimal Convergent

$$u_h := T\lambda_h + \hat{T}f \quad \text{and} \quad \boldsymbol{\sigma}_h := D\,u_h + \frac{1}{2}\boldsymbol{\alpha}\,u_h \in H(div, \Omega)$$

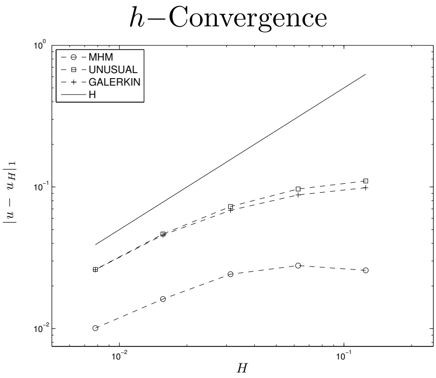
$$\int_K \nabla \cdot D\,u_h + \sigma\,u_h = \int_K f \quad \text{(Local Conservation)}$$

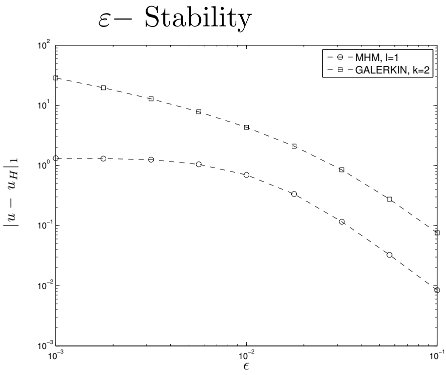
An (Face) A Posteriori Error Estimator + Adaptativity

INVITATION
TO
DIEGO PAREDES' TALK

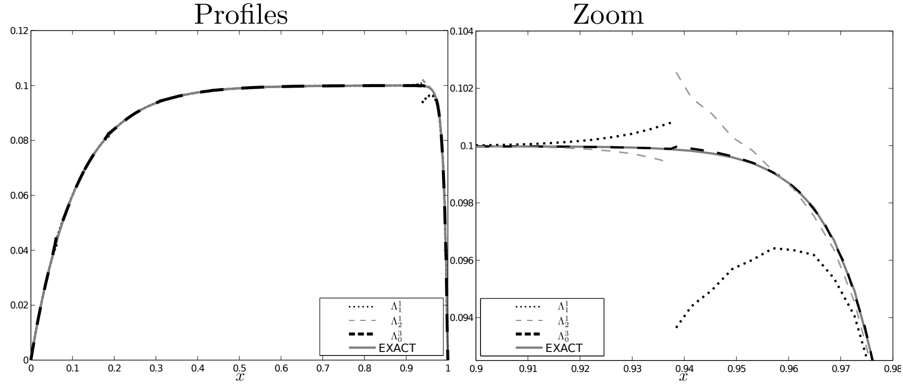
Numerical Validation

$\varepsilon = 10^{-2}$, $|\boldsymbol{\alpha}| = 1$ and $\sigma = 100$

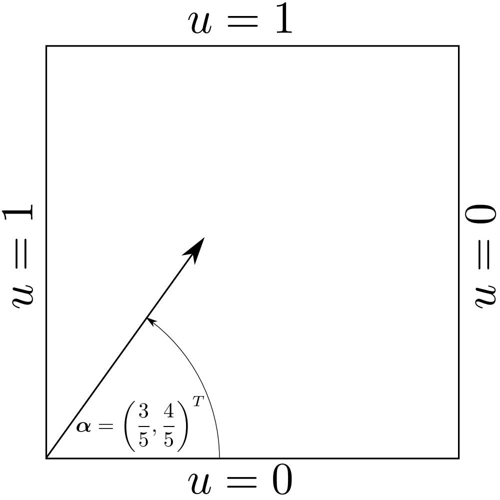




Space Λ_h: Piecewise Low-Order Λ_h is the Best

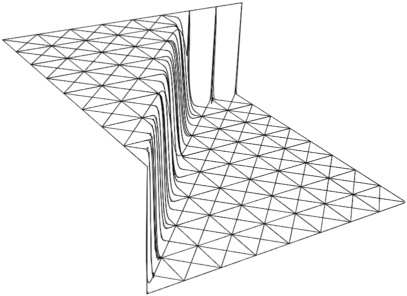


Skew Advection: $\varepsilon = 10^{-4}$, $|\boldsymbol{\alpha}| = 1$, $\sigma = 0$

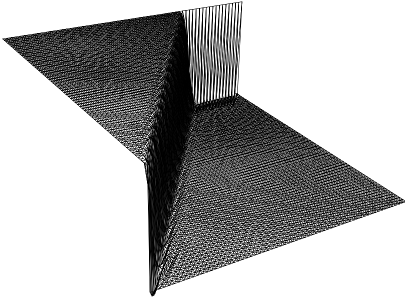


Same Order and D.O.Fs

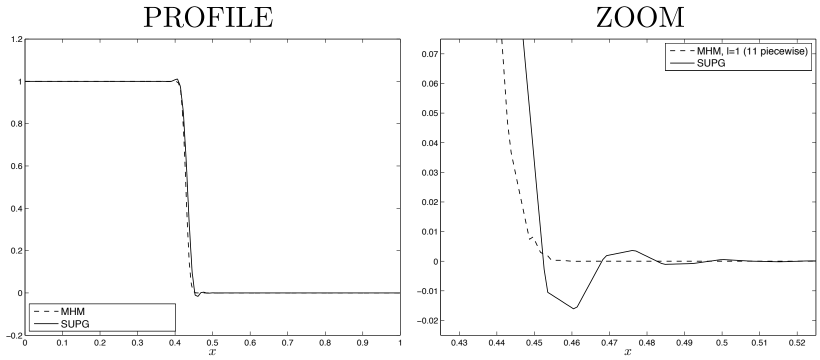
MHM



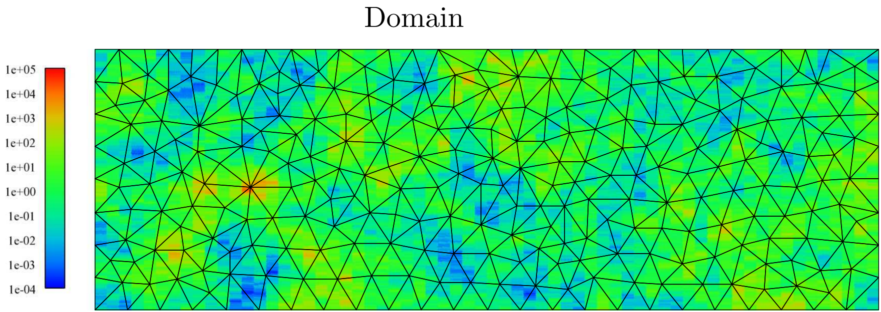
SUPG



Internal Layer: No Need of Shock-Capturing

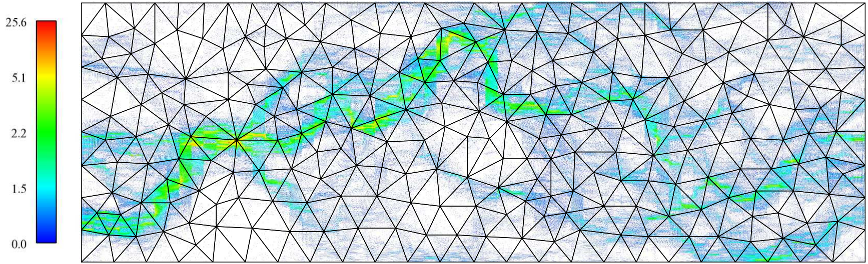


Transport in a Heterogenous Media



Transport in a Heterogenous Media

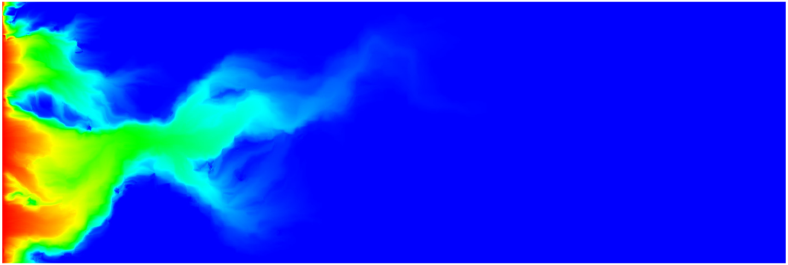
Darcy Velocity[3]



[3]Harder-Paredes-Valentin, JCP 2013

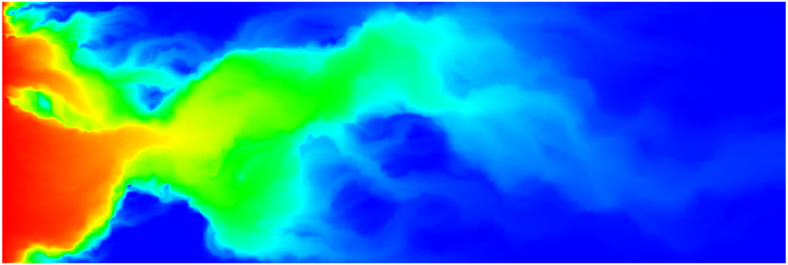
Transport in a Heterogenous Media

T = 1



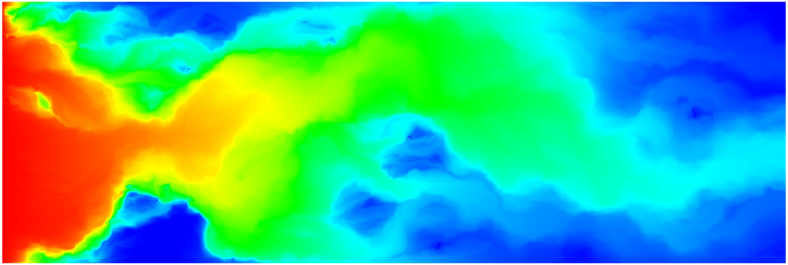
Transport in a Heterogenous Media

T = 2.5

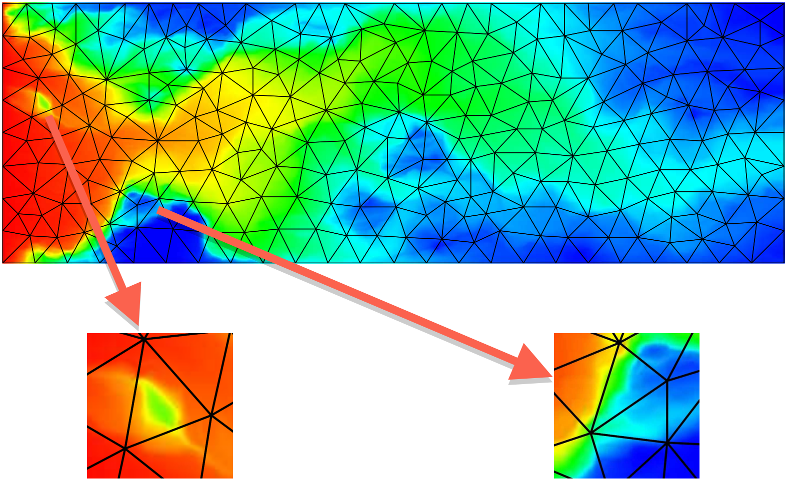


Transport in a Heterogenous Media

T = 5.0

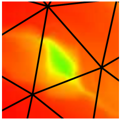


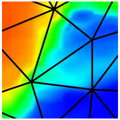
Transport in a Heterogenous Media



Conclusion

- New multiscale method for heterogenous RAD problems
- Include local upscaling and crossing interfaces
- Capture boundary layers on coarse meshes
- Locally conservative
- Highly adapted to parallel computation





Coming Up Soon

MHM Method for Wave Problems

on

Highly Heterogeneous Media

Acoustic Model

Elasto-Dynamics Model

One Year Visiting Position and (possible) Pos-Doctoral Position at
INRIA Sophia-Antipolis