Multiscale Hybrid-Mixed Method and Numerical Zoom

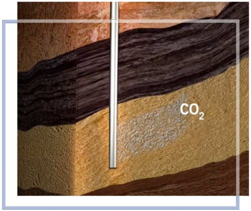
Frédéric Valentin[1]

Applied Mathematics Department, LNCC
BRAZIL

First Brazil-France Workshop
LNCC, September 10-13, 2012

[1] Joint work with R. Araya, C. Harder, D. Paredes and A. Madureira

Sequestration of CO_2



$$-\nabla \cdot (\mathcal{A}\,\mathcal{E}(\boldsymbol{u}_s)) = \alpha\,\nabla p$$

$$\boldsymbol{u}_f + \mathcal{K}(S)\,\nabla p = \mathbf{g}(S)$$

$$\nabla\cdot \boldsymbol{u}_f + \partial_t(\phi\,p + \alpha\,\nabla\cdot\boldsymbol{u}_s) = f$$

$$\partial_t\phi S - \epsilon(S)\Delta S + \beta(S,\boldsymbol{u}_f)\cdot\nabla S = g(S)$$

Darcy / Elasticity Model

Elliptic Models

Find u such that

$$-\nabla \cdot D\,u = f \quad \text{in } \Omega, \quad D\,u \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega$$

Laplace: *Find the pressure u where*

$$D\,u := \mathcal{K}\,\nabla u$$

Elasticity: *Find the displacement $\boldsymbol{u}$ where*

$$D\,u := \mathcal{K}\,\mathcal{E}(\boldsymbol{u})$$

Here $\mathcal{K}$ is a positive definite tensor with *multi-scales* features

Mixed Finite Element: u and $D\,u$

Stability + Precision + Conservation

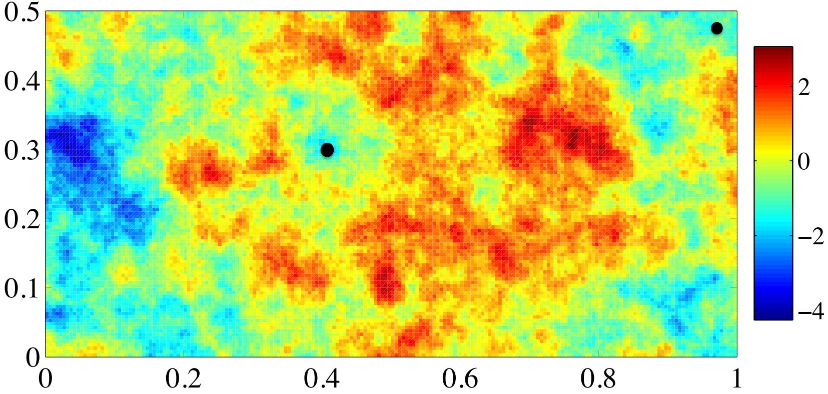
Laplace (Darcy): $RT_k/\mathbb{P}_k$ or $BDM_k/\mathbb{P}_{k-1}$ elements

Elasticity: Relax conformity or the symmetry of $D\,u$

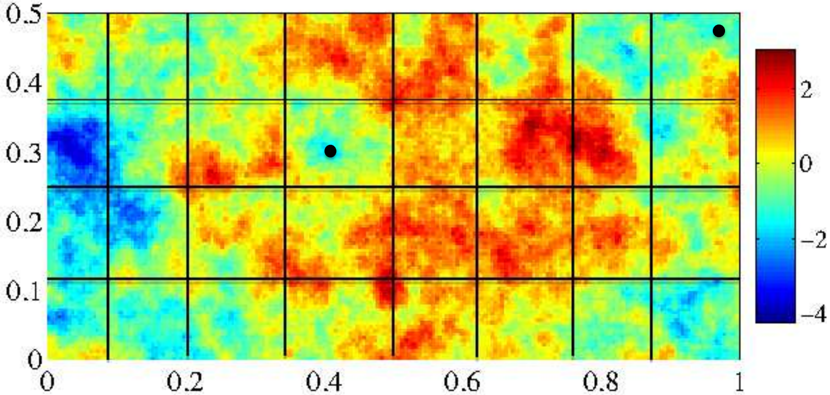
Arnold at al. (2008): 162 d.o.f. in 3D

Basis functions are *"ignorant" of multi-scales* structures

A typical permeability field (Courtesy M. Borges)



Upscalling and Numerical Zoom



A (incomplete) State-of-the-Art

Elasticity Equation Multi-Scale + Stable		Darcy Equation Multi-Scale + Stable	
??	??	$RT_0/\mathbb{P}_0$	Hou et al. (2002)
		$BDM_1/\mathbb{P}_0$	Arbogast (2004)

The MHM Formulation

Hybrid Formulation (Coarse Mesh $\mathcal{T}_h$)

Find u such that

$$-\nabla \cdot D\, u = f \quad \text{in } \Omega, \quad D\, u \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega$$

Find $(u, \lambda) \in V \times \Lambda$ s.t. for all $(v, \mu) \in V \times \Lambda$

$$\begin{aligned} (\mathcal{K}^{-1} D\, u,\, D\, v)_{\mathcal{T}_h} + (\lambda\, \boldsymbol{n},\, [v])_{\mathcal{E}_h} &= (f,\, v)_{\mathcal{T}_h} \\ ([u], \mu\, \boldsymbol{n})_{\mathcal{E}_h} &= 0 \end{aligned}$$

Observe that u satisfies the (ill-posed) local problem

$$(\mathcal{K}^{-1}\, D\, u, D\, v)_K = (f, v)_K - (\lambda, v)_{\partial K}$$

Space Decomposition

Split the space

$$V = V_0 \oplus V_0^{\perp}$$

such that

$$\bar{v} \in V_0 \quad \text{s.t.} \quad (\mathcal{K}^{-1} D\, \bar{v}, D\, v)_K = 0$$

Laplace problem: V_0 is the piecewise constant space

Elasticity problem: V_0 is the space of rigid body modes

A function $v \in V$ decomposes

$$v = \bar{v} + \tilde{v}, \qquad \bar{v} \in V_0 \quad \text{and} \quad \tilde{v} \in V_0^{\perp}$$

Weak Formulation - Hybrid

Test function $(\tilde{v}, 0)$: *Find* $\tilde{u} \in V_0^{\perp}$ *such that*

$$(\mathcal{K}^{-1}\, D\,\tilde{u}, D\,\tilde{v})_K = (f, \tilde{v})_K - (\lambda, \tilde{v})_{\partial K} \quad \forall \tilde{v} \in V_0^{\perp}$$

Split

$$\tilde{u} \;=\; u^{\lambda} + u^{f}$$

Test function $(\bar{v}, \mu)$: *Find* $(\bar{u},\, \lambda) \in V_0 \times \Lambda$ *such that*

$$(\mathcal{K}^{-1} D\, u,\, D\,\bar{v})_{\mathcal{T}_h} + (\lambda\, \boldsymbol{n},\, [\bar{v}])_{\mathcal{E}_h} = (f,\, \bar{v})_{\mathcal{T}_h} \quad \forall \bar{v} \in V_0$$

$$\Big([\bar{u} + u^{\lambda}], \mu\, \boldsymbol{n}\Big)_{\mathcal{E}_h} = -\Big([u^{f}], \mu\, \boldsymbol{n}\Big)_{\mathcal{E}_h} \quad \forall \mu \in \Lambda$$

Alternative Weak Formulation - Mixed

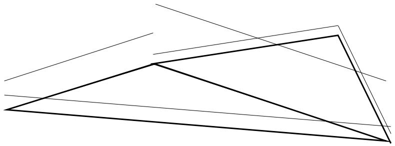
Find $(\bar{u},\, \lambda) \in V_0 \times \Lambda$ *such that*

$$\begin{aligned} \left(\nabla \cdot D\, u^{\lambda},\, \bar{v}\right)_{\mathcal{T}_h} &= -\,(f,\, \bar{v})_{\mathcal{T}_h} \quad && \forall \bar{v} \in V_0 \\ \left(\mathcal{K}^{-1} D\, u^{\lambda}, D\, u^{\mu}\right)_{\mathcal{T}_h} + (\bar{u}, \nabla \cdot D\, u^{\mu})_{\mathcal{T}_h} &= -\big(f, u^{\mu}\big)_{\mathcal{T}_h} \quad && \forall \mu \in \Lambda \end{aligned}$$

The MHM method

Finite Data Set

Space Λ_0



- Thus far, no discretization introduced
- The only missed choice is $\Lambda_l \subset \Lambda$
- Functions denoted $\lambda_l \in \Lambda_l$

Find u^{λ_l} *such that*

$$(\mathcal{K}^{-1}\, D\, u^{\lambda_l}, D\, \tilde{v})_K = -(\lambda_l, \tilde{v})_{\partial K}$$

Finite Element

Given $\lambda_l \in \Lambda_l$ by

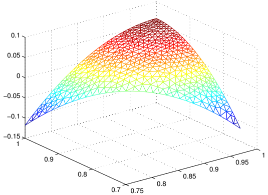
$$\lambda_l = \sum_{i=1}^{\dim \Lambda_l} \lambda_l(\mathbf{x}_i)\, \psi_i^l, \quad \psi_i^l \in \mathbb{P}_l(F)$$

Let $L = \{\lambda_l(\mathbf{x}_i)\}_{i=1}^{\dim(\Lambda_l|_{\partial K})}$ and $\mathcal{N} = \{\eta_i\}_{i=1}^{\dim(\lambda_l|_{\partial K})}$ where η_i

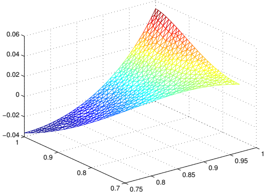
$$(\mathcal{K}^{-1}\, D\, \eta_i, D\, \tilde{v})_K = -(\psi_i^l, \tilde{v})_{\partial K}$$

The triplet $\{K, \mathcal{N}, L\}$ is a *finite element*

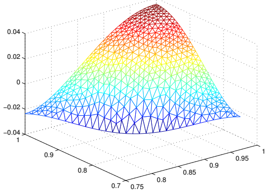
Space Λ_0



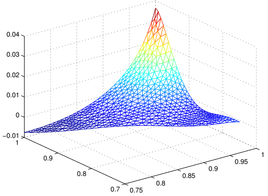
Space Λ_{-1}



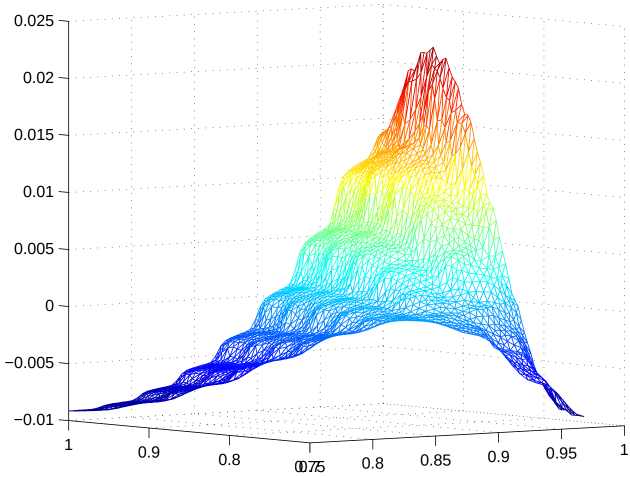
Space Λ_{-2}



Space Λ_{-3}



An Oscillatory Basis Function for Λ_{-3}



The MHM Method

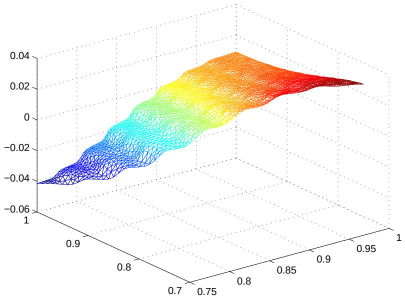
Find $(u_0,\, \lambda_l) \in V_0 \times \Lambda_l$ *such that*

$$\left(\nabla \cdot D\, u^{\lambda_l},\, v_0\right)_{\mathcal{T}_h} = -\,(f,\, v_0)_{\mathcal{T}_h} \quad \forall v_0 \in V_0$$

$$\left(\mathcal{K}^{-1} D\, u^{\lambda_l}, D\, u^{\mu_l}\right)_{\mathcal{T}_h} + (u_0, \nabla \cdot D\, u^{\mu_l})_{\mathcal{T}_h} = -(f, u^{\mu_l})_{\mathcal{T}_h} \quad \forall \mu_l \in \Lambda_l$$

u^{λ_l} solves

$$(\mathcal{K}^{-1}\, D\, u^{\lambda_l}, D\, \tilde{v})_K = -(\lambda_l, \tilde{v})_{\partial K}$$



Relationship with Other Methods: The Darcy Model

$l = 0,\ f \in \mathbb{R}$		$l = 0,\ 1$
$\mathcal{K} \equiv I$	Multi-Scale $\mathcal{K}$	Multi-Scale $\mathcal{K}$
$\mathrm{RT}_0/\mathbb{P}_0$	Hou at al. (2002)	Arbogast (2004)

Considerations on the Method

$$u_h := u_0 + u^{\lambda_l} + u^f \quad \text{and} \quad \boldsymbol{\sigma}_h := D\, u_h \in H(div, \Omega)$$

$V_0 \times \Lambda_l$ is inf-sup stable

Convergence driven only by $\Lambda_l \approx \Lambda$

Locally conservative

$$\int_K \nabla \cdot \boldsymbol{\sigma}_h\, v_0 = \int_K f\, v_0$$

An (Face) A Posteriori Error Estimator

$$R_F := \begin{cases} \frac{1}{2}\,[u_h], & F \in \text{internal edges} \\ u_h\,\boldsymbol{n}, & F \in \text{Dirichlet b.c.} \\ \mathbf{0}, & F \in \text{Neumann b.c.} \end{cases}$$

$$\eta := \Big[\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \frac{1}{h_F} \|R_F\|^2_{0,F} \Big]^{1/2}$$

Both reliability and efficiency hold

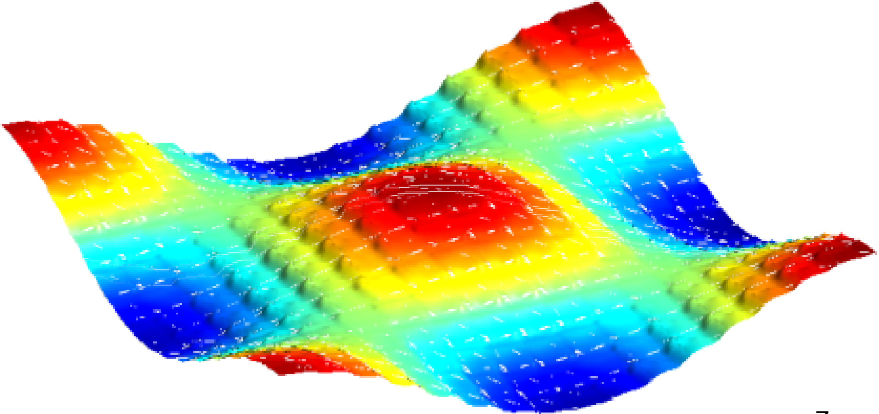
Numerical Validation
Darcy Equation

The Highly-Oscillatory Problem ($\varepsilon = \frac{1}{16}$)

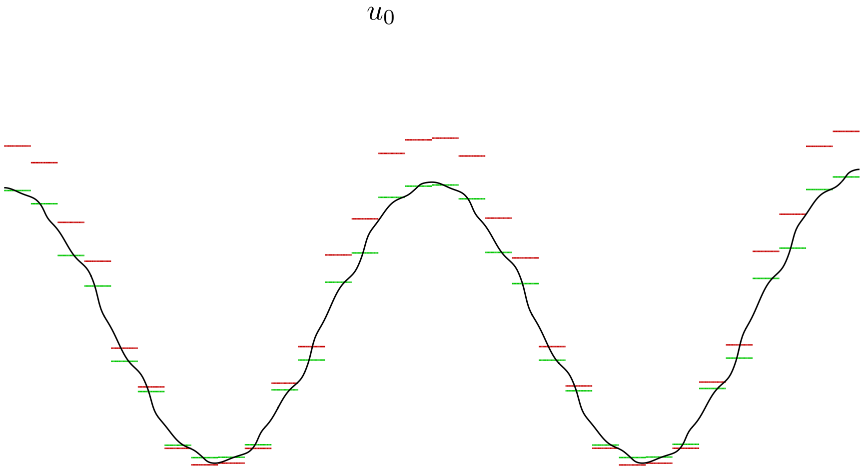
$$\mathcal{K}(\boldsymbol{x}) = \frac{2 + \frac{1.8 \sin 2\pi x}{\varepsilon}}{2 + \dfrac{1.8 \sin 2\pi y}{\varepsilon}} + \frac{2 + \dfrac{1.8 \sin 2\pi y}{\varepsilon}}{2 + \dfrac{1.8 \cos 2\pi x}{\varepsilon}},$$

Element $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$ (16 × 16 Elements)

$$u_0 + u^{\lambda_l} + u^f$$

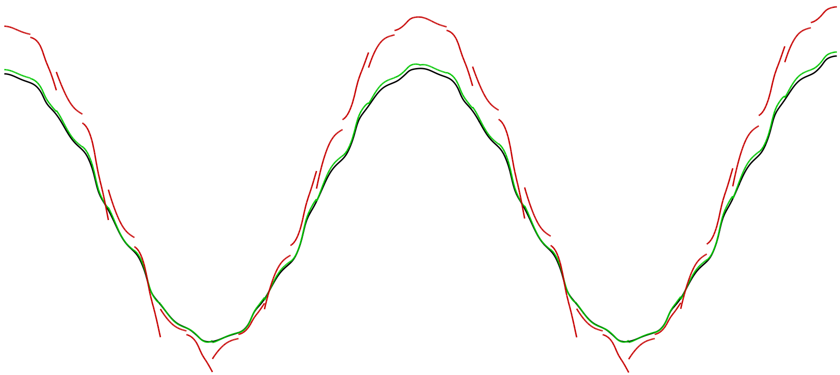


Elements $\mathbb{P}_0(K)/\mathbb{P}_0(F)$ and $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$



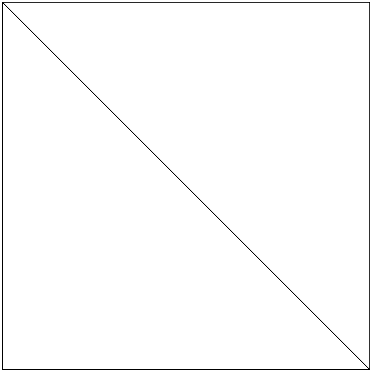
Elements $\mathbb{P}_0(K)/\mathbb{P}_0(F)$ and $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$

$$u_0 + u^{\lambda_l} + u^f$$



A Quarter Five-Spot Problem

- Two wells, in opposing corners.
- Permeability set to 1.
- Homogeneous Neumann boundary conditions



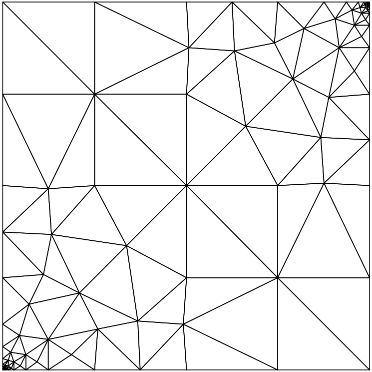
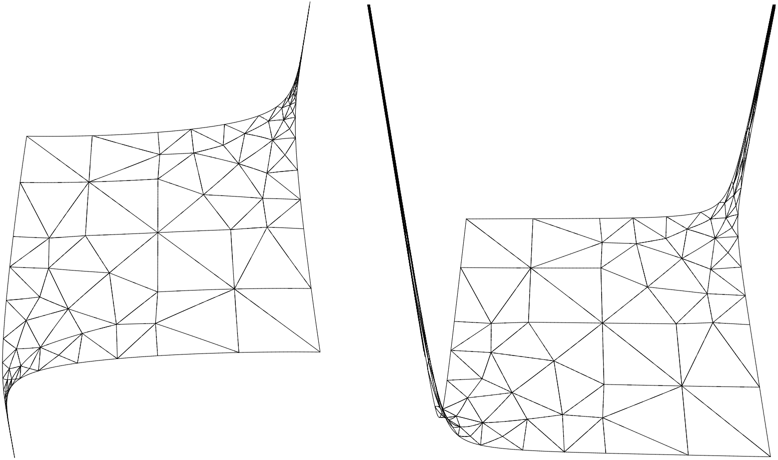
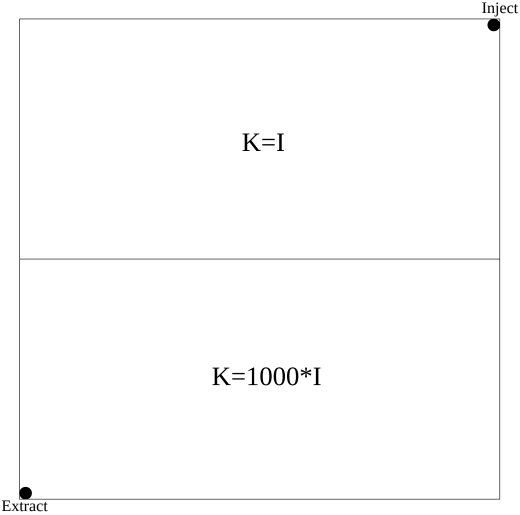


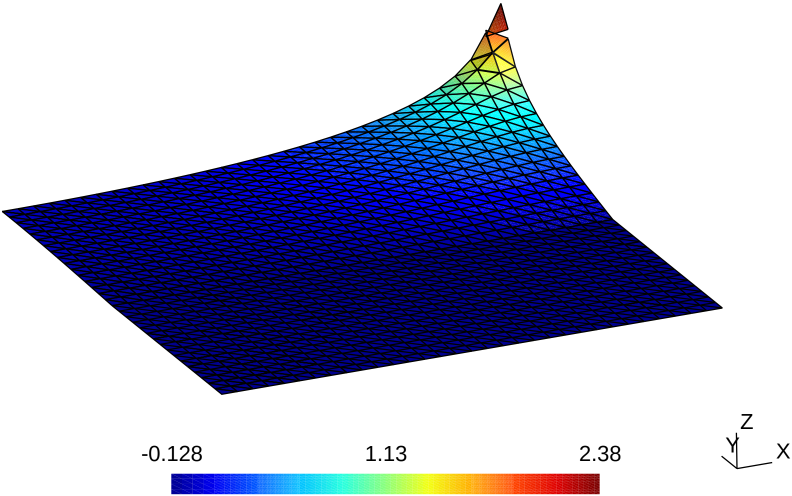
Figure: u_h and $|\boldsymbol{\sigma}_h|$



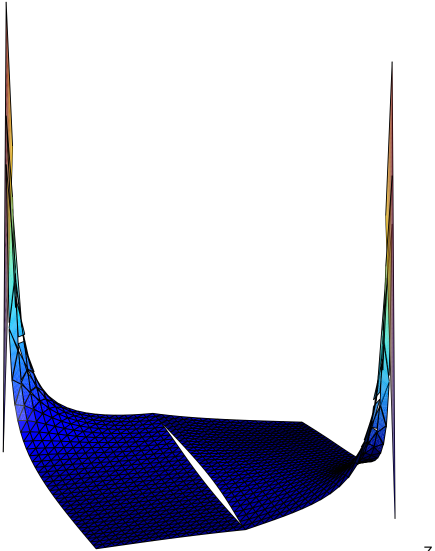
A High-Constrast Permeability Problem



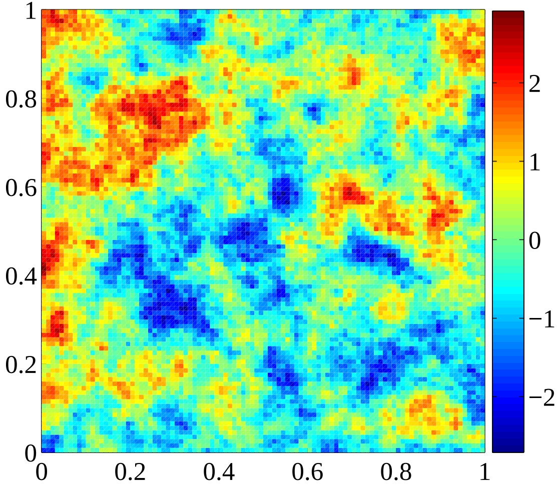
Element $\mathbb{P}_0(K)/\mathbb{P}_{-1}(F)$: u_h



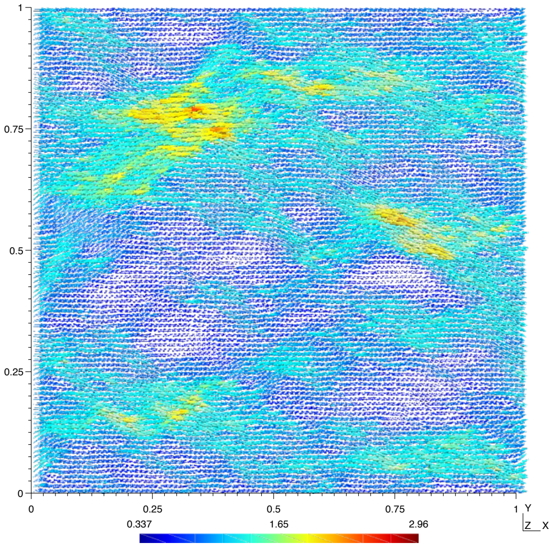
Element $\mathbb{P}_0(K)/\mathbb{P}_{-1}(F)$: $|\boldsymbol{\sigma}_h|$



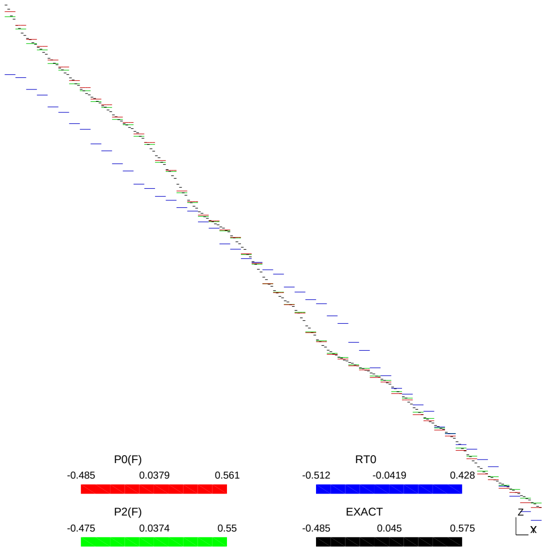
A Heterogenous Permeability Problem



Elements $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$ (25×25 elements)

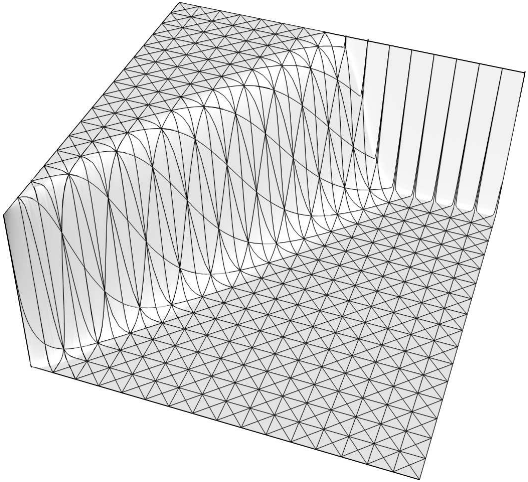


RT_0 Element Fails: Profiles of u_0



Upcoming: Advective Dominate Model

The MHM method avoids under and over shootings



Conclusion

- New stable elements for the Darcy and elasticity models
- Include: Local upscaling + Numerical zoom
- Highly adapted to paralel computation
- Elasticity: Conformity + Strong Symmetry + Equilibrium
- Elasticity: Simplest element = 12 (2D) or 30 (3D) d.o.f.

Perspectives: Possible Collaborations

- Porous media problems ⇒ Heterogenous media (S. Lanteri's Group) + Multiphase flows (R. Masson's Group) + Including fractures (J. Jaffré's Group)
- Extension of the MHM method: Electromagnetism in heterogenous media ⇒ S. Lanteri's Group
- Paralelization: CPU or GPU (or hybrid)