

Multiscale Hybrid-Mixed Method

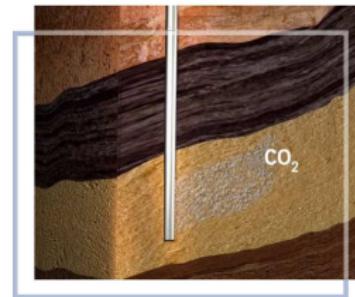
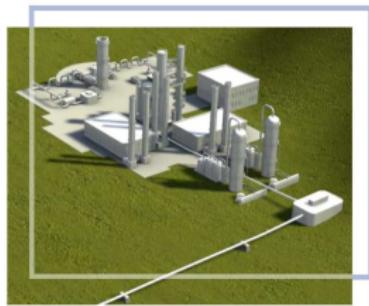
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INRIA Meeting, Sophia-Antipolis
July 25-27, 2012

¹Joint work with R. Araya, C. Harder, D. Paredes and A. Madureira

Sequestration of CO₂



$$\begin{aligned} -\nabla \cdot (\mathcal{A} \mathcal{E}(\boldsymbol{u}_s)) &= \alpha \nabla p \\ \boldsymbol{u}_f + \mathcal{K}(S) \nabla p &= \mathbf{g}(S) \\ \nabla \cdot \boldsymbol{u}_f + \partial_t (\phi p + \alpha \nabla \cdot \boldsymbol{u}_s) &= f \\ \partial_t \phi S - \epsilon(S) \Delta S + \beta(S, \boldsymbol{u}_f) \cdot \nabla S &= g(S) \end{aligned}$$

Darcy / Elasticity Model

Elliptic Models

Find u such that

$$-\nabla \cdot D u = f \quad \text{in } \Omega, \quad D u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

Laplace: Find the pressure u where

$$D u := \mathcal{K} \nabla u$$

Elasticity: Find the displacement \mathbf{u} where

$$D u := \mathcal{K} \mathcal{E}(\mathbf{u})$$

Here \mathcal{K} is a positive definite tensor with *multi-scales* features

Mixed Finite Element: u and $D u$

Stability + Precision + Conservation

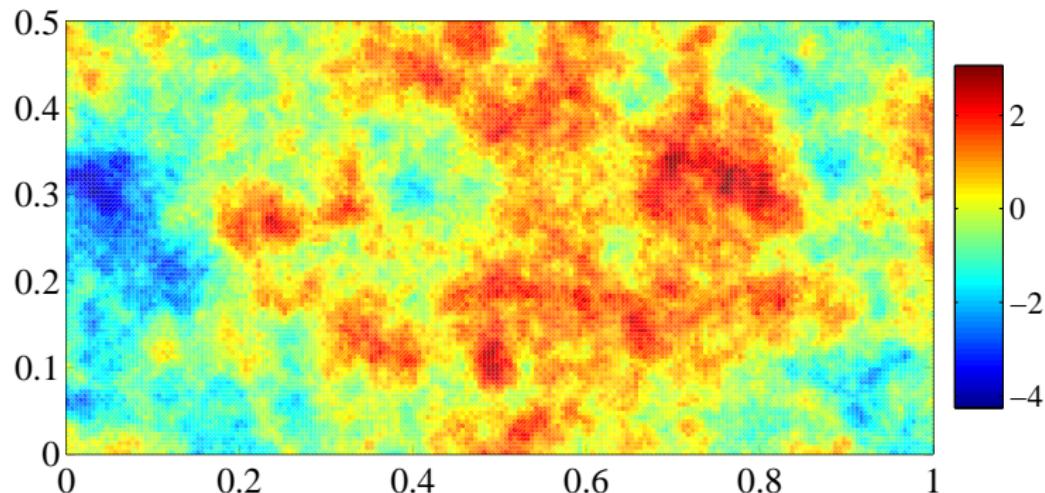
Laplace (Darcy): RT_k/\mathbb{P}_k or BDM_k/\mathbb{P}_{k-1} elements

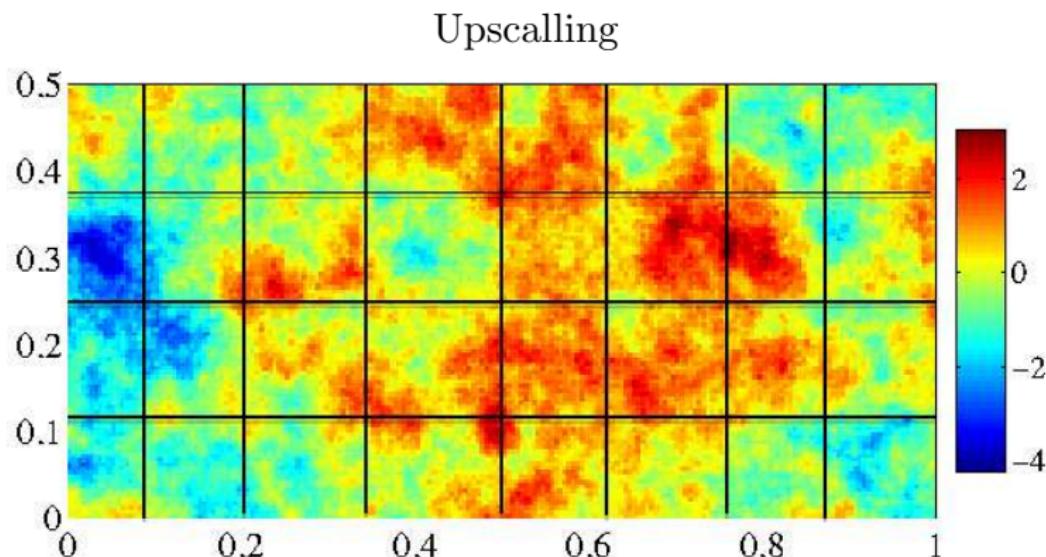
Elasticity: Relax conformity or the symmetry of $D u$

Arnold et al. (2008): 162 d.o.f. in 3D

Basis functions are “*ignorant*” of *multi-scales* structures

A typical permeability field (Courtesy M. Borges)





A (incomplete) State-of-the-Art

Elasticity Equation
Multi-Scale + Stable

?? ??

Darcy Equation
Multi-Scale + Stable

RT_0/\mathbb{P}_0 Hou et al. (2002)
 $\text{BDM}_1/\mathbb{P}_0$ Arbogast (2004)

Multiscale Hybrid-Mixed Method

└ MHM Method

The MHM Formulation

Hybrid Formulation (Coarse Mesh \mathcal{T}_h)

Find u such that

$$-\nabla \cdot D u = f \quad \text{in } \Omega, \quad D u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

Find $(u, \lambda) \in V \times \Lambda$ s.t. for all $(v, \mu) \in V \times \Lambda$

$$\begin{aligned} (\mathcal{K}^{-1} D u, D v)_{\mathcal{T}_h} + (\lambda \mathbf{n}, [v])_{\mathcal{E}_h} &= (f, v)_{\mathcal{T}_h} \\ ([u], \mu \mathbf{n})_{\mathcal{E}_h} &= 0 \end{aligned}$$

Observe that u satisfies the (ill-posed) local problem

$$(\mathcal{K}^{-1} D u, D v)_K = (f, v)_K - (\lambda, v)_{\partial K}$$

Space Decomposition

Split the space

$$V = V_0 \oplus V_0^\perp$$

such that

$$\bar{v} \in V_0 \quad \text{s.t.} \quad (\mathcal{K}^{-1} D \bar{v}, D v)_K = 0$$

Laplace problem: V_0 is the piecewise **constant** space

Elasticity problem: V_0 is the space of **rigid body modes**

Function $v \in V$ decomposes

$$v = \bar{v} + \tilde{v}, \quad \bar{v} \in V_0 \quad \text{and} \quad \tilde{v} \in V_0^\perp$$

Weak Formulation - Hybrid

Test function $(\tilde{v}, 0)$: Find $\tilde{u} \in V_0^\perp$ such that

$$(\mathcal{K}^{-1} D \tilde{u}, D \tilde{v})_K = (f, \tilde{v})_K - (\lambda, \tilde{v})_{\partial K} \quad \forall \tilde{v} \in V_0^\perp$$

Split

$$\tilde{u} = u^\lambda + u^f$$

Test function (\bar{v}, μ) : Find $(\bar{u}, \lambda) \in V_0 \times \Lambda$ such that

$$(\mathcal{K}^{-1} D u, D \bar{v})_{\mathcal{T}_h} + (\lambda \mathbf{n}, [\bar{v}])_{\mathcal{E}_h} = (f, \bar{v})_{\mathcal{T}_h} \quad \forall \bar{v} \in V_0$$

$$([\bar{u} + u^\lambda], \mu \mathbf{n})_{\mathcal{E}_h} = - ([u^f], \mu \mathbf{n})_{\mathcal{E}_h} \quad \forall \mu \in \Lambda$$

Alternative Weak Formulation - Mixed

Given $\mu \in \Lambda$, define $u^\mu \in V_0^\perp$ such that

$$(\mathcal{K}^{-1} D u^\mu, D \tilde{v})_K = -(\mu, \tilde{v})_{\partial K} \quad \forall \tilde{v} \in V_0^\perp$$

Find $(\bar{u}, \lambda) \in V_0 \times \Lambda$ such that

$$\left(\nabla \cdot D u^\lambda, \bar{v} \right)_{T_h} = - (f, \bar{v})_{T_h} \quad \forall \bar{v} \in V_0$$

$$\left(\mathcal{K}^{-1} D u^\lambda, D u^\mu \right)_{T_h} + (\bar{u}, \nabla \cdot D u^\mu)_{T_h} = - (f, u^\mu)_{T_h} \quad \forall \mu \in \Lambda$$

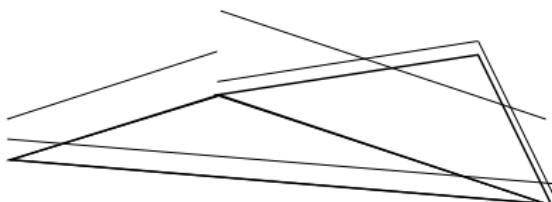
Multiscale Hybrid-Mixed Method

└ MHM Method

The MHM method

Finite Data Set

Space Λ_0



- ▶ Thus far, no discretization introduced
- ▶ The **only missed choice** is $\Lambda_l \subset \Lambda$
- ▶ Functions denoted $\lambda_l \in \Lambda_l$

Find u^{λ_l} such that

$$(\mathcal{K}^{-1} D u^{\lambda_l}, D \tilde{v})_K = -(\lambda_l, \tilde{v})_{\partial K}$$

Finite Element

Let $\lambda_l \in \Lambda_l$ be expanded with basis functions

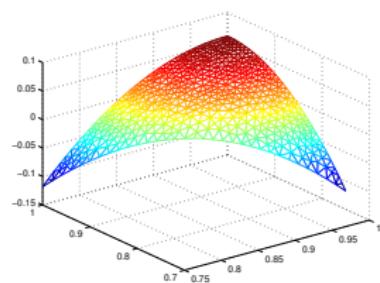
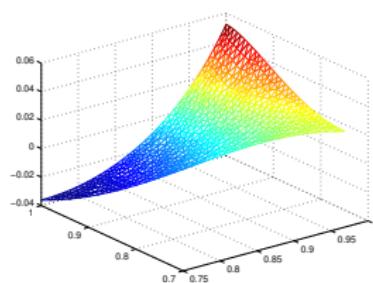
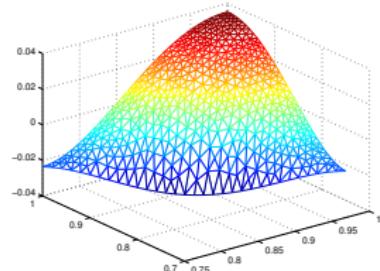
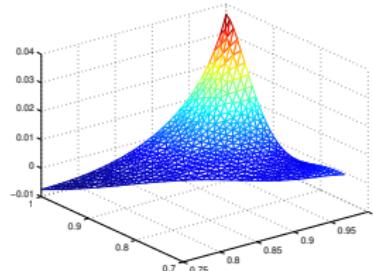
$$\lambda_l = \sum_{i=1}^{\dim \Lambda_l} \lambda_l(\mathbf{x}_i) \psi_i^l, \quad \psi_i^l \in \mathbb{P}_l(F)$$

Suppose $K \in \mathcal{T}_h$. Define $L = \{\lambda_l(\mathbf{x}_i)\}_{i=1}^{\dim(\Lambda_l|_{\partial K})}$ and

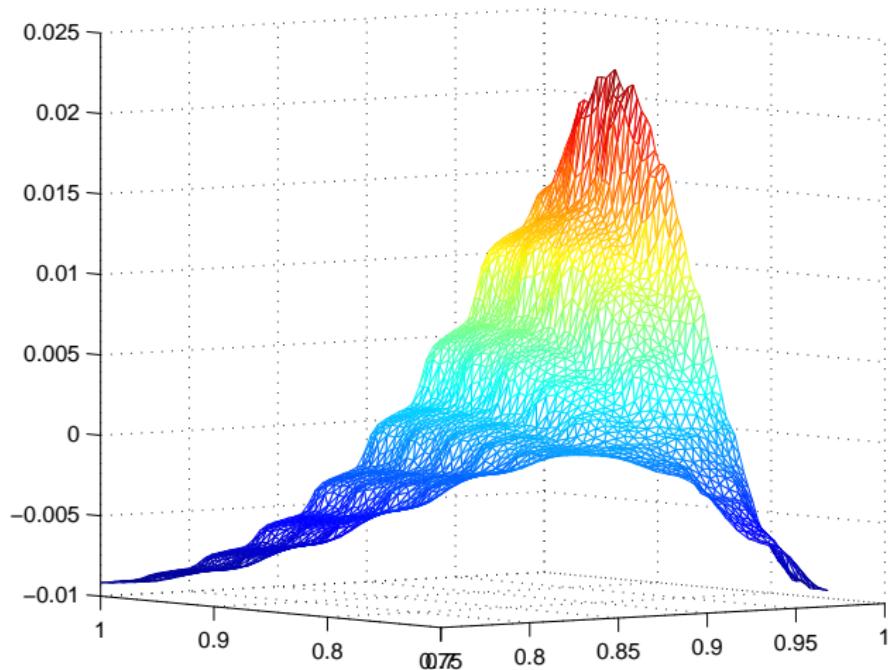
$\mathcal{N} = \{\eta_i\}_{i=1}^{\dim(\lambda_l|_{\partial K})}$ where η_i satisfy

$$(\mathcal{K}^{-1} D \eta_i, D \tilde{v})_K = -(\psi_i^l, \tilde{v})_{\partial K}$$

The triplet $\{K, \mathcal{N}, L\}$ is a *finite element*

Space Λ_0 Space Λ_{-1} Space Λ_{-2} Space Λ_{-3} 

An Oscillatory Basis Function for Λ_{-3}



The MHM Method

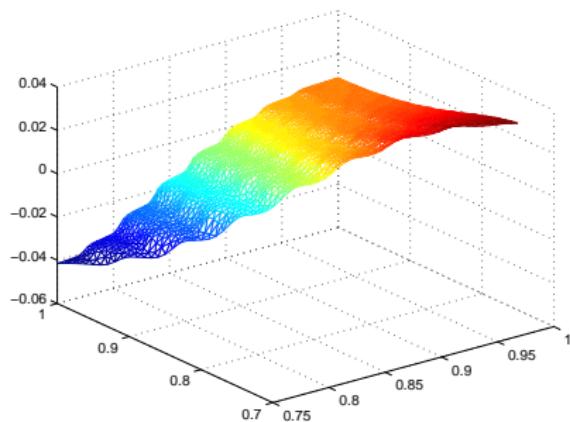
Find $(u_0, \lambda_l) \in V_0 \times \Lambda_l$ such that

$$\left(\nabla \cdot D u^{\lambda_l}, v_0 \right)_{\mathcal{T}_h} = - (f, v_0)_{\mathcal{T}_h} \quad \forall v_0 \in V_0$$

$$\left(\mathcal{K}^{-1} D u^{\lambda_l}, D u^{\mu_l} \right)_{\mathcal{T}_h} + (u_0, \nabla \cdot D u^{\mu_l})_{\mathcal{T}_h} = - (f, u^{\mu_l})_{\mathcal{T}_h} \quad \forall \mu_l \in \Lambda_l$$

u^{λ_l} solves

$$(\mathcal{K}^{-1} D u^{\lambda_l}, D \tilde{v})_K = - (\lambda_l, \tilde{v})_{\partial K}$$



Relationship with Other Methods: The Darcy Model

$$l = 0, f \in \mathbb{R}$$

$$l = 0, 1$$

$$\mathcal{K} \equiv I$$

$$\text{Multi-Scale } \mathcal{K}$$

$$\text{RT}_0/\mathbb{P}_0$$

$$\text{Multi-Scale } \mathcal{K}$$

$$\text{Hou et al. (2002)}$$

$$\text{Multi-Scale } \mathcal{K}$$

$$\text{Arbogast (2004)}$$

Considerations on the Method

$$u_h := u_0 + u^{\lambda_l} + u^f \quad \text{and} \quad \boldsymbol{\sigma}_h := D u_h \in H(\text{div}, \Omega)$$

- ▶ $V_0 \times \Lambda_l$ is **inf-sup stable**
- ▶ Convergence driven **only** by $\Lambda_l \approx \Lambda$
- ▶ Locally conservative

$$\int_K \nabla \cdot \boldsymbol{\sigma}_h = \int_K f$$

Primal

$$\|u - u_0\|_{0,\Omega} = O(h)$$

$$\|\bar{u} - u_0\|_{0,\Omega} = O(h^{l+2})$$

Updated

$$\|u - u_h\|_{0,\Omega} = O(h^{l+2})$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div},\Omega} = O(h^{l+1})$$

An (Face) A Posteriori Error Estimator

$$R_F := \begin{cases} -\frac{1}{2}[u_h], & F \in \text{internal edges} \\ -u_h \mathbf{n}, & F \in \text{Dirichlet b.c.} \\ \mathbf{0}, & F \in \text{Neumann b.c.} \end{cases}$$

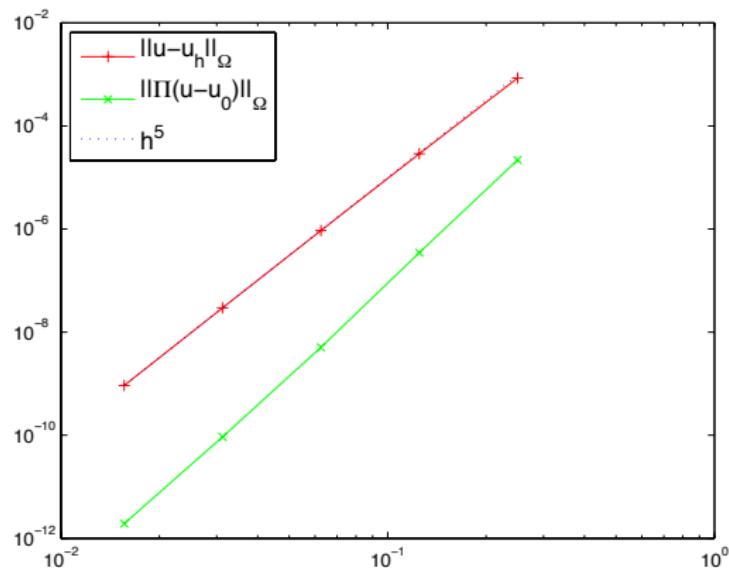
$$\eta := \left[\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \frac{C}{h_F} \|R_F\|_{0,F}^2 \right]^{1/2}$$

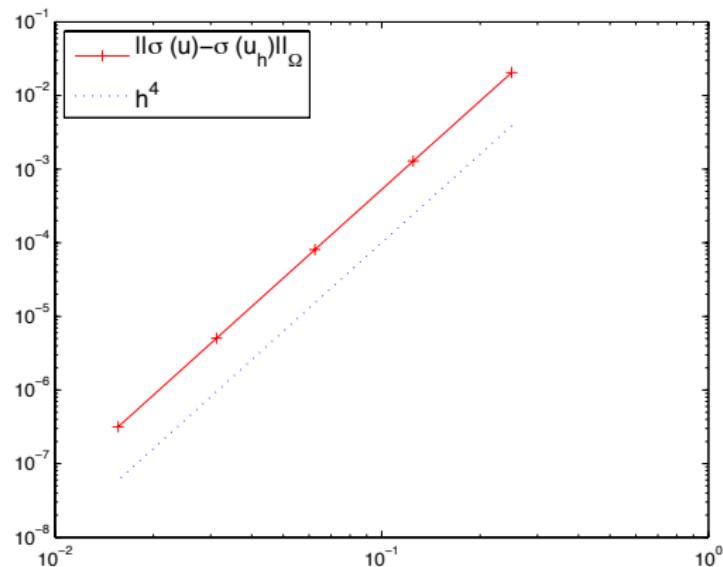
Both **reliability** and **efficiency** hold

Multiscale Hybrid-Mixed Method

└ MHM Method

Numerical Validation Darcy Equation

Element $\mathbb{P}_0(K)/\mathbb{P}_{-3}(F)$ Updated Pressure: Convergence h^5 

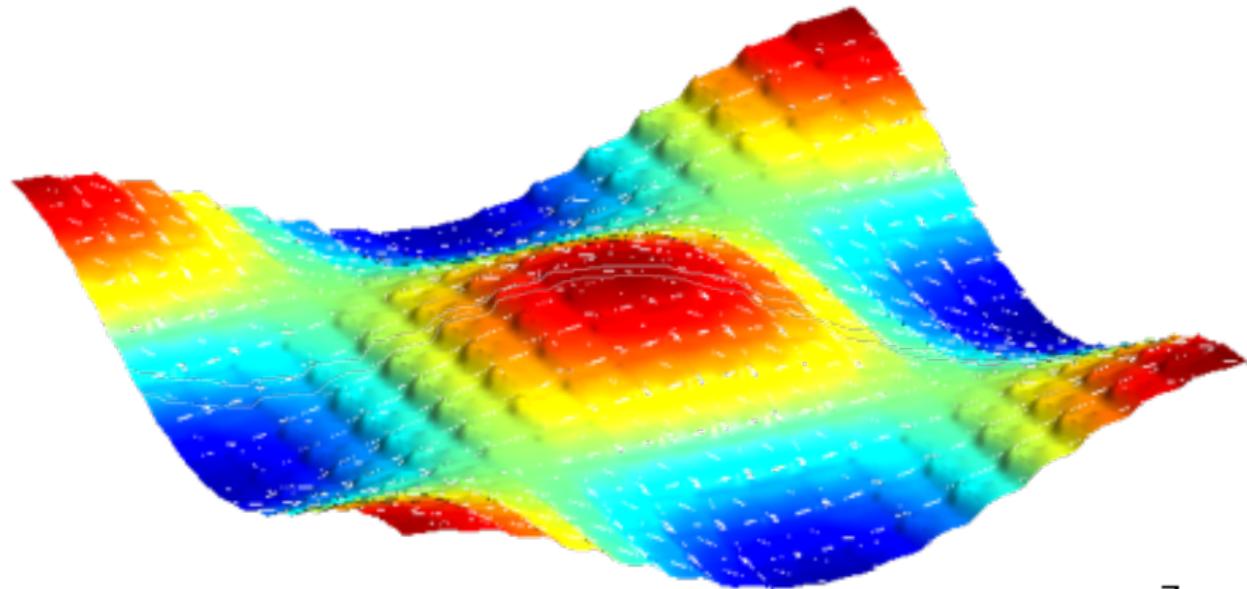
Element $\mathbb{P}_0(K)/\mathbb{P}_{-3}(F)$ Velocity: Convergence h^4 

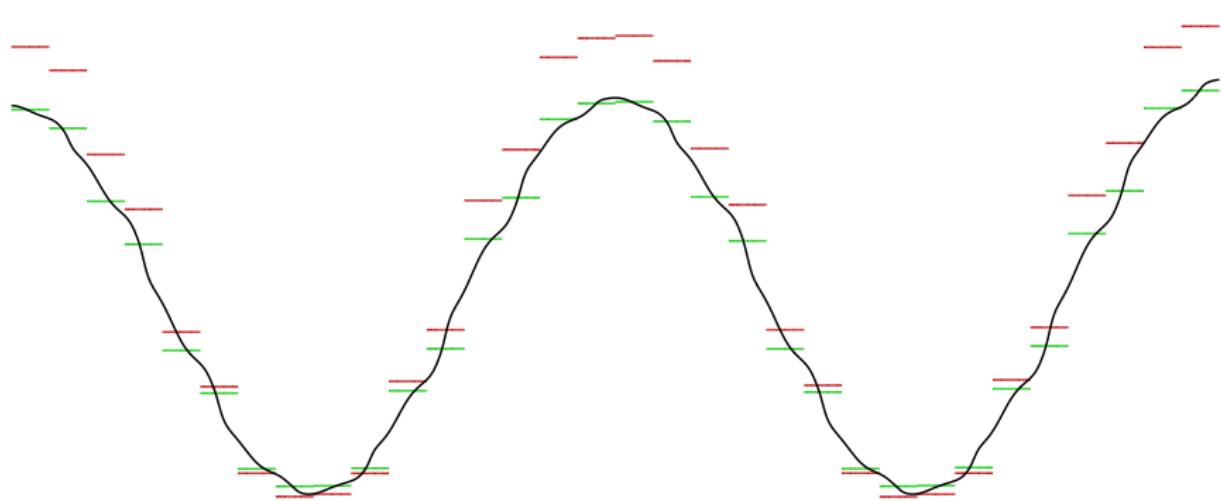
The Highly-Oscillatory Problem ($\varepsilon = \frac{1}{16}$)

$$\mathcal{K}(\boldsymbol{x}) = \frac{2 + \frac{1.8 \sin 2\pi x}{\varepsilon}}{2 + \frac{1.8 \sin 2\pi y}{\varepsilon}} + \frac{2 + \frac{1.8 \sin 2\pi y}{\varepsilon}}{2 + \frac{1.8 \cos 2\pi x}{\varepsilon}},$$

Element $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$ (16×16 Elements)

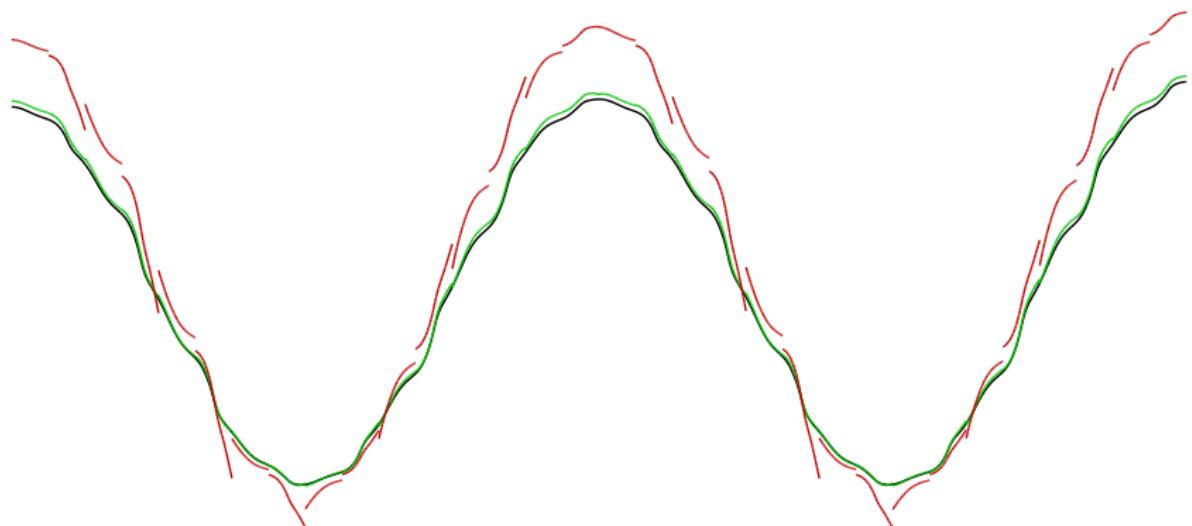
$$u_0 + u^{\lambda_l} + u^f$$



Elements $\mathbb{P}_0(K)/\mathbb{P}_0(F)$ and $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$ u_0 

Elements $\mathbb{P}_0(K)/\mathbb{P}_0(F)$ and $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$

$$u_0 + u^{\lambda_l} + u^f$$



A Quarter Five-Spot Problem

- ▶ Two wells, in opposing corners.
- ▶ Permeability set to 1.
- ▶ Homogeneous Neumann boundary conditions

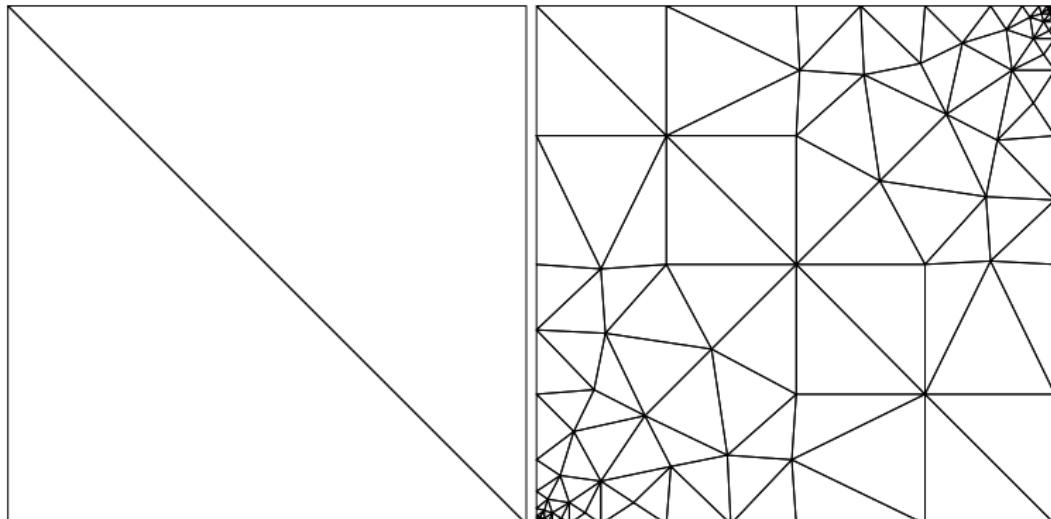
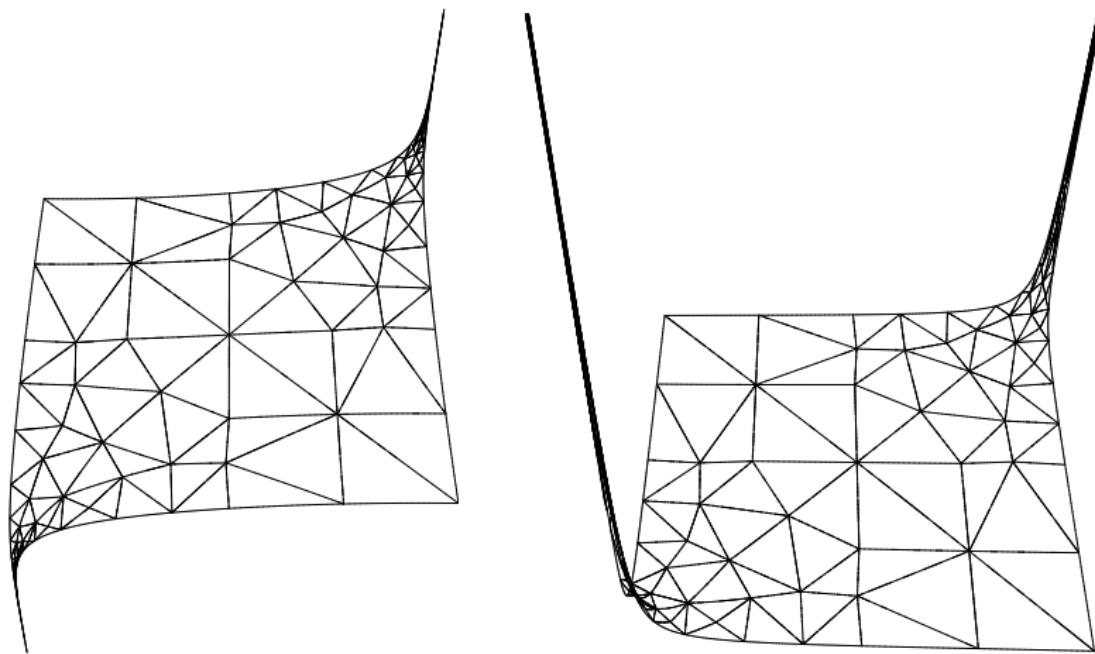
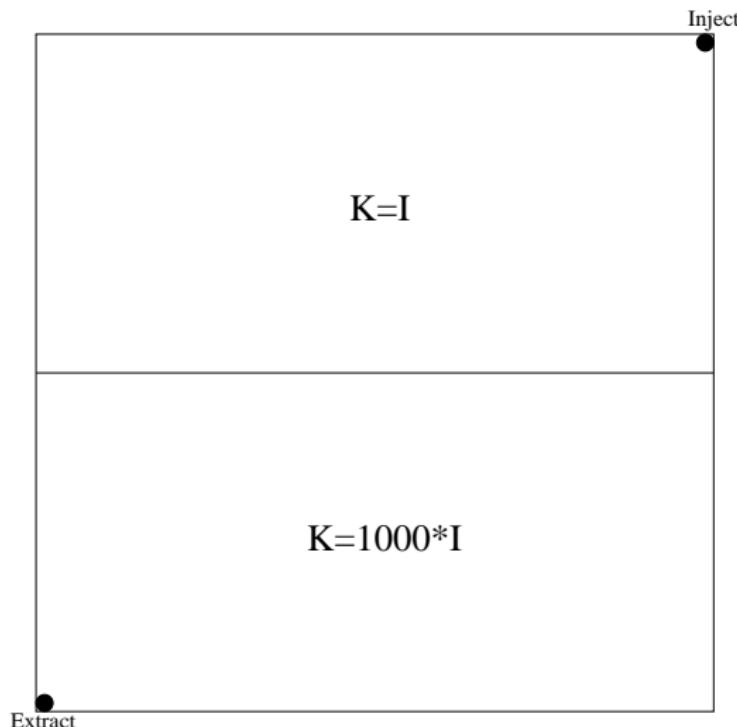


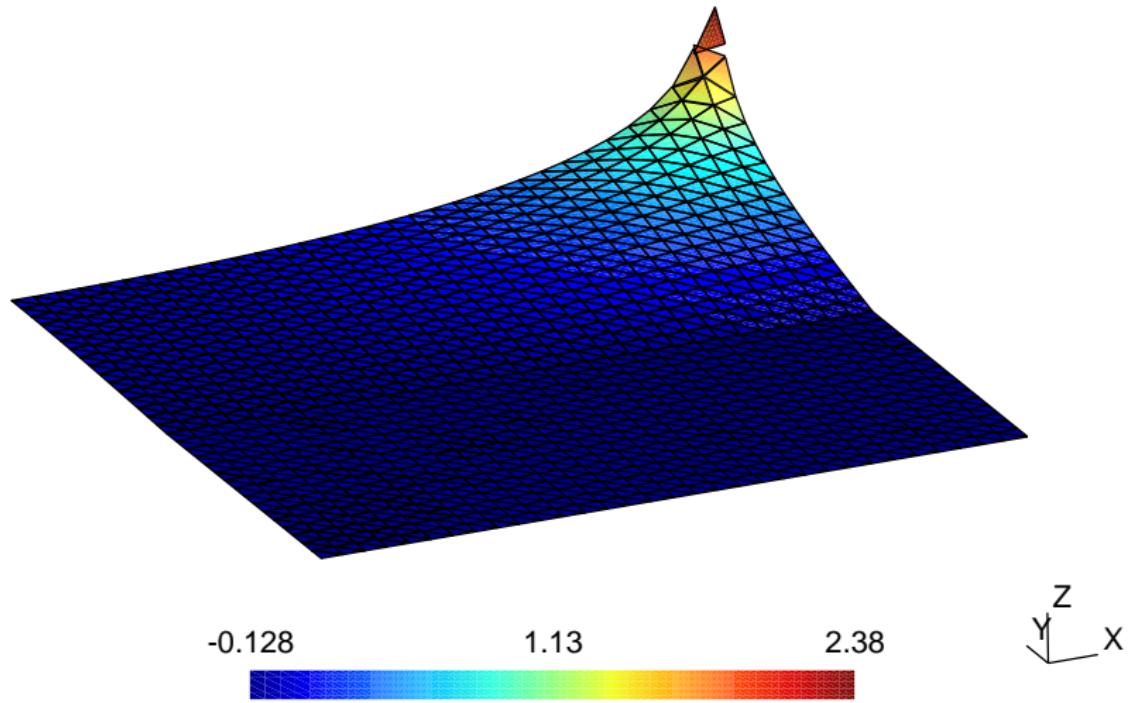
Figure: u_h and $|\sigma_h|$



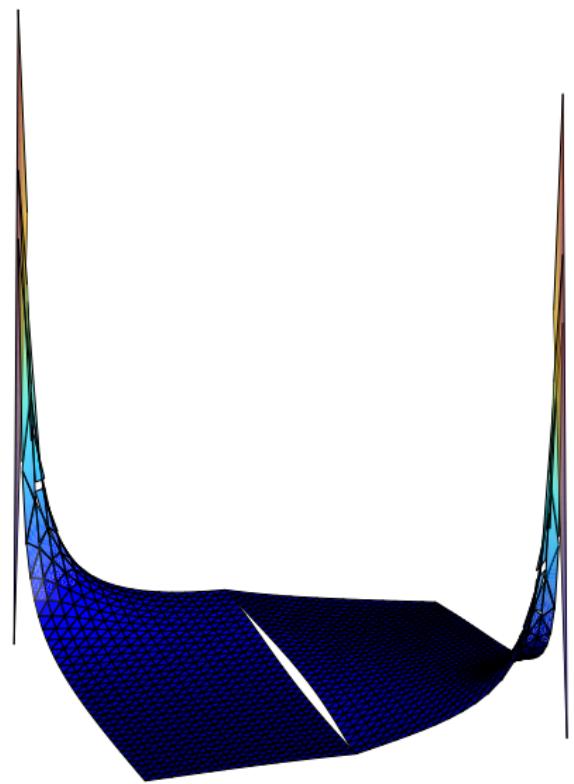
A High-Contrast Permeability Problem



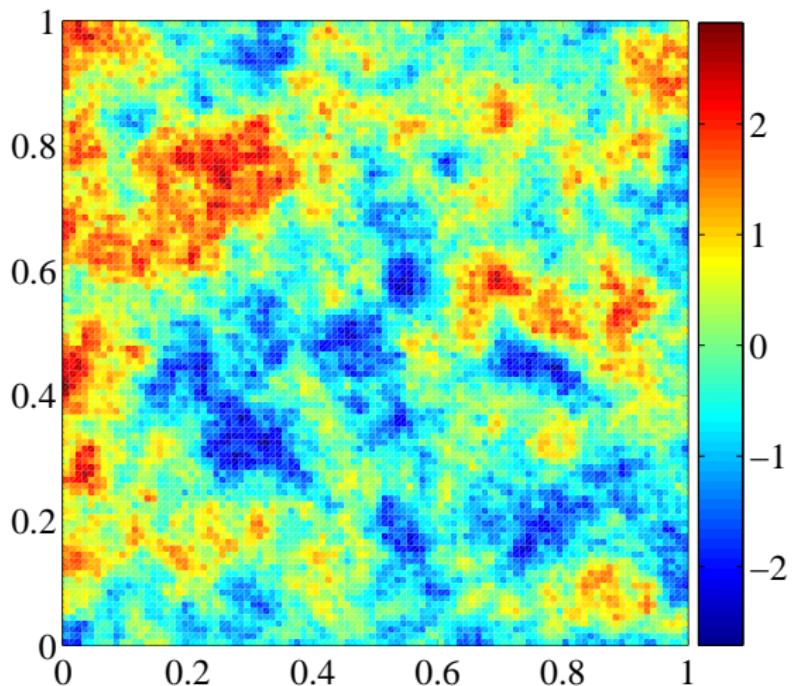
Element $\mathbb{P}_0(K)/\mathbb{P}_{-1}(F)$: u_h



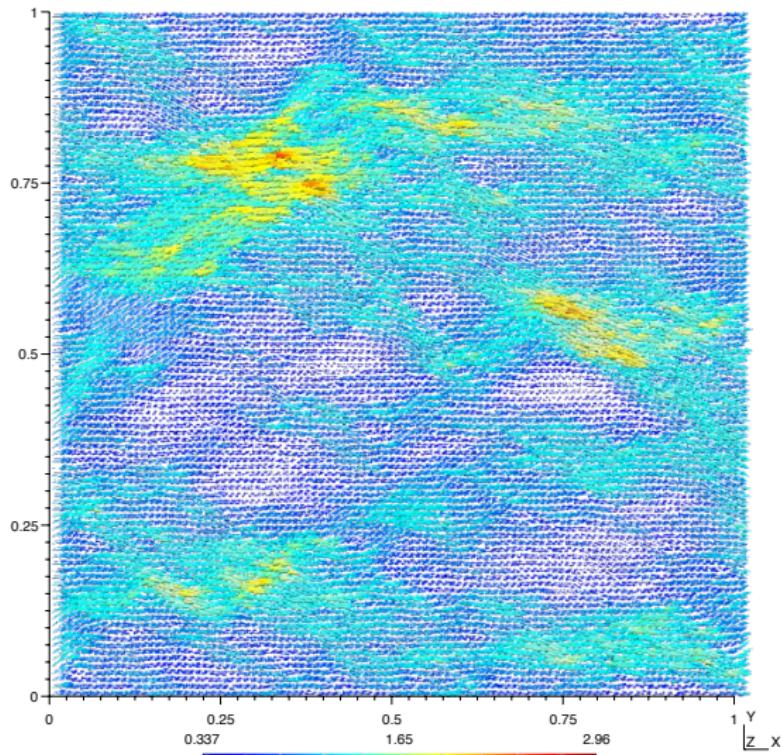
Element $\mathbb{P}_0(K)/\mathbb{P}_{-1}(F)$: $|\boldsymbol{\sigma}_h|$



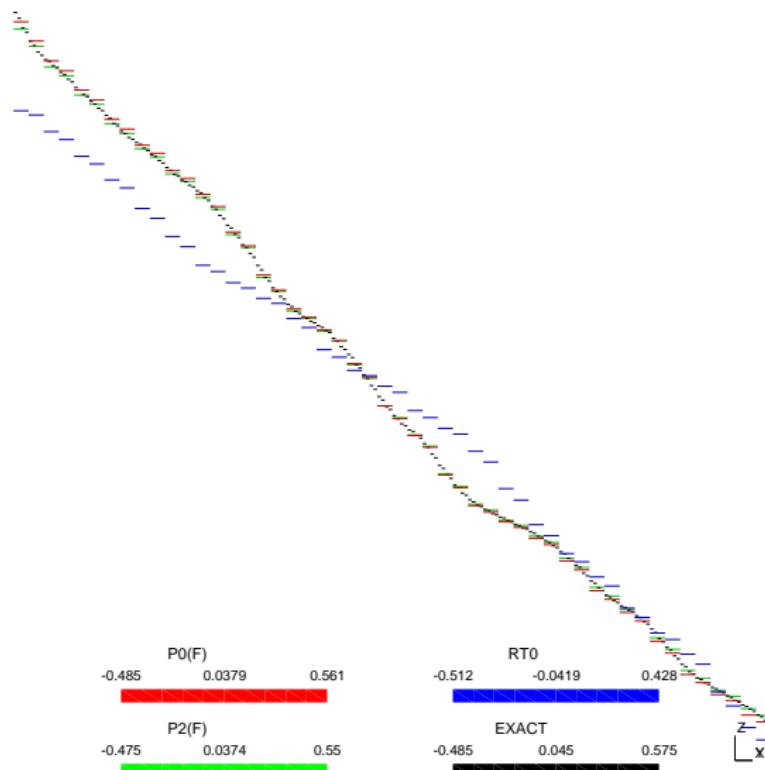
A Heterogenous Permeability Problem



Elements $\mathbb{P}_0(K)/\mathbb{P}_{-2}(F)$ (25×25 elements)

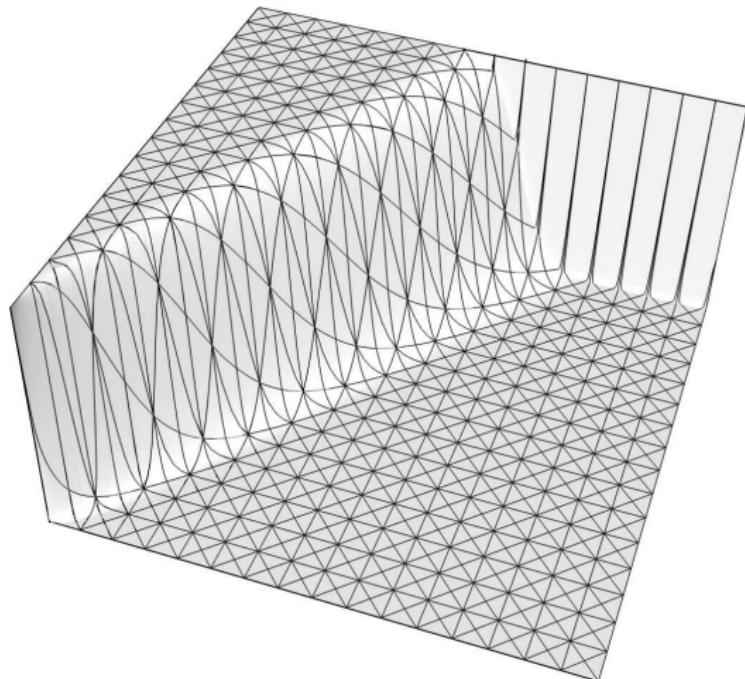


RT_0 Element Fails: Profiles of u_0



Upcoming: Advective Dominate Model

The MHM method avoids under and over shootings



Conclusion

- ▶ New stable elements for the Darcy and elasticity models
- ▶ Include local upscaling
- ▶ Highly adapted to parallel computation
- ▶ Elasticity: Conformity + Strong Symmetry + Equilibrium
- ▶ Elasticity: Simplest element = 12 (2D) or 30 (3D) d.o.f.

Perspectives: First Ideas for Collaboration

- ▶ The non-linear system: Linearization algorithm + Further complexities (fractures/faults ?)
- ▶ Paralelization: CPU or GPU (or hybrid) ?
- ▶ MHM method: Electromagnetism in heterogenous media