Multiscale Hybrid Method (MHM) Remarks and Applications

Alexandre L. Madureira
www.lncc.br/∼alm

Laboratório Nacional de Computação Científica – LNCC
Petrópolis - RJ

Third Workshop of the CNPq-Inria HOSCAR Project
Inria Bordeaux — September 3, 2013

Conteúdo

Contents

Linearized Elasticity

Assume

- Domain $\Omega \subset \mathbb{R}^2$ and load $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ are "nice"
- Rigidity tensor $\mathcal{A}(\boldsymbol{x}) : \mathbb{R}^{2\times2}_{\mathrm{sym}} \to \mathbb{R}^{2\times2}_{\mathrm{sym}}$ not so "nice" wrt $\boldsymbol{x} \in \Omega$

The problem:

$$\sigma = \mathcal{A}\, \boldsymbol{\nabla}^{\mathrm{s}} \boldsymbol{u}, \quad -\operatorname{\mathbf{div}} \sigma = \boldsymbol{f} \qquad \text{in } \Omega,$$
$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega,$$

where

- $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ is the displacement
- $\boldsymbol{\nabla}^{\mathrm{s}} \boldsymbol{u} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)$
- $\sigma : \Omega \to \mathbb{R}^{2\times2}_{\mathrm{sym}}$ is the stress

“Classical” Hybrid formulation

Definition

- *$\mathcal{P}$: partition of Ω into elements K*
- $\partial\mathcal{P} = \cup_{K\in\mathcal{P}} \partial K$
- $\boldsymbol{H}^1(\mathcal{P}) = \prod_{K\in\mathcal{P}} \boldsymbol{H}^1(K)$
- $\boldsymbol{\Lambda} = \boldsymbol{H}^{-1/2}(\partial\mathcal{P})$

Problem:

Find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbf{V} \times \boldsymbol{\Lambda}$ *such that*

$$\begin{aligned} (\mathcal{A}\,\boldsymbol{\nabla}^{\mathsf{s}}\boldsymbol{u}, \boldsymbol{\nabla}^{\mathsf{s}}\boldsymbol{v})_{\mathcal{P}} + (\boldsymbol{\lambda}, [\![\boldsymbol{v}]\!])_{\partial\mathcal{P}} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{P}} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{P}), \\ (\boldsymbol{\mu}, [\![\boldsymbol{u}]\!])_{\partial\mathcal{P}} &= 0 \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \end{aligned}$$

Also, $\boldsymbol{\lambda}$ is the traction on $\partial\mathcal{P}$.

MHM Recipe

- Decompose $\boldsymbol{H}^1(\mathcal{P}) = \mathbf{V}_{\rm rm} \oplus \mathbf{V}_{\rm rm}^{\perp}$
- $\mathbf{V}_{\rm rm} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{P}) : \boldsymbol{\nabla}^{\rm s}(\boldsymbol{v}) = 0\}$: piecewise rigid motions
- Then $\boldsymbol{u} = \boldsymbol{u}_{\rm rm} + \boldsymbol{u}_{\rm rm}^{\perp}$ and for all $K \in \mathcal{P}$,

$$-\,\mathbf{div}\,\mathcal{A}\,\boldsymbol{\nabla}^{\rm s}\,\boldsymbol{u}_{\rm rm}^{\perp} = \boldsymbol{f} \quad \text{in } K \qquad \mathcal{A}\,\boldsymbol{\nabla}^{\rm s}\,\boldsymbol{u}_{\rm rm}^{\perp}\boldsymbol{n} = \boldsymbol{\lambda} \quad \text{on } \partial K$$

- Write $\boldsymbol{u}_{\rm rm}^{\perp} = T\boldsymbol{\lambda} + \hat{T}\boldsymbol{f}$. Thus $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, $\boldsymbol{u}_{\rm rm} \in \mathbf{V}_{\rm rm}$ solve

$$\begin{aligned} (\boldsymbol{\mu}, [\![T\boldsymbol{\lambda}]\!])_{\partial\mathcal{P}} + (\boldsymbol{\mu}, [\![\boldsymbol{u}_{\rm rm}]\!])_{\partial\mathcal{P}} &= -(\boldsymbol{\mu}, [\![\hat{T}\boldsymbol{f}]\!])_{\partial\mathcal{P}} && \text{for all } \boldsymbol{\mu} \in \boldsymbol{\Lambda} \\ (\boldsymbol{\lambda}, [\![\boldsymbol{v}_{\rm rm}]\!])_{\partial\mathcal{P}} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{P}} && \text{for all } \boldsymbol{v}_{\rm rm} \in \mathbf{V}_{\rm rm} \end{aligned}$$

- Discretize $\boldsymbol{\Lambda}$ to obtain the MHM

Contents

General Mixed Problems

Definition

- *X, M Hilbert spaces*
- *$a \in \mathcal{L}(X \times X, \mathbb{R})$, $b \in \mathcal{L}(X \times M, \mathbb{R})$, $l \in X$ and $\sigma \in M$*

The problem: $(u, \eta) \in X \times M$ solve

$$\begin{aligned} a(u, v) + b(v, \eta) &= l(v) \quad \text{for all } v \in X \\ b(u, \mu) &= \sigma(\mu) \quad \text{for all } \mu \in M \end{aligned}$$

We could also consider the same problem in the form

$$\begin{aligned} Au + B^T \eta &= l \quad \text{in } X \\ Bu &= \sigma \quad \text{in } M \end{aligned}$$

where $A \in \mathcal{L}(X, X)$, $B \in \mathcal{L}(X, M)$ are associated with a, b

General Mixed Problems

Hypotheses

(a) $a(\cdot,\cdot)$ *is symmetric (for simplicity)*

(b) *Usual well-posedness assumptions:*

$$a(v,v) \gtrsim \|v\|_X^2 \text{ for } v \in \mathrm{Kern}(B), \qquad \inf_{\mu \in M} \sup_{v \in X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_M} \gtrsim 1$$

(c) *Korn-like inequality:*

$$a(v,v) \gtrsim \|v\|_X^2 \quad \text{for all } v \in \mathrm{Kern}(A)$$

MHM formalism

- $X_{\mathrm{rm}} = \mathrm{Kern}\, A$: "rigid motions"
- $X = X_{\mathrm{rm}} \oplus \tilde{X}$ where $\tilde{X} = X_{\mathrm{rm}}^{\perp}$ wrt X-inner-product
- Then $u = u_{\mathrm{rm}} + \tilde{u}$ and

$$A\tilde{u} = -B^T \eta + l$$

- Write $\tilde{u} = T\eta + \hat{T}l$
- Thus $\eta \in M$, $u_{\mathrm{rm}} \in X_{\mathrm{rm}}$ solve

$$\begin{aligned} b(T\eta, \mu) - b(u_{\mathrm{rm}}, \mu) &= -\sigma(\mu) + b(\hat{T}l, \mu) \quad \text{for all } \mu \in M, \\ b(v_{\mathrm{rm}}, \eta) &= l(v_{\mathrm{rm}}) \quad \text{for all } v_{\mathrm{rm}} \in X_{\mathrm{rm}}. \end{aligned}$$

MHM formalism

In terms of operators,

$$\begin{aligned} \tilde{A}\tilde{u} \qquad\qquad + \tilde{B}^T\eta &= \tilde{l}, \\ B_{\mathrm{rm}}^T\eta &= l_{\mathrm{rm}}, \\ \tilde{B}\tilde{u} + B_{\mathrm{rm}} u_{\mathrm{rm}} \qquad\quad &= \sigma. \end{aligned}$$

With such notation, $T = \tilde{A}^{-1}\tilde{B}^T$ and $\hat{T} = \tilde{A}^{-1}$. Thus

$$\begin{aligned} \tilde{B}T\eta - B_{\mathrm{rm}} u_{\mathrm{rm}} &= -\sigma + \tilde{B}\hat{T}\tilde{l}, \\ B_{\mathrm{rm}}^T\eta &= l_{\mathrm{rm}}. \end{aligned}$$

Mathematical properties

- Well-posedness follow easily using the original problem
- It can also be obtained from “first principles”, but not easily

Discretization: split spaces first, discretize latter

Assumptions

- $M_h \subset M$ and $\tilde{X}_h \subset \tilde{X}$
- X_{rm} is available
- T_h and $\hat{T}_h$ "approximates" T and $\hat{T}$

Discrete Problem

$u^h_{\mathrm{rm}} \in X_{\mathrm{rm}}$ and $\eta^h \in M_h$ solve

$$\begin{aligned} -b(T_h\eta^h, \mu^h) + b(u^h_{\mathrm{rm}}, \mu^h) &= \sigma(\mu^h) - b(\hat{T}_h l, \mu^h) \quad && \text{for all } \mu^h \in M_h, \\ b(v_{\mathrm{rm}}, \eta^h) &= l(v_{\mathrm{rm}}) && \text{for all } v_{\mathrm{rm}} \in X_{\mathrm{rm}}. \end{aligned}$$

Error estimate as usual (under conditions for $\tilde{X}_h$ and M_h):

$$\|\eta - \eta^h\|_M + \|u - u^h\|_X \lesssim \inf_{\tilde{v}^h \in \tilde{X}_h} \|\tilde{u} - \tilde{v}^h\|_X + \inf_{\mu^h \in M_h} \|\eta - \mu^h\|_M$$

Discretization: discretize first, split spaces latter

Assumptions

- $X_h \subset X$ and $M_h \subset M$
- $(u^h, \eta^h) \in X_h \times M_h$ such that

$$\begin{aligned} a(u^h, v^h) + b(v^h, \eta) &= l(v^h) \quad \text{for all } v^h \in X_h \\ b(u^h, \mu^h) &= \sigma(\mu^h) \quad \text{for all } \mu \in M_h. \end{aligned}$$

- $X_{\mathrm{rm}} \subset X_h$, and let $X_h = \tilde{X}_h \oplus X_{\mathrm{rm}}$ with $\tilde{X}_h \subset \tilde{X}$

If $u^h = \tilde{u}^h + u^h_{\mathrm{rm}}$, then

$$\begin{aligned} -b(T_h \eta^h, \mu^h) + b(u^h_{\mathrm{rm}}, \mu^h) &= \sigma(\mu^h) - b(\hat{T}_h l, \mu^h) \quad \text{for all } \mu^h \in M_h, \\ b(v_{\mathrm{rm}}, \eta^h) &= l(v_{\mathrm{rm}}) \quad \text{for all } v_{\mathrm{rm}} \in X_{\mathrm{rm}}. \end{aligned}$$

Conclusion: discretization and MHM formalism commute

Contents

Dual hybrid formulation for Poisson equation

Poisson problem

$$\boldsymbol{\sigma} = \mathcal{C}\,\boldsymbol{\nabla}\, u, \quad -\operatorname{div}\boldsymbol{\sigma} = f \quad \text{in } \Omega, \qquad \boldsymbol{u} = 0 \quad \text{on } \partial\Omega,$$

Hybrid formulation

- Fix $\sigma^f \in \mathbf{W}^f = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) : -\operatorname{div}\boldsymbol{\tau} = f \text{ in } K \in \mathcal{P}\}$
- $\boldsymbol{\sigma} \in \mathbf{W}^0$ and $u \in M = \prod_{K\in\mathcal{P}} H^{1/2}(\partial K)$ solve

$$\int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,d\boldsymbol{x} - \sum_{K\in\mathcal{P}}\int_{\partial K} u\boldsymbol{\tau}\cdot\boldsymbol{n}\,d\boldsymbol{x} = -\int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\sigma}^f\cdot\boldsymbol{\tau}\,d\boldsymbol{x}$$

$$-\sum_{K\in\mathcal{P}}\int_{\partial K} v\boldsymbol{\sigma}\cdot\boldsymbol{n}\,d\boldsymbol{x} = 0$$

for all $\boldsymbol{\tau} \in \mathbf{W}^0$ and $v \in M$.

MHM version

Definition of T and $\hat{T}$

For $v \in M$ and $K \in \mathcal{P}$:

$$\int_K \mathcal{C}^{-1}(Tv) \cdot \boldsymbol{\tau}\, d\boldsymbol{x} = -\int_{\partial K} v\boldsymbol{\tau} \cdot \boldsymbol{n}\, d\boldsymbol{x}, \qquad \hat{T}l = -\boldsymbol{\sigma}^f$$

MHM method

$u \in M$ solves

$$-\sum_{K \in \mathcal{P}} \int_{\partial K} (Tu) \cdot \boldsymbol{n} v\, d\boldsymbol{x} = \int_\Omega fv\, d\boldsymbol{x} + \sum_{K \in \mathcal{P}} \int_{\partial K} \boldsymbol{\sigma}^f \cdot \boldsymbol{n} v\, d\boldsymbol{x}$$

for all $v \in M$.

What happens under discretization?

Solving discrete MHM is equivalent to find $u^h \in V_{Ms}(\Omega)$

$$\int_\Omega \mathcal{C}\,\nabla\, u^h\, \nabla\, v^h\, d\boldsymbol{x} = \int_\Omega \sigma^f\, \nabla\, v^h\, d\boldsymbol{x} \qquad \text{for all} v^h \in V_{Ms}(\Omega),$$

where

$$V_{Ms}(\Omega) = \{v \in H_0^1(\Omega) : \operatorname{div} \mathcal{C}\,\nabla\, v = 0 \text{ in } K, v \text{ is linear over } \partial K\}.$$

Nice:

- This is the Residual Free Bubbles formulation of [Brezzi, Franca and Russo, 96, 98], and [Sangalli, 03]
- Also related to the Multiscale Finite Element Method of Tom Hou and collaborarors

Contents

Conclusions

- MHM yields a powerful to way to solve hybrid formulations through parallel computation
- It's possible to frame MHM in a more general formulation, maybe allowing alternative analyses and implementations
- In particular, maybe two level estimates become (easily) within reach
- Also, it connects the RFB and Hybrid methods

Merci!