A discontinuous Galerkin method for viscoelastic wave propagation

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Motivation

- Simulation of seismic wave propagation at regional scale, especially site effects
- We solve the direct problem with a discontinuous Galerkin (DG) method with the possibilities of using:
  - meshes adapted to complex geometries
  - coarser meshes thanks to high order polynomial interpolation
- Objective is to take into account realistic physical phenomena – like attenuation

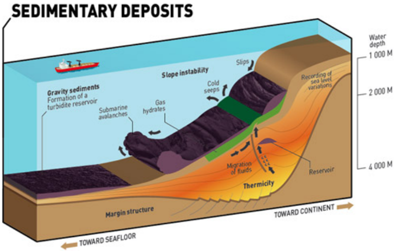


Table of contents

Outline

1. Modeling wave attenuation
2. A DG scheme for viscoelastic attenuation
3. Numerical results
4. Conclusion

- Attenuation: decrease in amplitude of the seismic wave
- Several processes are involved
- One is not described within the theory of elasticity: *intrinsic attenuation*
- Implies conversion of energy into heat by permanent deformation of the medium

Modeling wave attenuation

1D Boltzmann's principle

- Stress at time t depends of the entire strain history until t:

$$\sigma = \int_{-\infty}^{t} \phi(t-\tau)\, \partial_\tau \epsilon(\tau)\, d\tau = \partial_t \phi * \epsilon$$

  where ϕ is the relaxation function

- Convolution: nearly intractable in a numerical computation

- Successful approach is to consider frequency-domain relations using rheological models [Day and Minster, Geophys. J. R. astr. Soc. (1984)]

Modeling wave attenuation

Frequency-domain relations

- The convolution is replaced by a multiplication using a Fourier Transform:

$$\sigma = \partial_t \phi * \epsilon \xrightarrow{\mathcal{F}} \widehat{\sigma}(\omega) = M(\omega)\,\widehat{\epsilon}(\omega)$$

where $\quad M = \mathcal{F}\left[\partial_t \phi\right] = i\omega \mathcal{F}\left[\phi\right] \quad$ is the complex modulus

- Now, if M is a rational fraction of $i\omega$:

$$M(\omega) = \frac{\sum_{l=1}^{m} p_l (i\omega)^l}{\sum_{l=1}^{n} q_l (i\omega)^l}$$

then

$$\sum_{l=1}^{m} q_l \frac{d^l}{dt^l}\sigma(t) = \sum_{l=1}^{n} p_l \frac{d^l}{dt^l}\epsilon(t),$$

and the convolution has been replaced by differential equations

- Linear rheological models are made of springs and dashpots connected in series and/or parallel
- Time and frequency-domain rules (k elastic modulus, η viscosity):

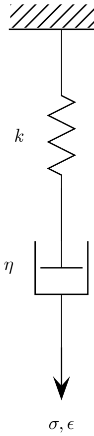
Element	Hooke (spring)	Stokes (dashpot)
Time relation	$\sigma(t) = k\,\epsilon(t)$	$\sigma(t) = \eta\,\partial_t \epsilon(t)$
Frequency relation	$\widehat{\sigma}(\omega) = k\,\widehat{\epsilon}(\omega)$	$\widehat{\sigma}(\omega) = i\omega\eta\,\widehat{\epsilon}(\omega)$

Connection	In series	In parallel
σ	equal	additive
ϵ	additive	equal

- Spring: $\widehat{\sigma}_{HB}(\omega) = k\,\widehat{\epsilon}_{HB}(\omega)$
- Dashpot: $\widehat{\sigma}_{SB}(\omega) = i\omega\eta\,\widehat{\epsilon}_{SB}(\omega)$
- In series: $\widehat{\sigma} = \widehat{\sigma}_{HB} = \widehat{\sigma}_{SB}$ and $\widehat{\epsilon} = \widehat{\epsilon}_{HB} + \widehat{\epsilon}_{SB}$
- Finally: $\widehat{\sigma} = M\,\widehat{\epsilon} = \left(\dfrac{k\,i\omega}{\omega_0 + i\omega}\right)\widehat{\epsilon}$

where $\omega_0 = \dfrac{k}{\eta}$ is the relaxation frequency

Maxwell body



Modeling wave attenuation

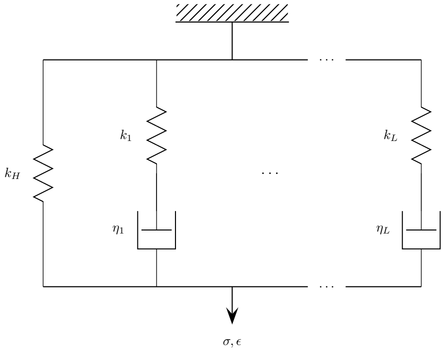
Generalized Maxwell Body

- One spring and L Maxwell Bodies in parallel
- $$\widehat{\sigma} = \left(k_H + \sum_{l=1}^{L} \frac{k_l \, i\omega}{\omega_l + i\omega} \right) \widehat{\epsilon}$$

$\omega_l = k_l / \eta_l$ relaxation frequency of the l-th Maxwell Body

[Emmerich and Korn, Geophysics (1987)]

Generalized Maxwell body



Modeling wave attenuation

Generalized Maxwell Body

- GMB modulus: $$M(\omega) = \left(k_H + \sum_{l=1}^{L} \frac{k_l \, i\omega}{\omega_l + i\omega} \right)$$
- Unrelaxed modulus: $$M^U = \lim_{\omega \to +\infty} M(\omega) = \lim_{t \to 0} M(t) = k_H + \sum_{l=1}^{L} k_l$$
- $M(\omega) = M^U \left(1 - \sum_{l=1}^{L} \frac{\Upsilon^{M,l} \, \omega_l}{\omega_l + i\omega} \right)$ where $\Upsilon^{M,l} = k_l / M^U$ are the anelastic coefficients

- Stress-strain time relation:

$$\sigma(t) = M^U \left(\epsilon(t) - \sum_{l=1}^{L} \Upsilon^{M,l} \zeta^l(t) \right)$$

where $\zeta^l(t) = \omega_l \int_{-\infty}^{t} e^{-\omega_l(t-\tau)}\, \epsilon(\tau)\, d\tau$ are the anelastic functions

- These anelastic functions verify differential equations:

$$\partial_t \zeta^l(t) + \omega_l\, \zeta^l(t) = \omega_l\, \epsilon(t)$$

- In 1D, we had for one GMB: $M(\omega) = M^U \left(1 - \sum_{l=1}^{L} \frac{\Upsilon^{M,l}\,\omega_l}{\omega_l + i\omega}\right)$

- The 3D generalization is obtained by introducing as many GMB as needed

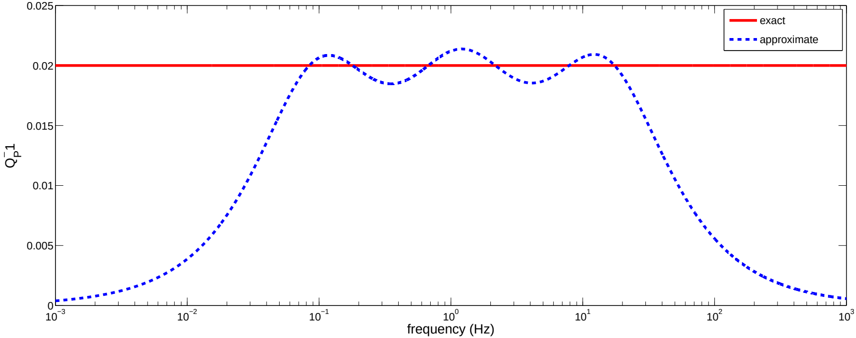
- For example, in the isotropic case, we consider one for λ and one for μ:

$$\lambda(\omega) = \lambda^U \left(1 - \sum_{l=1}^{L_1} \frac{\Upsilon^{\lambda,l}\,\omega_l}{\omega_l + i\omega}\right) \quad \text{and} \quad \mu(\omega) = \mu^U \left(1 - \sum_{l=1}^{L_2} \frac{\Upsilon^{\mu,l}\,\omega_l}{\omega_l + i\omega}\right)$$

where λ^U and μ^U are the unrelaxed Lamé parameters

Modeling wave attenuation

Attenuation law



Attenuation law ($Q = 50$) approximated by 3 mechanisms

Step-by-step modeling for the isotropic case:

- Consider two GMB, one for λ and one for μ, each composed of L MB
- Choose relaxation frequencies for these GMB so that they cover efficiently the frequency range of interest
- Determine anelastic coefficients $\Upsilon^{\lambda,l}$ and $\Upsilon^{\mu,l}$ using measured/desired attenuation laws

It is then possible to solve the global system by adding the differential equations verified by the anelastic functions

Outline

- Initially introduced to solve neutron transport problems (*Reed and Hill, 1973*)
- Became popular as a framework for solving hyperbolic systems (especially Maxwell equations)
- Increasing number of contributions in seismic wave propagation
- Somewhere between a finite element and a finite volume method
- Main properties:
  - can easily deal with discontinuous coefficients and solutions
  - can handle unstructured or non-conforming meshes
  - yield local finite element mass matrices
  - high-order accurate methods with compact stencils
  - naturally suited for *p*-adaptivity
  - amenable to efficient parallelization
- Main drawback: CPU and memory cost

- Stress-displacement linear viscoelastic relation:

$$\sigma = \lambda\,(\nabla\cdot u)\,I + \mu\,(\nabla u + \nabla u^T) - \sum_{l=1}^{L}\left(\lambda\,\Upsilon^{\lambda,l}\,tr(\zeta^l)\,I + 2\mu\,\Upsilon^{\mu,l}\,\zeta^l\right)$$

- Velocity-stress viscoelastic system ($9+6L$ equations in 3D, $5+3L$ in 2D):

$$\begin{cases} \rho\,\dfrac{\partial v}{\partial t} & = & \nabla\cdot\sigma \\ \dfrac{\partial \sigma}{\partial t} & = & \lambda\,(\nabla\cdot v)\,I + \mu\,(\nabla v + \nabla v^T) - \displaystyle\sum_{l=1}^{L}\left(\lambda\,\Upsilon^{\lambda,l}\,tr(\xi^l)\,I + 2\mu\,\Upsilon^{\mu,l}\,\xi^l\right) \\ \dfrac{\partial \xi^l}{\partial t} & = & \dfrac{\omega_l}{2}(\nabla v + \nabla v^T) - \omega_l\,\xi^l, \qquad l = 1,\dots,L \end{cases}$$

- Stress-displacement linear viscoelastic relation:

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A DG scheme for viscoelastic attenuation

Viscoelastic velocity-stress system

- Stress-displacement linear viscoelastic relation:

$$\sigma = \lambda\,(\nabla \cdot u)\,I + \mu\,(\nabla u + \nabla u^T) - \sum_{l=1}^{L} \left(\lambda\,\Upsilon^{\lambda,l}\,tr(\zeta^l)\,I + 2\mu\,\Upsilon^{\mu,l}\,\zeta^l\right)$$

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- Velocity-stress viscoelastic system ($9+6L$ equations in 3D, $5+3L$ in 2D):

$$\left\{ \begin{array}{rcl} \rho \, \dfrac{\partial v}{\partial t} & = & \nabla \cdot \sigma \\ \dfrac{\partial \sigma}{\partial t} & = & \lambda \, (\nabla \cdot v) \, I + \mu \, (\nabla v + \nabla v^T) - \displaystyle\sum_{l=1}^{L} \left(\lambda \, \Upsilon^{\lambda,l} \, tr(\xi^l) \, I + 2\mu \, \Upsilon^{\mu,l} \, \xi^l \right) \\ \dfrac{\partial \xi^l}{\partial t} & = & \dfrac{\omega_l}{2} (\nabla v + \nabla v^T) - \omega_l \, \xi^l, \qquad l = 1, \ldots, L \end{array} \right.$$

- Let $\overline{\mathbb{W}} = \left(v, \overline{\sigma}, \overline{\xi}\right)^T$, where $\overline{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$ and

  $\overline{\xi} = (\overline{\xi}^1, \ldots, \overline{\xi}^l, \ldots, \overline{\xi}^L)^T$ with $\overline{\xi}^l = (\xi_{11}^l, \xi_{22}^l, \xi_{12}^l)^T$

- Compact form:

$$\frac{\partial \overline{\mathbb{W}}}{\partial t} + \sum_{\alpha \in \{1,2\}} \overline{\overline{A}}_\alpha \, \partial_\alpha \overline{\mathbb{W}} = \overline{\overline{E}} \, \overline{\mathbb{W}}$$

where $\overline{\overline{A}}_\alpha$ and $\overline{\overline{E}}$ are $5 + 3L$ block matrices

- Finite element type discretization in triangles
- Lagrange nodal interpolation – degree 1 to 4 – on simplicial elements
- Boundary conditions: free surface, absorbing, periodic
- Centered fluxes for the internal faces combined with a leap-frog time scheme
  - Stability [Delcourte et al., ESAIM: Proceedings, 2009]
  - A priori convergence (submitted)

After spatial and leap-frog time discretization, we obtain:

$$
\left\{
\begin{array}{rcl}
\overline{\overline{M}}^{\mathcal{T}_i} \left(\overline{\mathcal{W}_{v_\alpha}^{n+1}} - \overline{\mathcal{W}_{v_\alpha}^{n}} \right) & = & \Delta t \, \overline{\mathcal{F}_\alpha^{\mathcal{T}_i} \left(\overline{\sigma}^{n+\frac{1}{2}} \right)} \\
\overline{\overline{M}}^{\mathcal{T}_i} \left(\overline{\mathcal{W}_{\sigma_{\alpha\beta}}^{n+\frac{3}{2}}} - \overline{\mathcal{W}_{\sigma_{\alpha\beta}}^{n+\frac{1}{2}}} \right) & = & \Delta t \, \overline{\mathcal{G}_{\alpha,\beta}^{\mathcal{T}_i} \left(v^{n+1}, \overline{\xi}^{l,n+\frac{1}{2}} \right)} \\
\overline{\overline{M}}^{\mathcal{T}_i} \left(\overline{\mathcal{W}_{\xi^l_{\alpha\beta}}^{n+\frac{3}{2}}} - \overline{\mathcal{W}_{\xi^l_{\alpha\beta}}^{n+\frac{1}{2}}} \right) & = & \Delta t \, \overline{\mathcal{H}_{\alpha,\beta}^{\mathcal{T}_i} \left(v^{n+1}, \overline{\xi}^{l,n+\frac{1}{2}} \right)}
\end{array}
\right.
$$

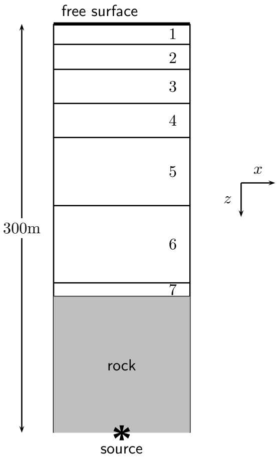
where:

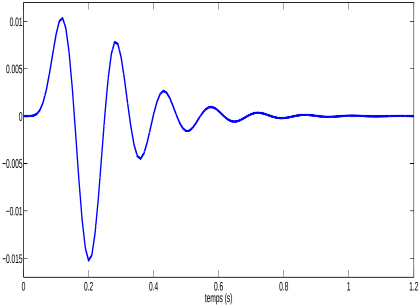
- $\overline{\mathcal{W}_{v_\alpha}}$, $\overline{\mathcal{W}_{\sigma_{\alpha\beta}}}$ and $\overline{\mathcal{W}_{\xi^l_{\alpha\beta}}}$ contain respectively the values of v_α, $\sigma_{\alpha\beta}$ and $\xi^l_{\alpha\beta}$ $(\alpha, \beta = 1, 2$ and $l = 1, \ldots, L)$ on an element $\mathcal{T}_i$
- $\mathcal{F}_\alpha^{\mathcal{T}_i}$, $\mathcal{G}_{\alpha,\beta}^{\mathcal{T}_i}$ and $\mathcal{H}_{\alpha,\beta}^{\mathcal{T}_i}$ $(\alpha, \beta = 1, 2)$ are operators collecting the integrals on $\mathcal{T}_i$ and $\partial\mathcal{T}_i$

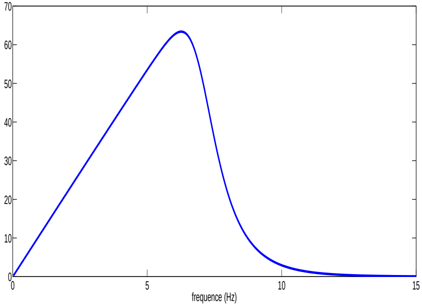
Outline

- A vertical S plane wave generated by a source propagates in a 1D column made of 7 layers (with differents parameters) above a bedrock
- An absorbing condition is applied at the bottom and periodic conditions on the sides

	v_p	v_s	ρ	Q_p	Q_s
1	1500	130	2050	75	15
2	1500	200	2150	75	20
3	1650	300	2075	83	30
4	2050	450	2100	103	40
5	2450	600	2155	123	60
6	2550	700	2200	140	70
7	3500	1250	2500	200	100
rock	4500	2600	2600	50000	50000





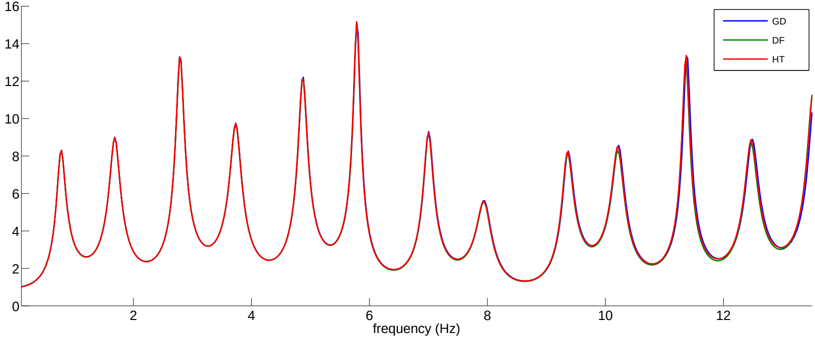


Source in time (l.) and in frequency (r.)

- We compare our 2D discontinuous Galerkin code with:
  - a 2D finite difference (DF) code from C. Gélis (IRSN) using the Liu/Archuleta method (*Liu and Archuleta, 2006*)
  - the Haskell-Thomson method (HT), with solutions also provided by C. Gélis
- We compute the ratios between the surface spectrum and twice the source spectrum of v_x, in order to know the amplification versus frequency curve

Numerical results

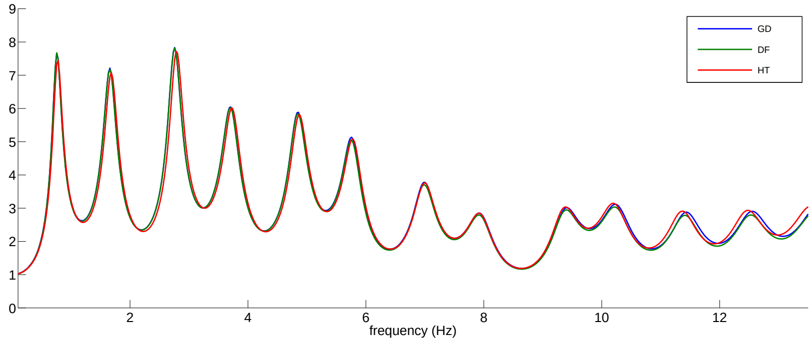
Layered media



Spectral ratios in the elastic case

Numerical results

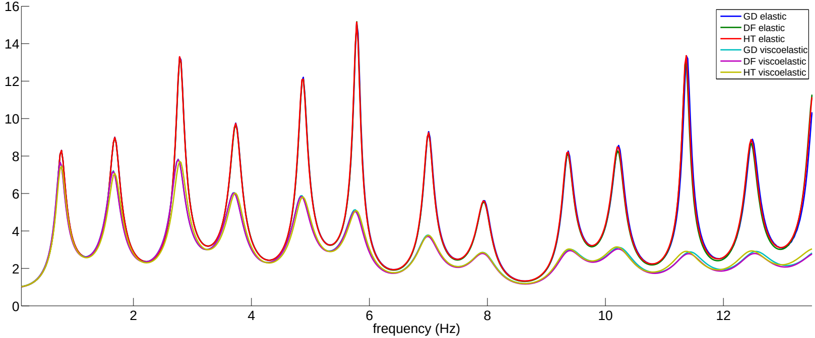
Layered media



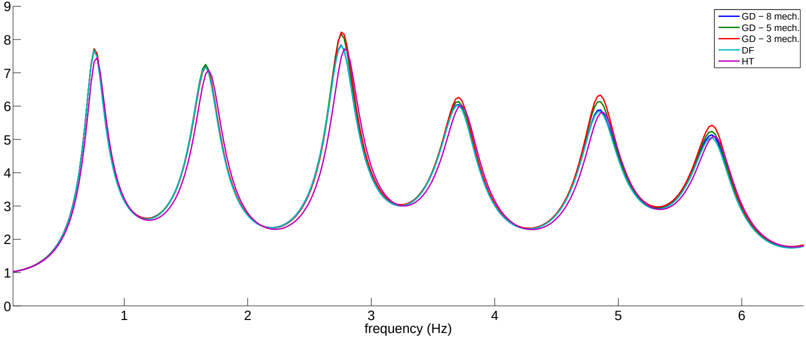
Spectral ratios in the viscoelastic case

Numerical results

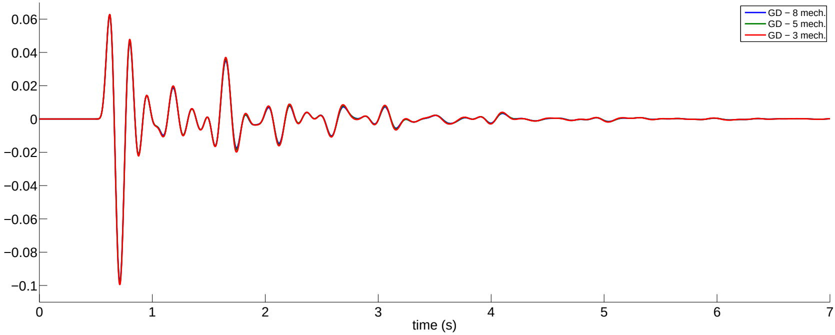
Layered media



Spectral ratios in the elastic and viscoelastic cases



Spectral ratios in the viscoelastic case – influence of the number of mechanisms

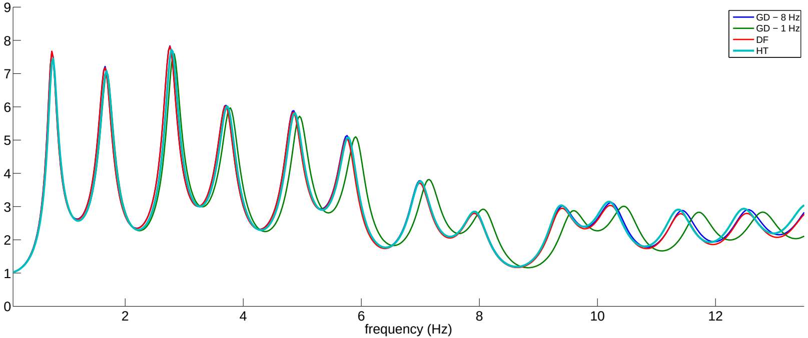


Solutions in time – influence of the number of mechanisms

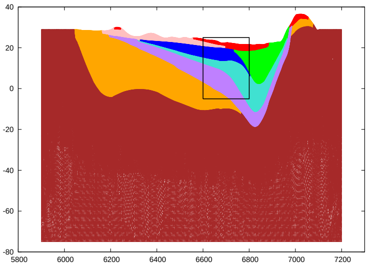
Numerical results

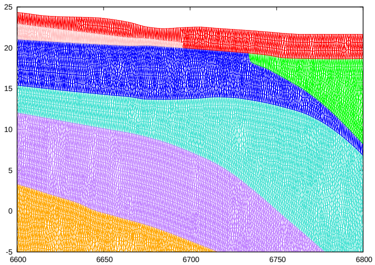
Layered media

The velocity is given at a reference frequency f_r and a different choice gives very different results because λ and μ are modified too



Spectral ratios in the viscoelastic case – influence of the reference frequency

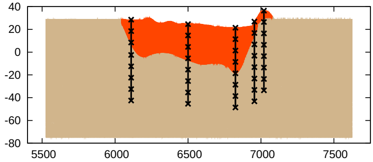




Nice basin 2D model (7 media + bedrock) – data are from the Seismic Risk team of CETE Méditerranée de Nice

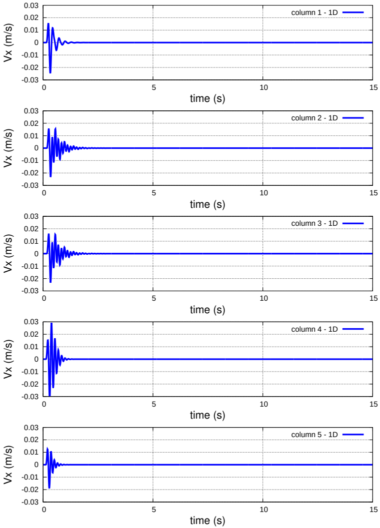
- Application to a simplified 2D model of the Nice basin
- A vertical S plane wave propagates in the domain (homogeneous basin)
- The boundary conditions and the source characteristics are unchanged from the previous test-case
- We compare 2D simulations and 1D simulations for 5 columns

	v_p	v_s	ρ	Q_p	Q_s
basin	730	300	2000	73	30
rock	2450	1000	2100	245	100

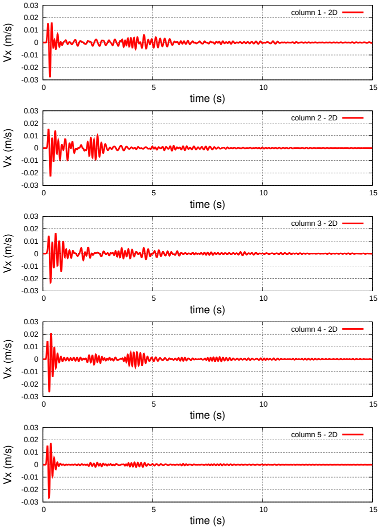


Numerical results

2D test-case – viscoelastic time solutions



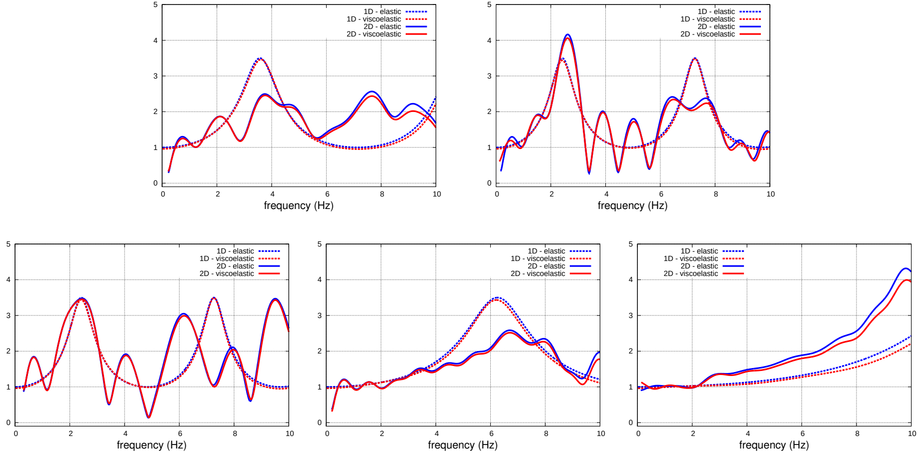
1D – viscoelastic



2D – viscoelastic

Numerical results

2D test-case – spectral ratios



Outline

Conclusion

- Linear viscoelastic attenuation can be modeled using rheological models
- Resulting system generalizes the elastic case
- Our DG method has been applied to the viscoelastic case
- Ongoing and future work:
  - heterogeneous Nice basin: comparisons with the FD code and observational data using higher-degree polynomials on coarser meshes
  - investigation of the best compromise between accuracy and memory/CPU cost
  - other models for viscoelastic attenuation

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