Upscaling for the Laplace problem using a Discontinuous Galerkin method

Théophile CHAUMONT FRELET

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Plan

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Seismic imaging in higthly heterogeneous media

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- We say that $\mathcal{T}_h^i$ is the fine submesh of the coarse cell K^i.

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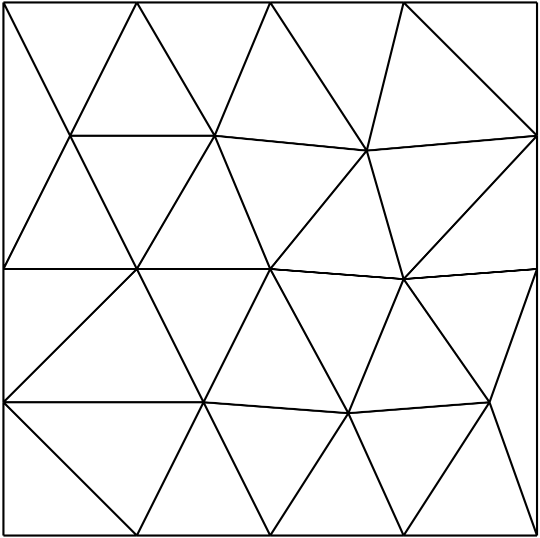
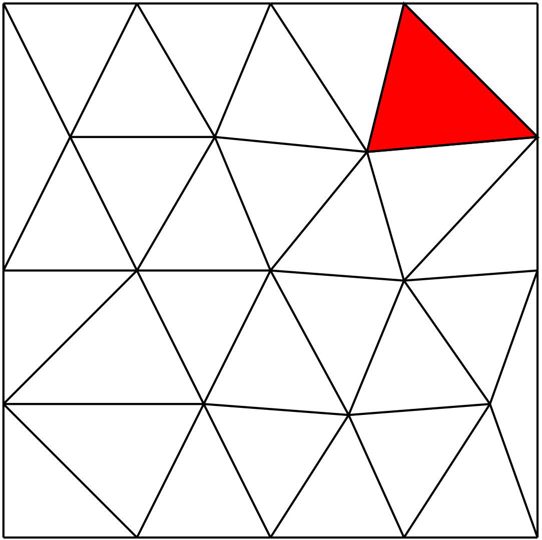


$K^1 \in \mathcal{T}_H$



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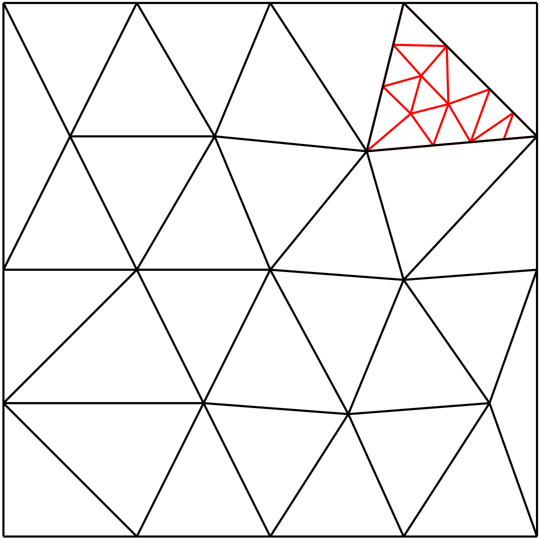

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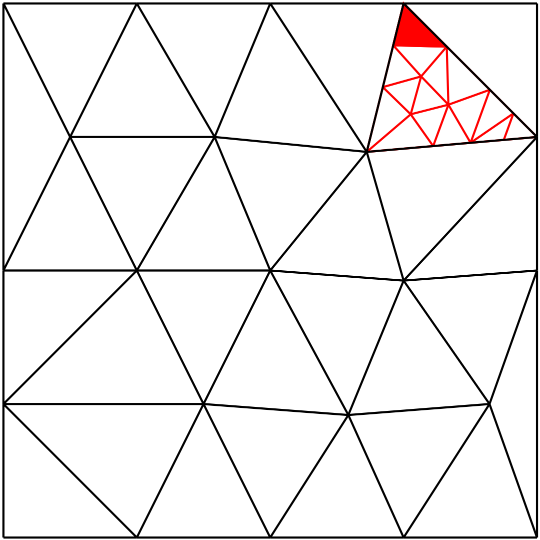

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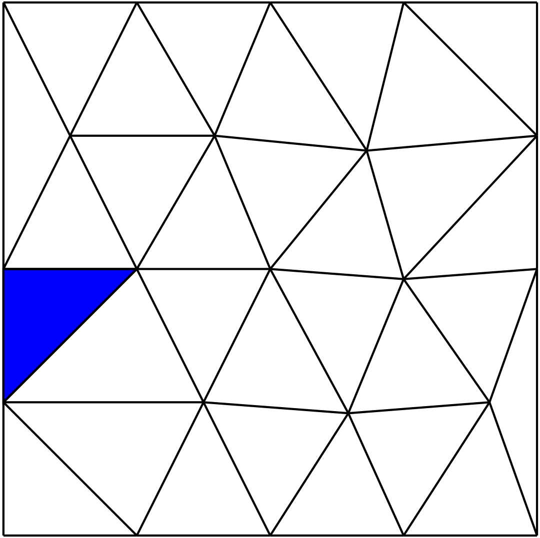

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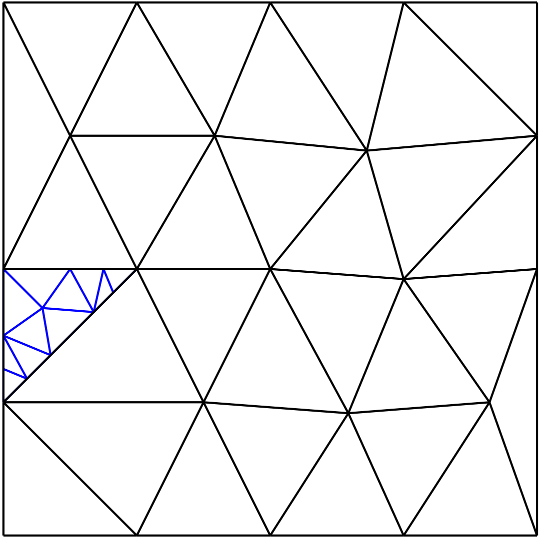

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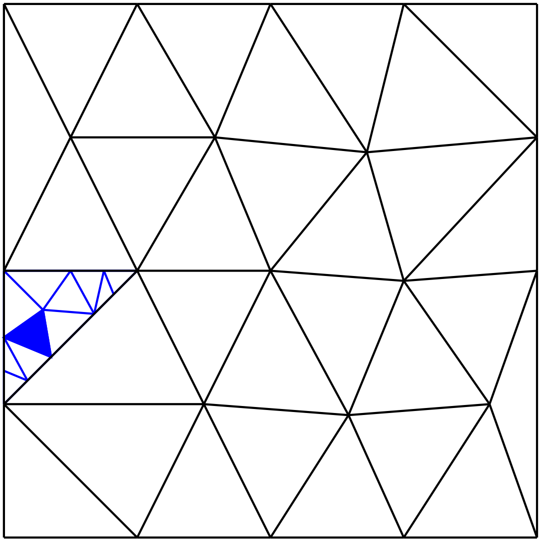

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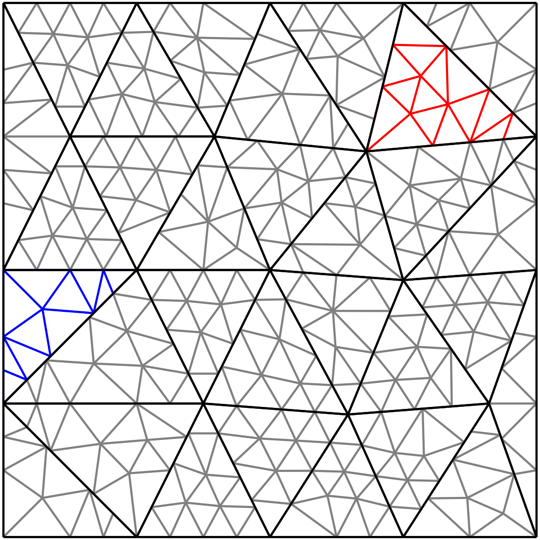

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- Note that $\mathbb{P}_{1,0}(\mathcal{T}) \subset H_0^1(\omega)$.

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- The fonctions $v \in \hat{V}_h^i$ are extended by 0 on $\Omega \setminus K^i$.
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- We say that $\hat{V}_h^i$ contains artificial boundary contidition, because the trace condition in the definition is not imposed by the PDE ($v|_{\partial K^i} = 0$).

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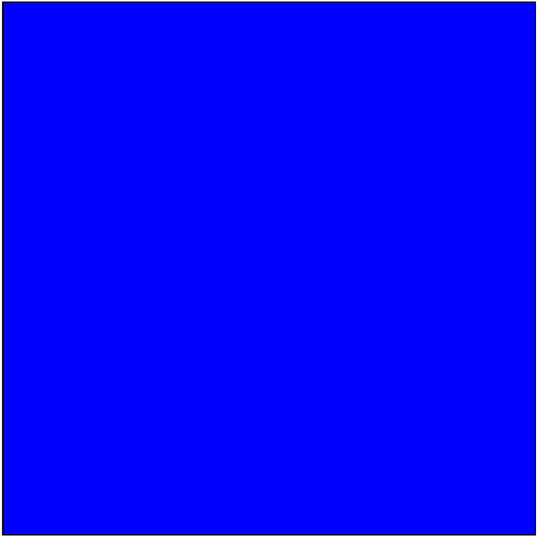

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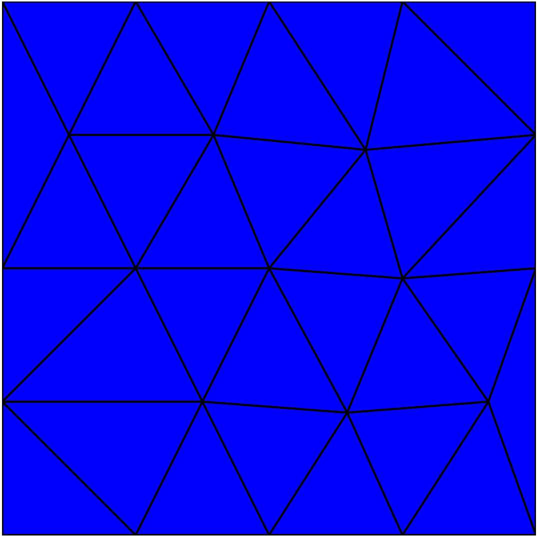

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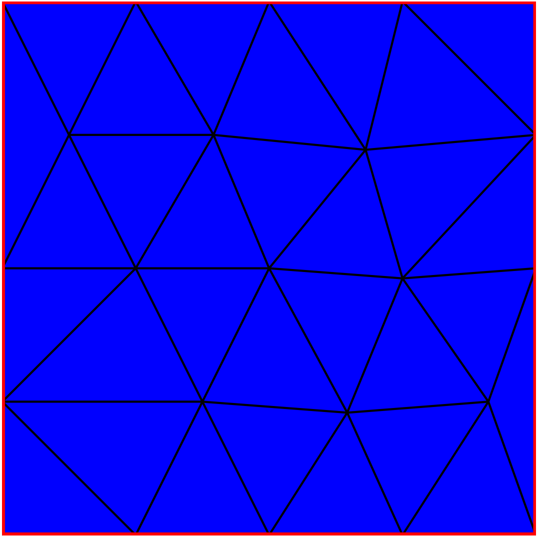

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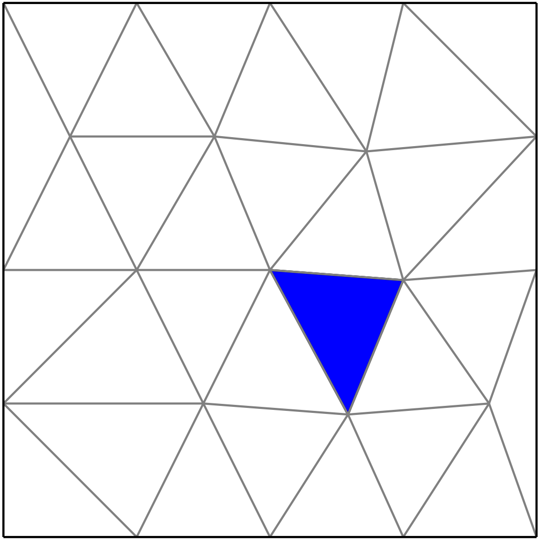

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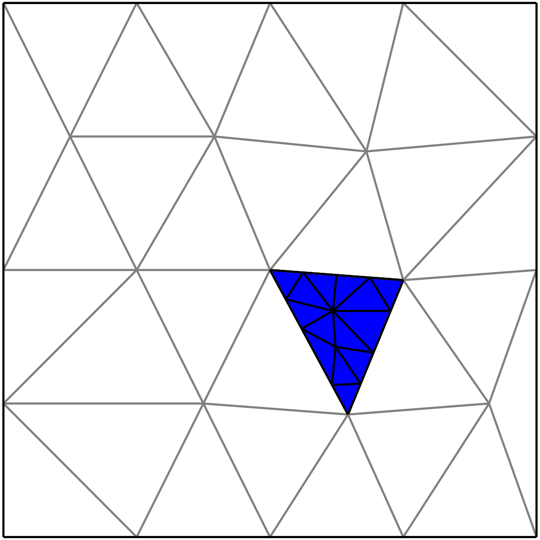

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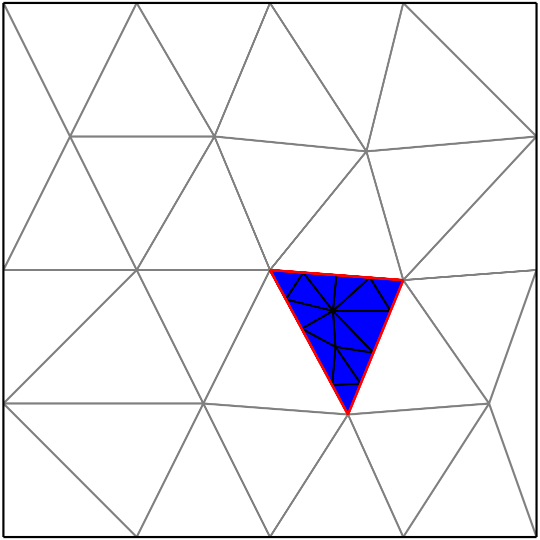

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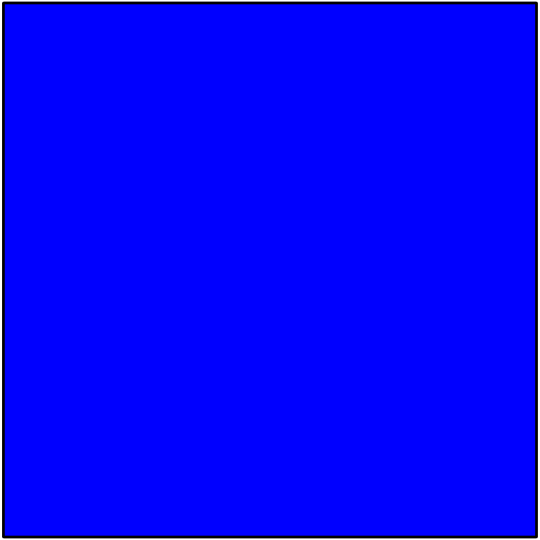

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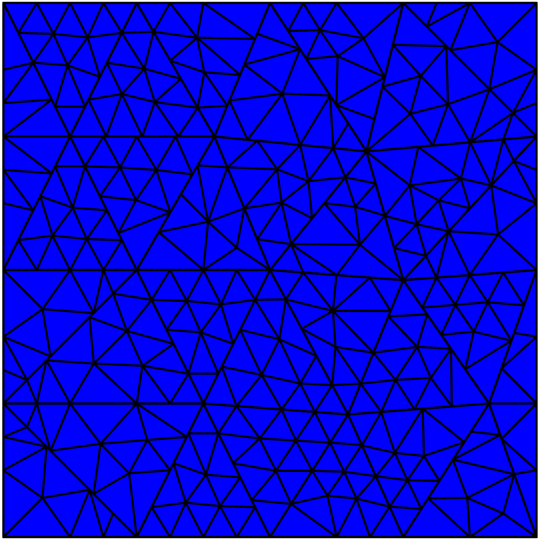

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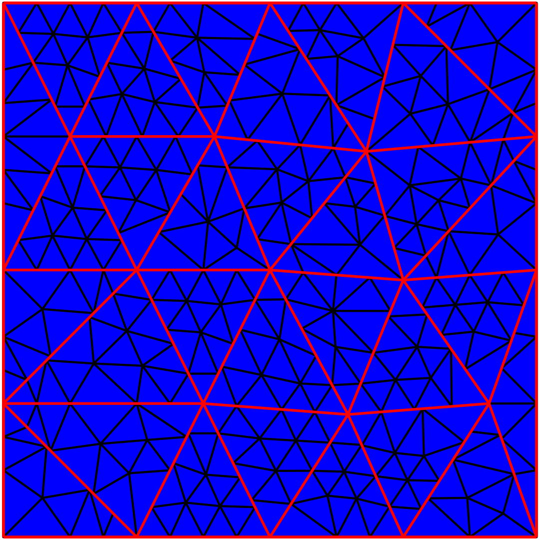

Proposition

Let $\hat{u}^i \in \hat{V}_h^i$ and $\hat{v}^j \in \hat{V}_h^j$, with $i \neq j$. Then $a(\hat{u}^i, \hat{v}^j) = 0$.

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Let $\hat{u}^i \in \hat{V}_h^i$ *and* $\hat{v}^j \in \hat{V}_h^j$, *with* $i \neq j$. *Then* $a(\hat{u}^i, \hat{v}^j) = 0$.

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For all $\bar{u} \in \bar{V}_H$, there is a unique $\hat{u} \in \hat{V}_h$, solution to

$$a(\hat{u}, \hat{v}) = L(\hat{v}) - a(\bar{u}, \hat{v}) \quad \forall \hat{v} \in \hat{V}_h. \tag{4}$$

For $\bar{u} \in V_H$, we note $\hat{U}(\bar{u})$ the associated solution. That way, we define an affine operator $\hat{U} : \bar{V}_H \to \hat{V}_h$.

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Proof.

- Since $\hat{V}_h = \bigoplus_{i=1}^{N_h} \hat{V}_h^i$, we set

$$\hat{u} = \sum_{i=1}^{N_h} \hat{u}^i, \quad \hat{v} \sum_{j=1}^{N_h} \hat{v}^j,$$

with $\hat{u}^i \in \hat{V}_h^i$, $\hat{v}^j \in \hat{V}_h^j$.

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- Therefore, by linearity, $\hat{u} \in \hat{V}_h$ is solution iff

$$a(\hat{u}^i, \hat{v}^i) = L(\hat{v}^i) - a(\bar{u}, \hat{v}^i) \quad \forall \hat{v}^i \in \hat{V}_h^i,$$

for $i \in \{1, \dots, N_h\}$.

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- Remarking that functions of $\hat{V}_h^i$ satisfy a dirichlet condition on ∂K^i, we have $\hat{V}_i^h \subset H_0^1(K^i)$. Therefore, the bilinear form a is coercive on $\hat{V}_i^h$.

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- We also see that the function defined by

$$\tilde{L}_{\bar{u}}^i(\hat{v}^i) = L(\hat{v}^i) - a(\bar{u}, \hat{v}^i) \quad \forall \hat{v}^i \in \hat{V}_h^i,$$

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- Therefore, the Lax-Migram theorem shows that there is a unique $\hat{u}^i \in \hat{V}_h^i$ satisfying

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- Then, $\hat{u} = \sum_{i=1}^{N_h} \hat{u}^i$ is the unique solution of (4).

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- Then, using linearity, we have

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Remark

- In fact, we have shown that $\hat{U}(\bar{u}) - \hat{U}(\bar{v}) = P_{\hat{V}_h}(\bar{u} - \bar{v})$, in the sense of the scalar product $a(.,.)$.

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- Rewriting the second equation as

$$a(\hat{u},\hat{v}) = L(\hat{v}) - a(\bar{u},\hat{v}) \quad \forall \hat{v} \in \hat{V}_h,$$

we see that $\hat{u} = \hat{U}(\bar{u})$.

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What we need to do is

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Remark that since $\hat{U}$ is affine, the coarse equation is linear.

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- $K_{cc} = \{a(\bar{\phi}, \bar{\psi})\},\ \bar{\phi}, \bar{\psi} \in \bar{B}_H$ is the coarse stiffness matrix.

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- $K_{cf} = \{a(\bar{\phi}, \hat{\psi})\}, \; \bar{\phi} \in \bar{B}_H, \hat{\psi} \in \hat{B}_h$ is the cross stiffness matrix.
- $K_{ff}^i = \{a(\hat{\phi}^i, \hat{\psi}^i)\}, \; \hat{\phi}, \hat{\psi} \in \hat{B}_h^i$ are the fine stiffness submatrices.

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- We set $K_{ff}^{i,j} = \{a(\hat{\phi}^i, \hat{\psi}^j)\}$, $\phi^i \in B_h^i, \psi^j \in B_h^j$. Then $K_{ff}^{i,i} = K_{ff}^i$, and $K_{ff}^{i,j} = 0$ if $i \neq j$. So

$$K_{ff} = \begin{pmatrix} K_{ff}^{1,1} & \dots & K_{ff}^{1,N_h} \\ \vdots & \ddots & \vdots \\ K_{ff}^{N_h,1} & \dots & K_{ff}^{N_h,N_h} \end{pmatrix} = \begin{pmatrix} K_{ff}^1 & & 0 \\ & \ddots & \\ 0 & & K_{ff}^{N_h} \end{pmatrix}.$$

$\square$

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- We are to solve $K_{ups}U_{ups} = F_{ups}$. Using decomposition, we get

$$\begin{pmatrix} K_{cc} & K_{cf} \\ K_{cf}^T & K_{ff} \end{pmatrix} \begin{pmatrix} U_c \\ U_f \end{pmatrix} = \begin{pmatrix} F_c \\ F_f \end{pmatrix}.$$

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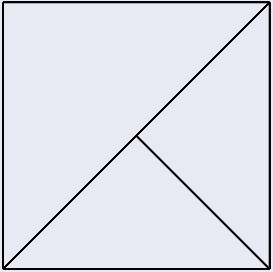

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$$\mathcal{F}_h^{int} = \{\partial K \cap \partial J \mid K, J \in \mathcal{T}_h\},\ \mathcal{F}_h^{ext} = \{\partial K \cap \partial \Omega \mid K \in \mathcal{T}_h\}.$$

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- Let $u \in \mathbb{D}_1(\mathcal{T}_h)$ and $v \in \mathbb{D}_1(\mathcal{T}_h)^2$.
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- Note that, since $V_{ups} \subset \mathbb{D}_1(\mathcal{T}_h)$, $a_h(u,v)$ is well defined for $u, v \in V_{ups}$.

Proposition

Let $\hat{u}^i \in \hat{V}_h^i$ *and* $\hat{v}^j \in \hat{V}_h^j$ *with* $i \neq j$*. Then* $a_h(\hat{u}^i, \hat{v}^j) = 0$*.*

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- We also see that

$$I_h(\hat{u}^i, \hat{v}^j) = \int_e [[\hat{u}^i]]\{\{\nabla \hat{v}^j\}\}, \quad I_h(\hat{v}^j, \hat{u}^i) = \int_e [[\hat{v}^j]]\{\{\nabla \hat{u}^i\}\},$$

and

$$J_h^\sigma(\hat{u}^i, \hat{v}^j) = \int_e \sigma[[\hat{u}^i]][[\hat{v}^j]].$$

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Proposition

For all $\bar{u} \in \bar{V}_H$, there is a unique $\hat{u} \in \hat{V}_h$ satisfying

$$a_h(\hat{u}, \hat{v}) = L(\hat{v}) - a_h(\bar{u}, \hat{v}) \quad \forall v \in \hat{V}_h.$$

We define an affine operator $\hat{U} : \bar{V}_h \to \hat{V}_h$ by $\hat{U}(\bar{u}) = \hat{u}$.

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- Therefore $\hat{U}$ is an affine application.

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- The inversion of the classical stiffness matrix requires $\mathcal{O}(\alpha^3 N^6 M^6)$ operations.
- The upscaling algorithm is N^4 times less expensive.

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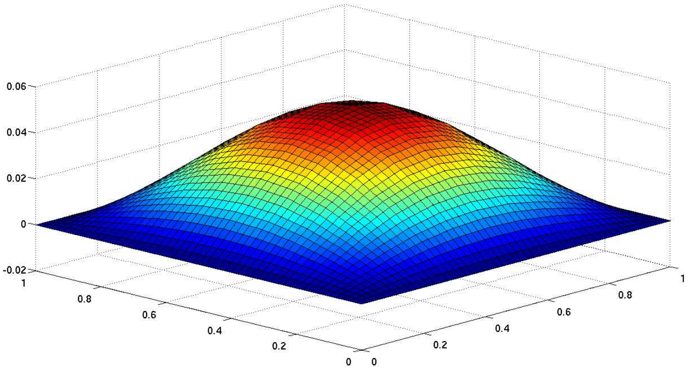
- Each coarse cell is subdivide with a 5×5 fine cartesian grid.

Coarse componant $\bar{u} \in \bar{V}_H$



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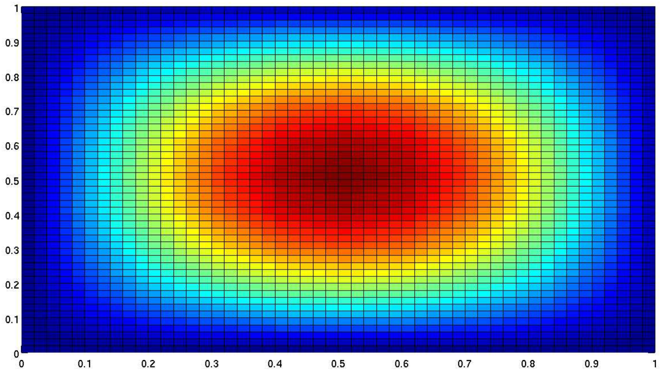

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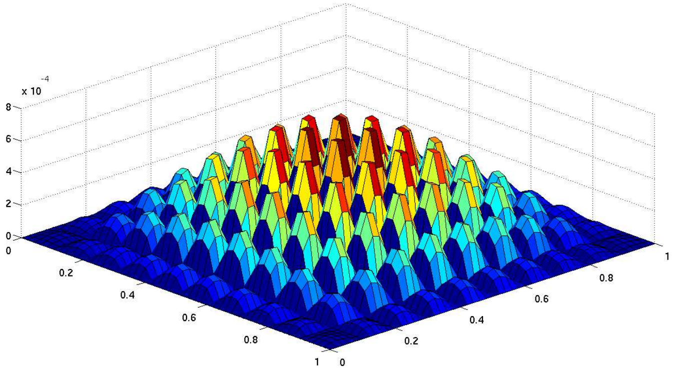

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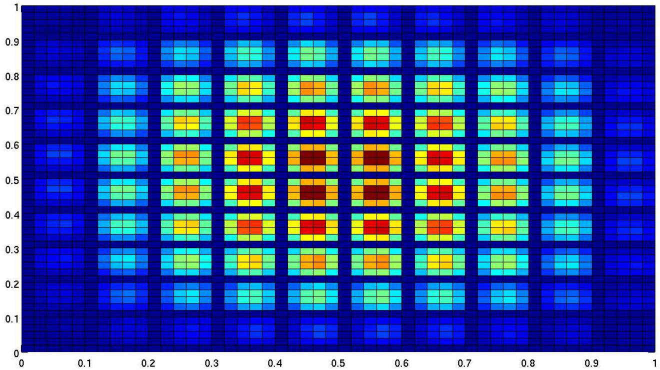
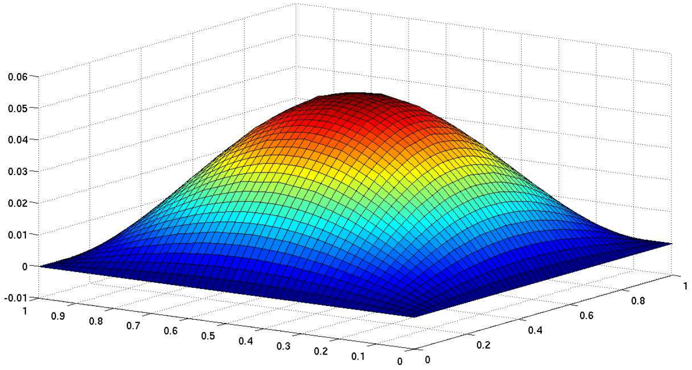


Upscaled solution $u \in V_{ups}$



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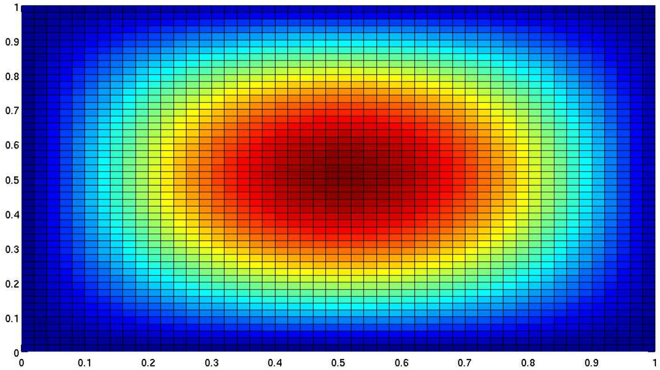
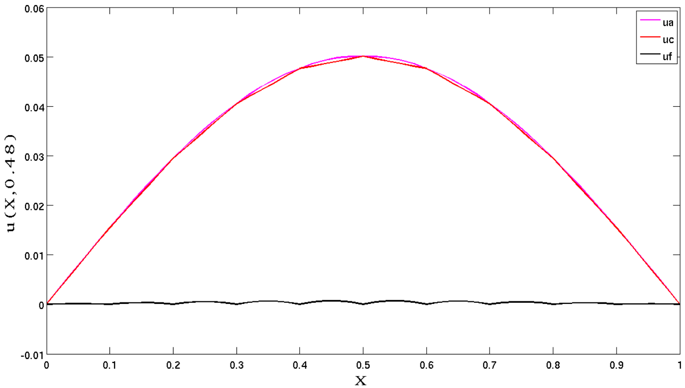


Slide at $Y = 0.48$



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Questions

- Do you have any question?