A Novel Adaptivity Process for Reaction-Advection-Diffusion Problems

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[1] Joint work with C. Harder and F. Valentin.

Advection-Diffusion Equation

Find $u = u(\mathbf{x})$ *such that*

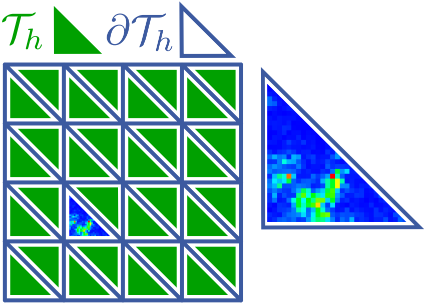
$$\begin{cases} \mathcal{L}u := \operatorname{div}(-\epsilon\nabla u + \boldsymbol{\alpha} u) + \sigma u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{in } \Gamma \end{cases}$$

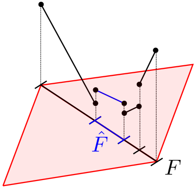
- ϵ is positive definite tensor
- advection and/or reaction may *dominate*
- $\epsilon(\mathbf{x})$, $\boldsymbol{\alpha}(\mathbf{x})$ and $f(\mathbf{x})$ may have *multi-scale* features

Settings

$$V = \bigoplus_{K \in \mathcal{T}_h} H^1(K)$$

$$\Lambda_h \subset \Lambda = \prod_{K \in \mathcal{T}_h} H^{-\frac{1}{2}}(\partial K)$$





polynomial $\lambda_h \in \Lambda_h$

The Method

Global problem

Find $\lambda_h \in \Lambda_h$ *such that*

$$(u^{\lambda_h}, \mu_h)_{\partial \mathcal{T}_h} = -(u^f, \mu_h)_{\partial \mathcal{T}_h}$$

for all $\mu_h \in \Lambda_h$

Local problem for λ_h

Find $u^{\lambda_h} \in H^1(K)$ *such that*

$$\begin{cases} \mathcal{L}u^{\lambda_h} = 0 \\ -\epsilon \nabla u^{\lambda_h} \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n}) u^{\lambda_h} = \lambda_h \end{cases}$$

Local problem for f

Find $u^f \in H^1(K)$ *such that*

$$\begin{cases} \mathcal{L}u^f = f \\ -\epsilon \nabla u^f \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n}) u^f = 0 \end{cases}$$

- Captures multiscale features and/or boundary layers
- Parallelization in a natural way
- A face-based posteriori estimator and a new adaptivity idea

Outline

Variational Hybrid-Mixed Formulation

$$a(u,v) := (\epsilon\nabla u, \nabla v)_{\mathcal{T}_h} + \frac{1}{2}(\boldsymbol{\alpha}\cdot\nabla u, v)_{\mathcal{T}_h} - \frac{1}{2}(u, \boldsymbol{\alpha}\cdot\nabla v)_{\mathcal{T}_h} + (\sigma\, u, v)_{\mathcal{T}_h}$$

Hybrid Formulation

Find $(u,\lambda) \in V \times \Lambda$

$$\begin{aligned} a(u,v) + (\lambda, v)_{\partial\mathcal{T}_h} &= (f,v)_{\mathcal{T}_h} \\ (u,\mu)_{\partial\mathcal{T}_h} &= 0 \end{aligned}$$

for all $(v,\mu) \in V \times \Lambda$

Global and local problems for $u = u^\lambda + u^f$

Global problem

Find $\lambda \in \Lambda$

$$(u^\lambda, \mu)_{\partial\mathcal{T}_h} = -(u^f, \mu)_{\partial\mathcal{T}_h}$$

for all $\mu \in \Lambda$

Local problem for λ

Find $u^\lambda \in H^1(K)$ *such that*

$$\begin{cases} \mathcal{L}u^\lambda = 0 \\ -\epsilon\nabla u^\lambda \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n})u^\lambda = \lambda \end{cases}$$

Local problem for f

Find $u^f \in H^1(K)$ *such that*

$$\begin{cases} \mathcal{L}u^f = f \\ -\epsilon\nabla u^f \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n})u^f = 0 \end{cases}$$

Discrete Space $\Lambda_h \subset \Lambda$

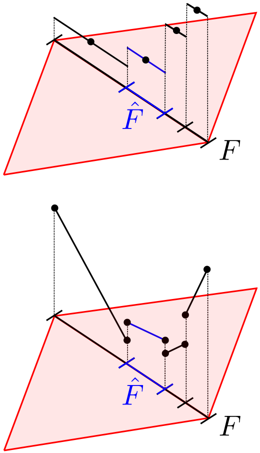
$$\lambda_h = \sum_{i=1}^{N_h} c_i \psi_i$$

Basis functions

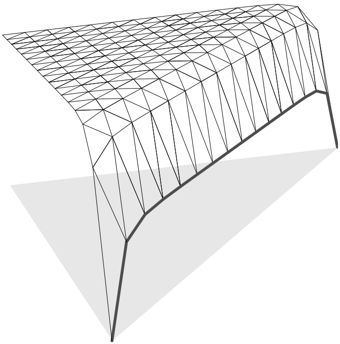
Find $\eta_i \in H^1(K)$ such that

$$\begin{cases} \mathcal{L}\eta_i = 0 \\ -\epsilon \nabla \eta_i \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n})\eta_i = \psi_i \end{cases}$$

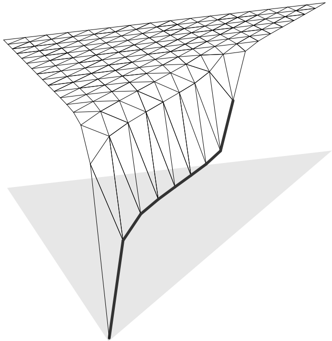
$$u_h = {\color{red}u^{\lambda_h}} + u^f = \sum_{i=1}^{N_h} c_i {\color{red}\eta_i} + u^f$$



Discrete Space $\Lambda_h \subset \Lambda$

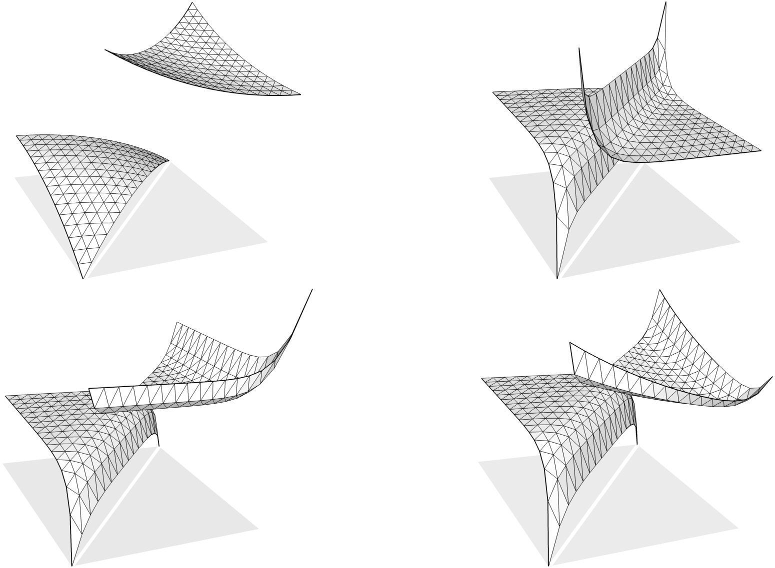


constant



piece-wise constant

Discrete Space $\Lambda_h \subset \Lambda$



Multiscale Hybrid Method

$$u_h = u^{\lambda_h} + u^f$$

Global problem

Find $\lambda_h \in \Lambda_h$

$$(u^{\lambda_h}, \mu_h)_{\partial \mathcal{T}_h} = -(u^f, \mu_h)_{\partial \mathcal{T}_h}$$

for all $\mu_h \in \Lambda_h$

Basis functions

Find $\eta_i \in H^1(K)$ *such that*

$$\begin{cases} \mathcal{L}\eta_i = 0 \\ -\epsilon \nabla \eta_i \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n})\eta_i = \psi_i \end{cases}$$

$$u^{\lambda_h} = \sum_{i=1}^{N_h} c_i \eta_i$$

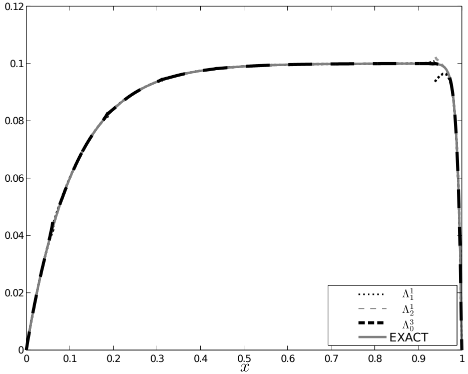
Local problem for f

Find $u^f \in H^1(K)$ *such that*

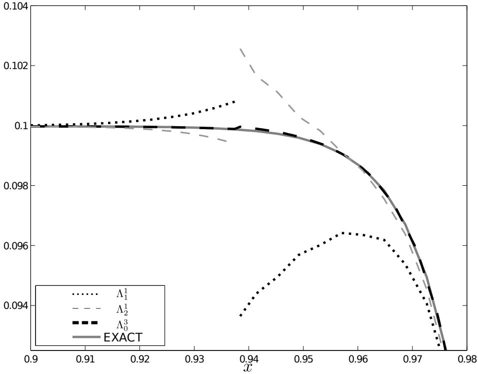
$$\begin{cases} \mathcal{L}u^f = f \\ -\epsilon \nabla u^f \cdot \mathbf{n} + \frac{1}{2}(\boldsymbol{\alpha} \cdot \mathbf{n})u^f = 0 \end{cases}$$

Space Adaptivity (inspiration)

Source problem: $\epsilon = 10^{-2}, \boldsymbol{\alpha} = (1,0)^T, \sigma = 10$ and $f = 1$

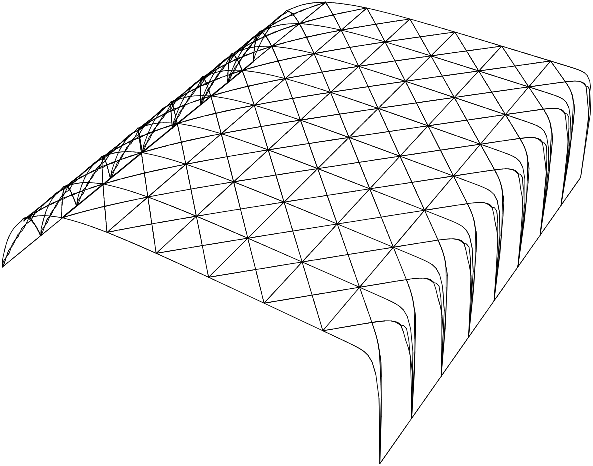


Profiles



Zoom

Space Adaptivity (inspiration)



$l = 0$, 3 pieces for each face

Face-Based Estimator

$$F = \partial K_1 \cap \partial K_2 \neq \emptyset$$

$$[\![u_h]\!]|_F = u_h^{K_1}|_F \, \mathbf{n}^{K_1} + u_h^{K_2}|_F \, \mathbf{n}^{K_2}$$

$$\eta_{\hat{F}} = C_{\epsilon,\alpha,\sigma} \frac{1}{\sqrt{h_F}} || \, [\![u_h]\!] \, ||_{0,\hat{F}}$$

$$\eta = \left[\sum_K \sum_{F \in \partial K} \eta_F^2 \right]^{\frac{1}{2}}$$

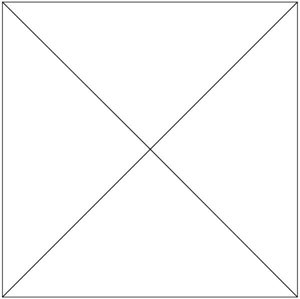
Face-Based Estimator

$$||u - u_h||_V \leq C_1 \ \eta$$

$$C_2 \ \eta_F \leq ||u - u_h||_{V,F}$$

Space Adaptivity

- Reduce the number of DOF
- No re-meshing needed
- The matrix can be "partially" modified (step by step)
- Avoid external software



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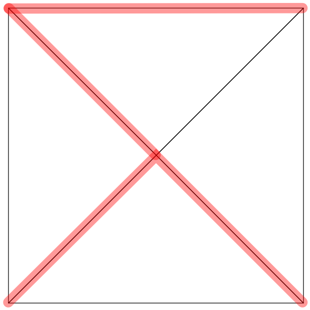

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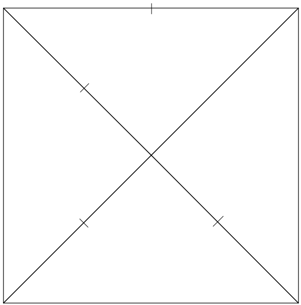

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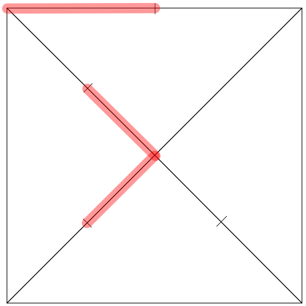

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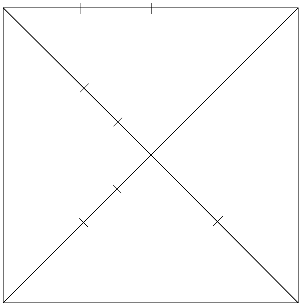

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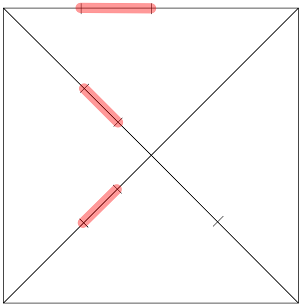

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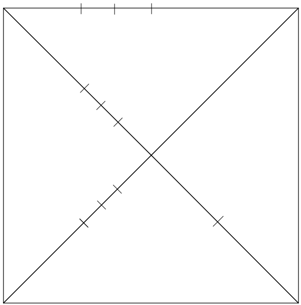
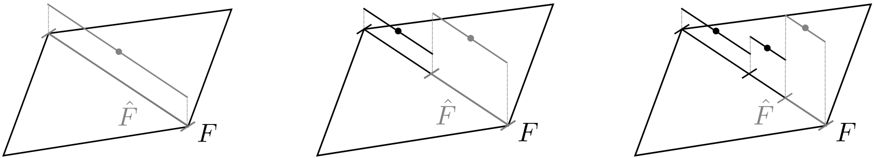
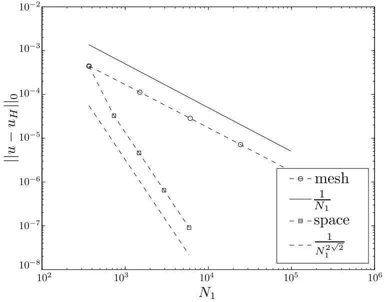
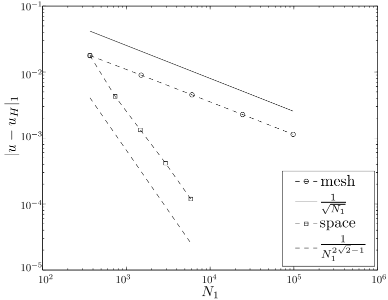


Space Adaptivity

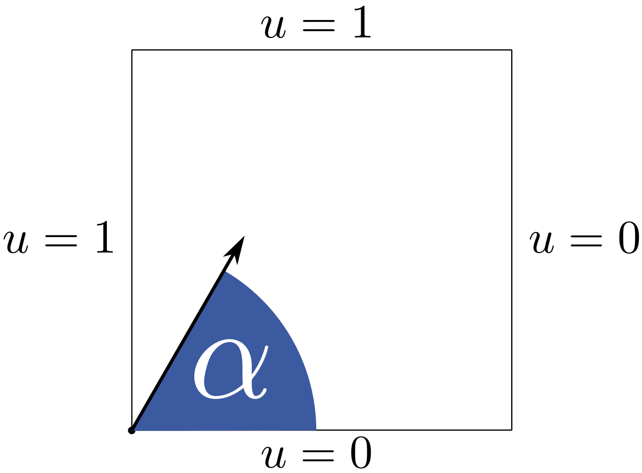


Space Adaptivity

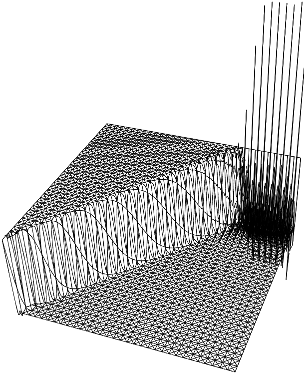




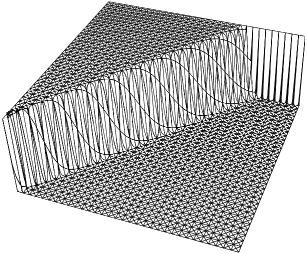
Skew-advection



Skew-advection: $\varepsilon = 10^{-4}$, $\sigma = 0$

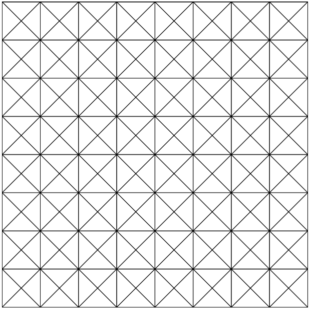


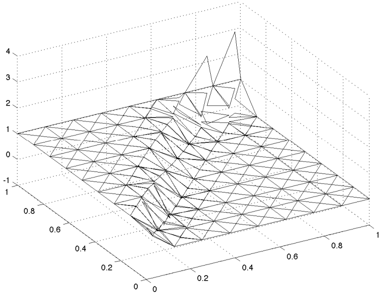
around 3.3×10^4 d.o.f.
$(k = 4)$



around 3.0×10^4 d.o.f.
$(l = 4)$

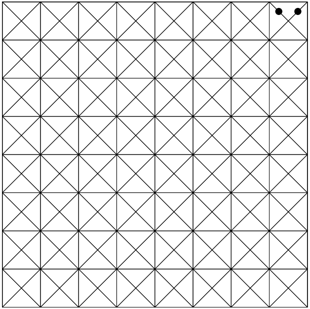
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

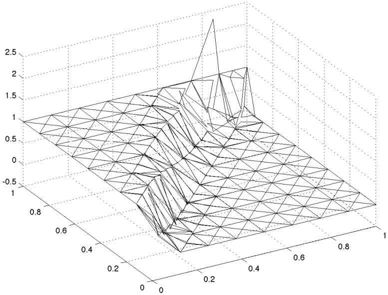




Iteration 1, ($l = 1$)

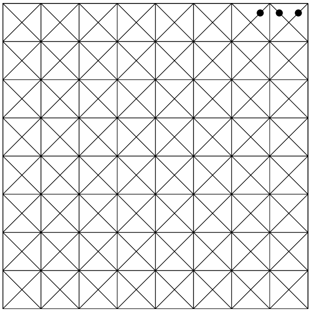
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

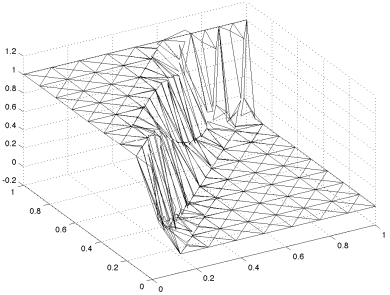




Iteration 2, ($l = 1$)

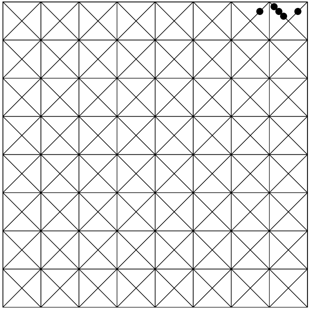
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

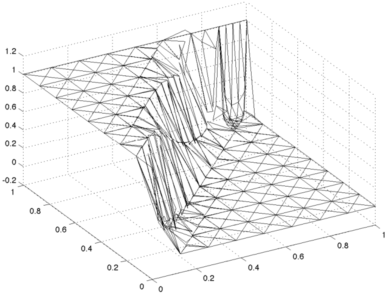




Iteration 3, ($l = 1$)

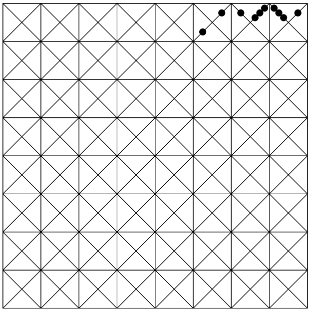
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

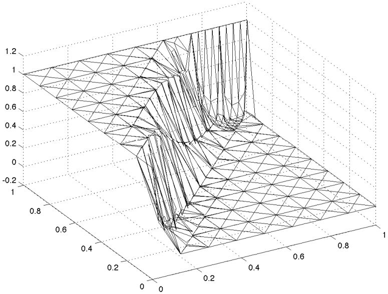




Iteration 4, ($l = 1$)

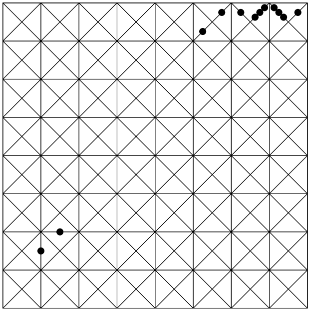
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

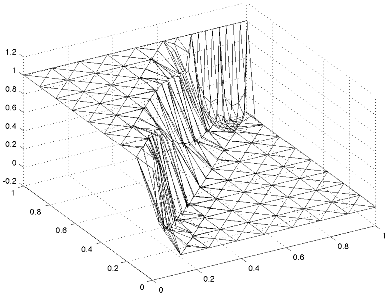




Iteration 5, ($l = 1$)

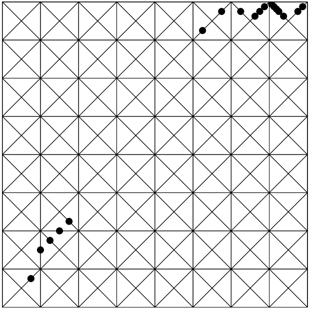
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

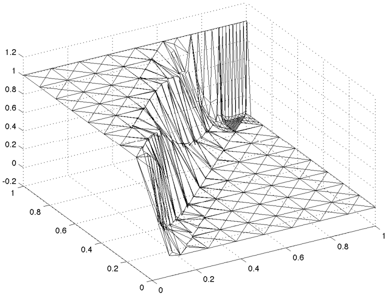




Iteration 6, ($l = 1$)

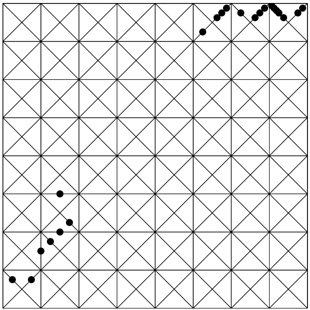
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

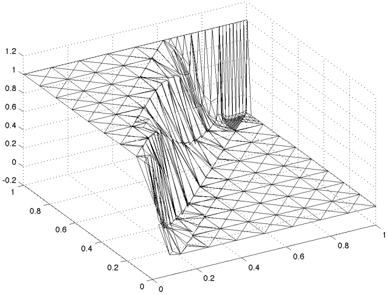




Iteration 7, ($l = 1$)

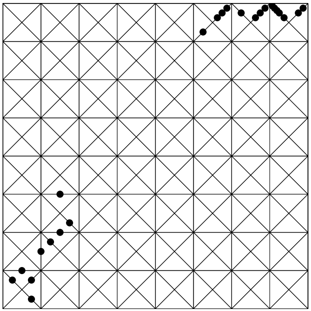
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

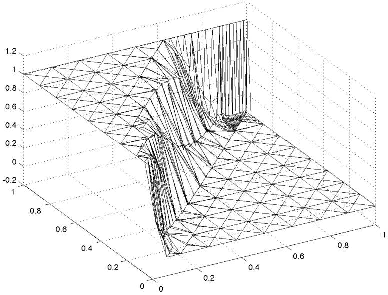




Iteration 8, ($l = 1$)

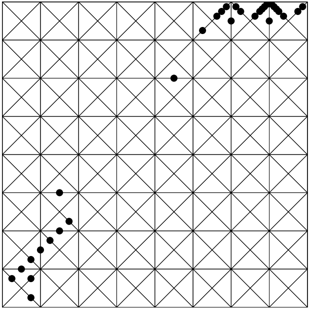
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

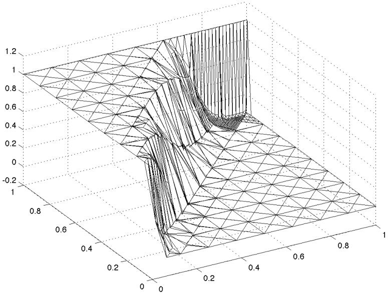




Iteration 9, ($l = 1$)

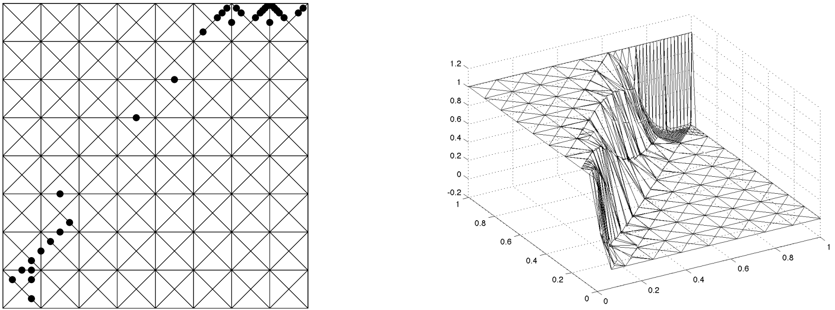
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$





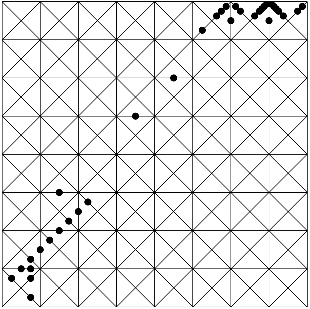
Iteration 10, ($l = 1$)

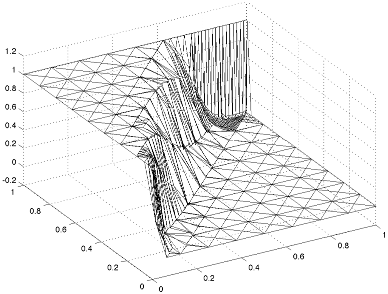
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$



Iteration 11, $(l = 1)$

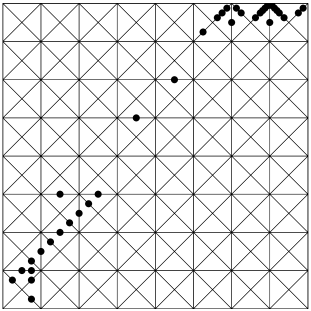
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

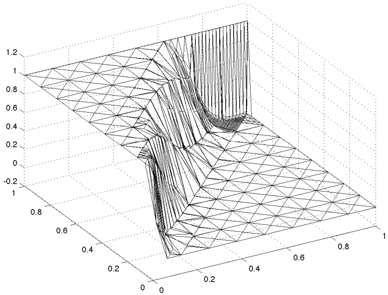




Iteration 12, ($l = 1$)

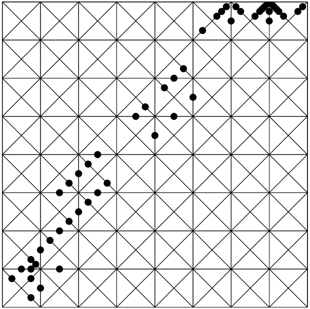
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

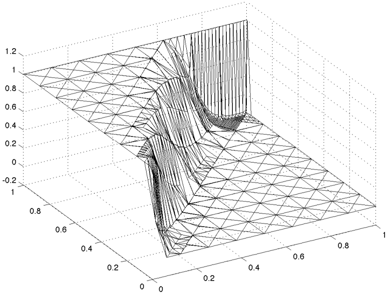




Iteration 13, ($l = 1$)

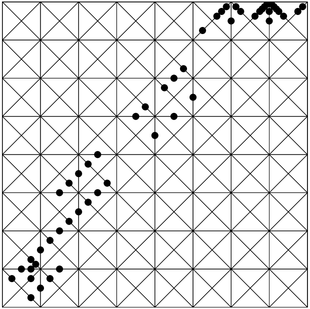
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

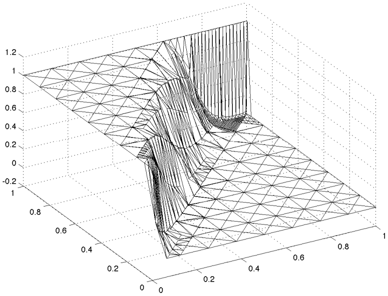




Iteration 14, ($l = 1$)

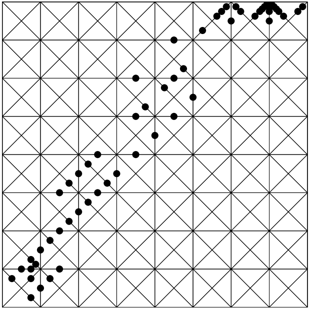
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

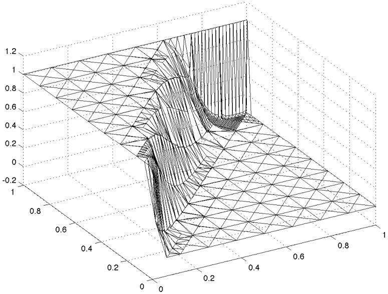




Iteration 15, ($l = 1$)

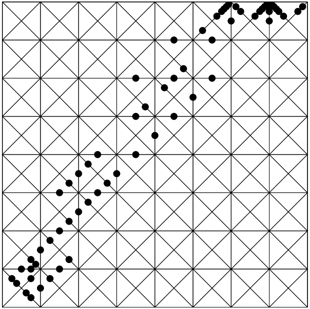
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

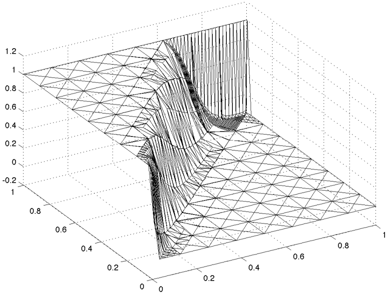




Iteration 16, ($l = 1$)

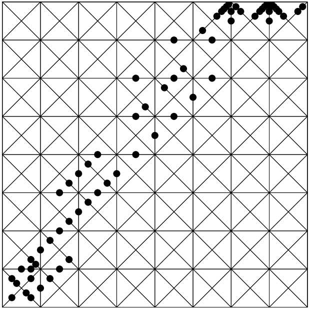
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

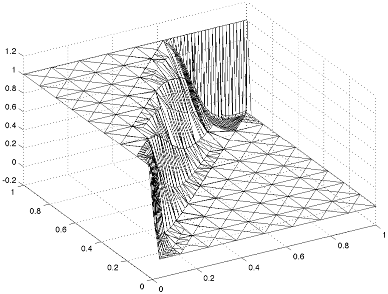




Iteration 17, ($l = 1$)

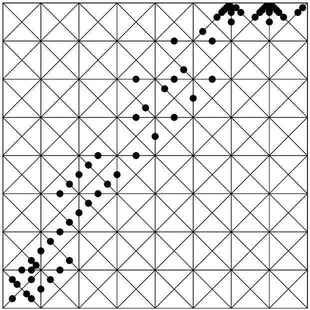
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

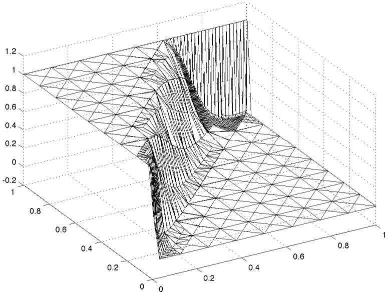




Iteration 18, ($l = 1$)

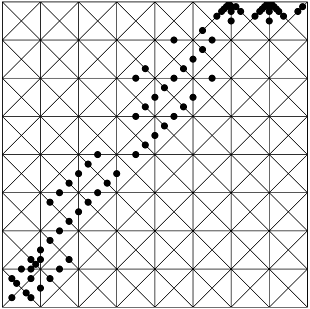
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

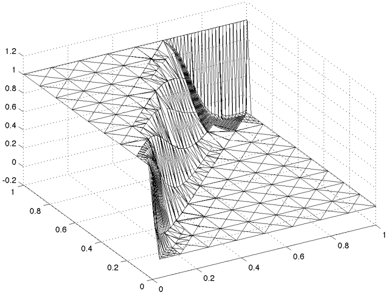




Iteration 19, ($l = 1$)

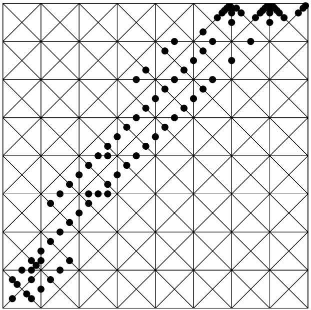
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

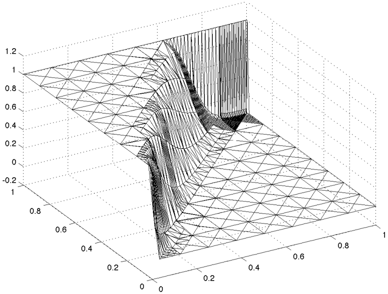




Iteration 20, ($l = 1$)

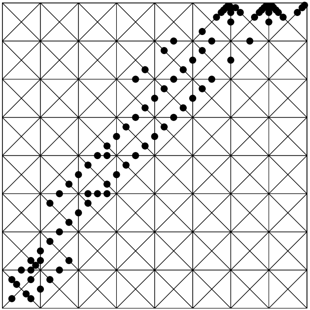
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

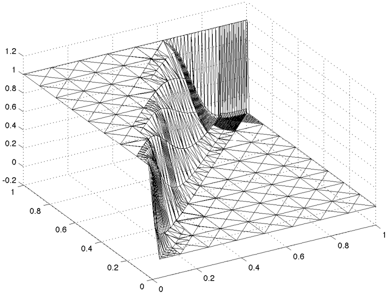




Iteration 21, ($l = 1$)

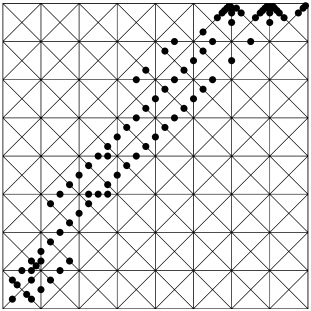
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

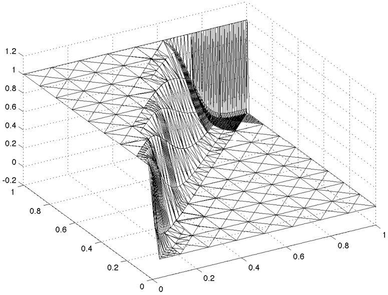




Iteration 22, $(l = 1)$

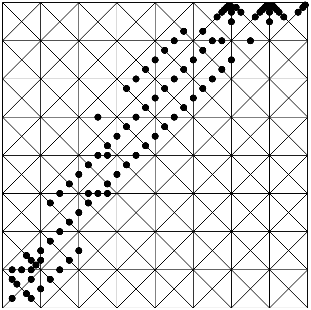
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

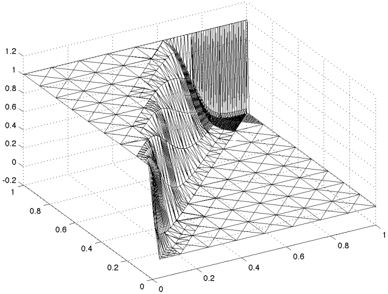




Iteration 23, ($l = 1$)

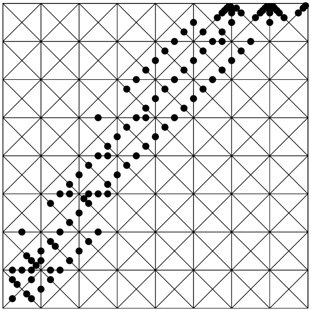
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

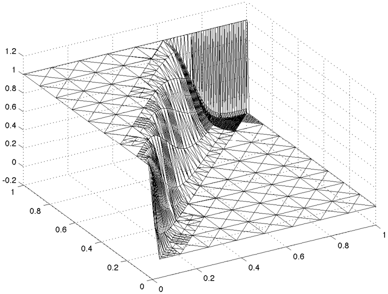




Iteration 24, ($l = 1$)

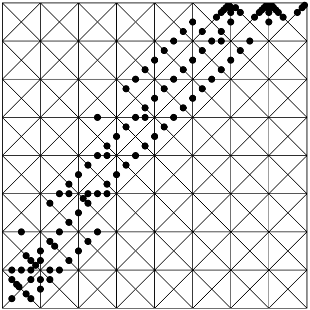
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

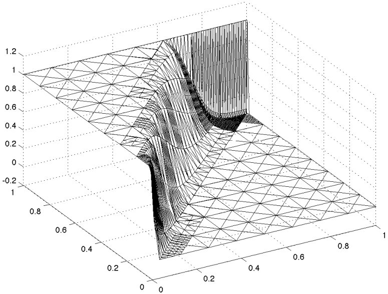




Iteration 25, ($l = 1$)

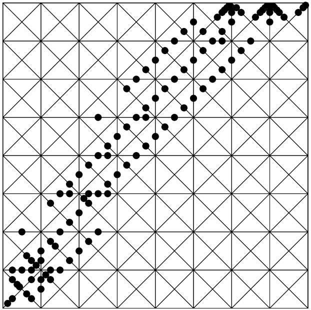
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

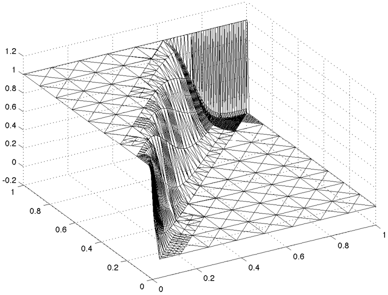




Iteration 26, ($l = 1$)

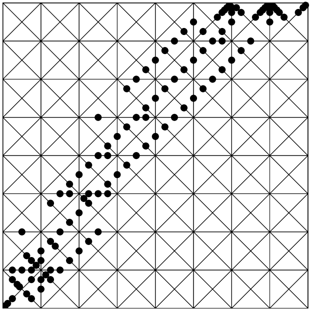
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

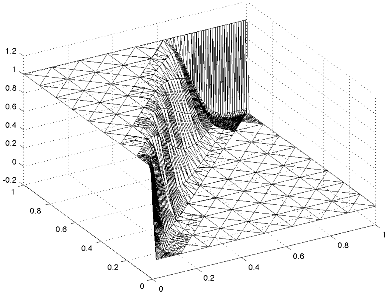




Iteration 27, ($l = 1$)

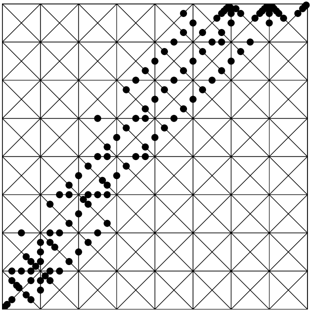
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

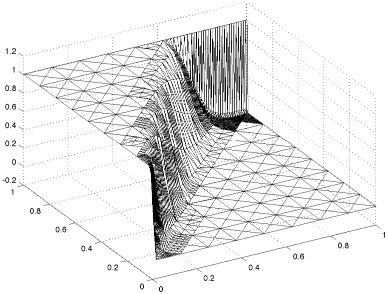




Iteration 28, ($l = 1$)

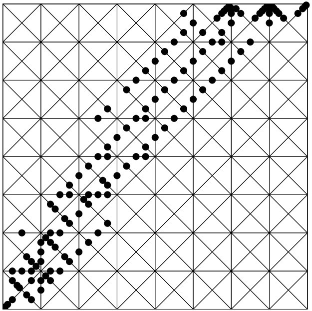
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

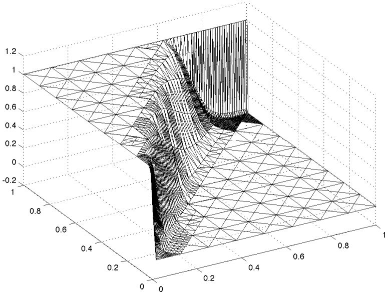




Iteration 29, ($l = 1$)

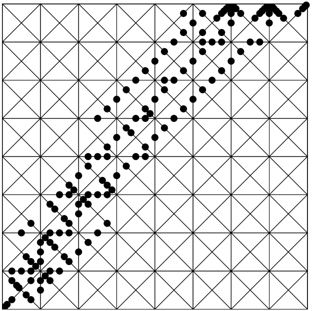
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

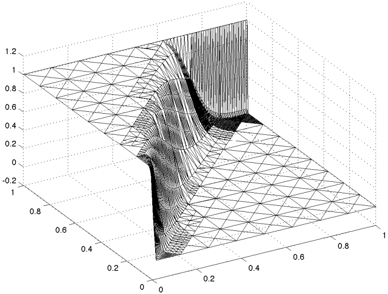




Iteration 30, ($l = 1$)

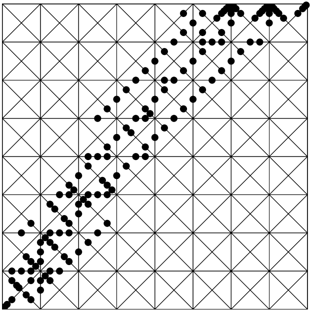
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

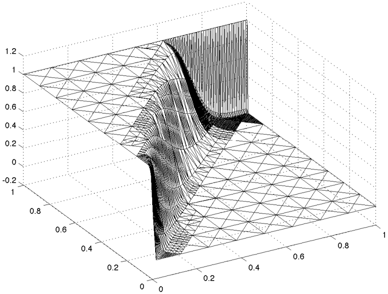




Iteration 31, $(l = 1)$

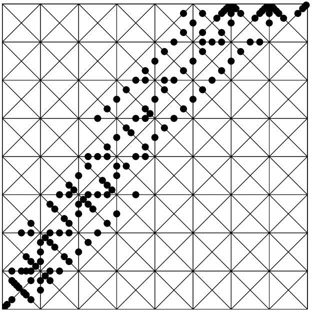
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

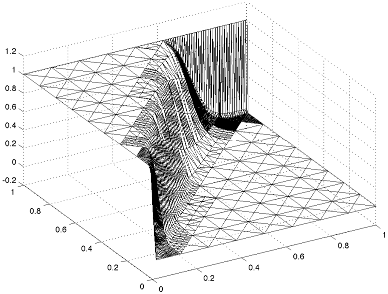




Iteration 32, $(l = 1)$

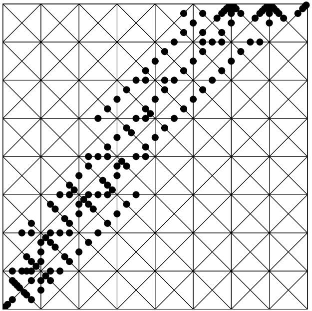
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

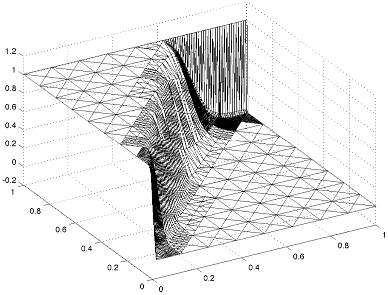




Iteration 33, ($l = 1$)

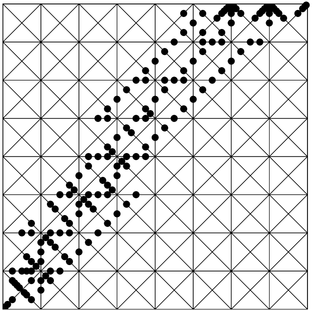
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

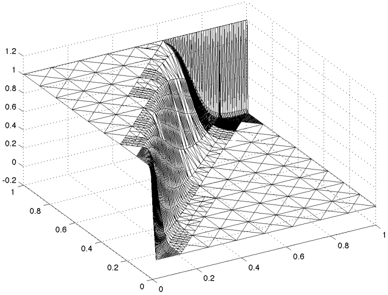




Iteration 34, ($l = 1$)

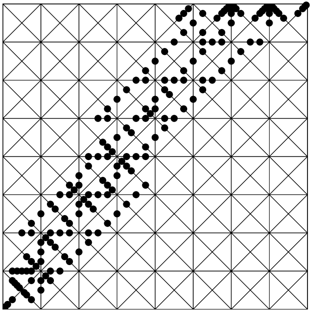
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$

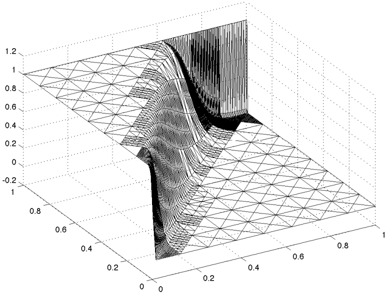




Iteration 35, ($l = 1$)

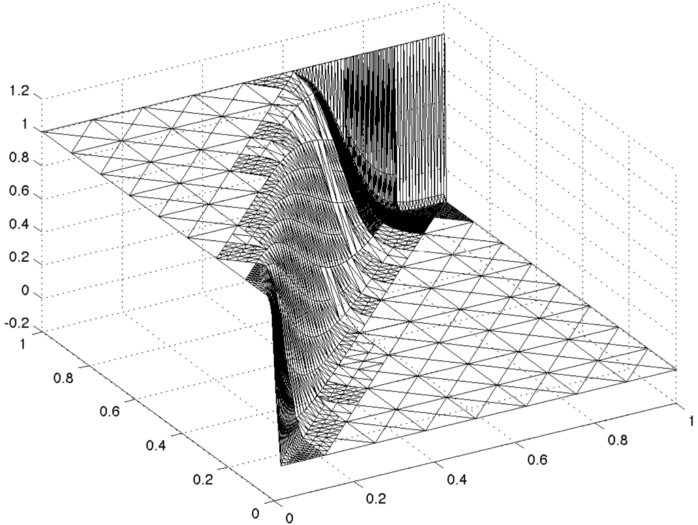
Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$



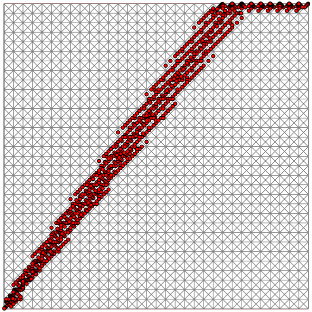


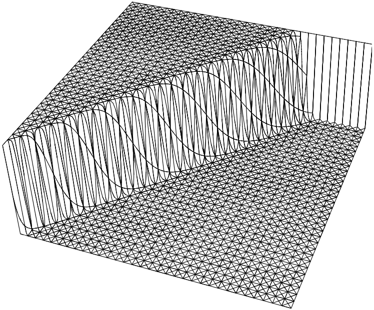
Iteration 36, ($l = 1$)

Skew-advection: $\varepsilon = 10^{-3}$, $\sigma = 0$



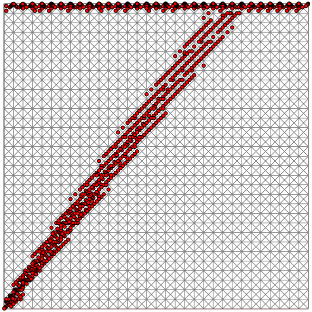
Skew-advection: $\varepsilon = 10^{-4}$, $\sigma = 0$

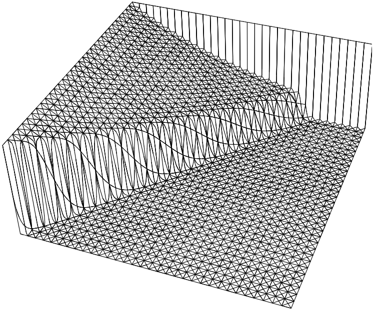




around 2.2×10^4 d.o.f.
($l = 2$)

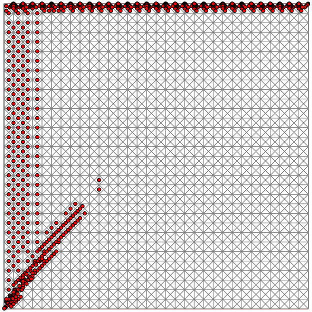
Skew-advection: $\varepsilon = 10^{-4}$, $\sigma = 1$

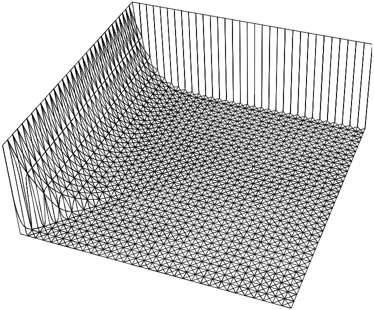




around 2.2×10^4 d.o.f.
($l = 2$)

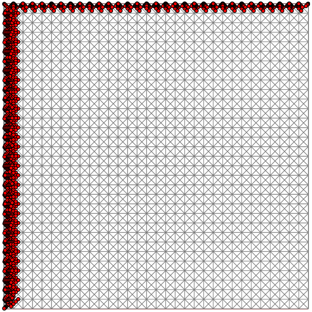
Skew-advection: $\varepsilon = 10^{-4}$, $\sigma = 10$

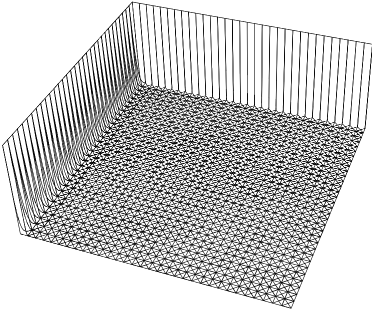




around 2.2×10^4 d.o.f.
($l = 2$)

Skew-advection: $\varepsilon = 10^{-4}$, $\sigma = 100$



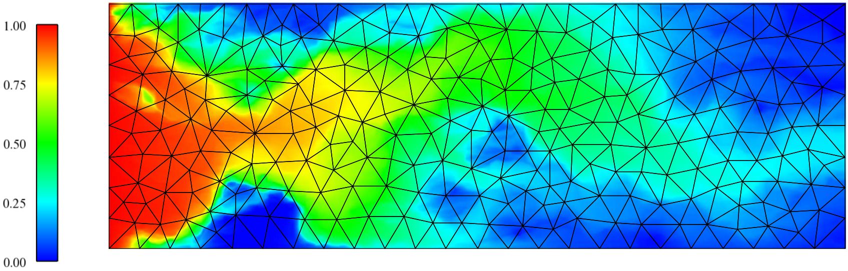


around 2.2×10^4 d.o.f.
$(l = 2)$

Conclusions

- New space-based adaptivity based on a posteriori estimator
- Improve accuracy on coarse meshes
- A "non-conform" type mesh as a result of local adaptivity
- A very efficient algorithm: re-calculate just the necessary
- Natural parallel process: method + adaptivity (Erlang, MPI, CUDA, etc.) $\rightarrow$ HPC opportunities!!!

In progress...



Next: wave propagation problems in a highly heterogeneous media

Thank you!