Vertex discretization of two phase Darcy flows: convergence analysis and discontinuous capillary pressures

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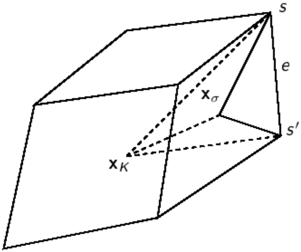
Vertex Approximate Gradient (VAG) scheme [Eymard et al 2010]

- Tetrahedral submesh $\mathcal{T}$
- Interpolation at the face centres $\boldsymbol{x}_\sigma$ using the face nodal values
- $\mathbb{P}_1$ finite element discretization on $\mathcal{T}$ with interpolation at the face centres
- Nodal basis: $\eta_\kappa, \eta_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{V}_\kappa, \kappa \in \mathcal{M}$

$$\mathbf{x}_\sigma = \sum_{s \in \mathcal{V}_\sigma} \frac{1}{\text{Card}\mathcal{V}_\sigma} \mathbf{x}_{\mathbf{s}}, \quad u_\sigma = \sum_{\mathbf{s} \in \mathcal{V}_\sigma} \frac{1}{\text{Card}\mathcal{V}_\sigma} u_{\mathbf{s}}$$



Variational formulation and fluxes

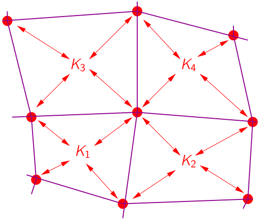
$$a(u_{\mathcal{T}}, v_{\mathcal{T}}) = \int_\Omega K(\boldsymbol{x})\nabla u_{\mathcal{T}}(\boldsymbol{x}) \cdot \nabla v_{\mathcal{T}}(\boldsymbol{x})\ d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\ v_{\mathcal{T}}(\boldsymbol{x})\ d\boldsymbol{x}$$

$$\begin{aligned} a(u_{\mathcal{T}}, v_{\mathcal{T}}) &= \sum_{\kappa\in\mathcal{M}} \sum_{s\in\mathcal{V}_\kappa} \Big(\int_\kappa -K(\boldsymbol{x})\nabla u_{\mathcal{T}}(\boldsymbol{x}) \cdot \nabla \eta_{\mathbf{s}}(\boldsymbol{x}) d\boldsymbol{x}\Big)\Big(v_\kappa - v_{\mathbf{s}}\Big), \\ &= \sum_{\kappa\in\mathcal{M}} \sum_{s\in\mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_{\mathcal{T}})\ \Big(v_\kappa - v_{\mathbf{s}}\Big) \end{aligned}$$

with the fluxes $F_{\kappa,\mathbf{s}}(u_{\mathcal{T}}) = -F_{\mathbf{s},\kappa}(u_{\mathcal{T}}) = \displaystyle\int_\kappa -K(\boldsymbol{x})\nabla u_{\mathcal{T}} \cdot \nabla \eta_{\mathbf{s}}(x) d\boldsymbol{x}$.

Equivalent discrete conservation laws

$$\begin{cases} \displaystyle\sum_{s \in \mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_\mathcal{T}) = \int_\kappa f(\boldsymbol{x})\eta_\kappa(\boldsymbol{x})\, d\boldsymbol{x} \text{ for all } \kappa \in \mathcal{M}, \\ \displaystyle\sum_{\kappa \in \mathcal{M}_\mathbf{s}} F_{\mathbf{s},\kappa}(u_\mathcal{T}) = \int_\Omega f(\boldsymbol{x})\eta_\mathbf{s}(\boldsymbol{x})\, d\boldsymbol{x} \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$



Mass lumping: $$\begin{cases} \displaystyle\sum_{s \in \mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_\mathcal{T}) = m_\kappa f(\boldsymbol{x}_\kappa) \text{ for all } \kappa \in \mathcal{M}, \\ \displaystyle\sum_{\kappa \in \mathcal{M}_\mathbf{s}} F_{\mathbf{s},\kappa}(u_\mathcal{T}) = m_\mathbf{s} f(\boldsymbol{x}_\mathbf{s}) \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$

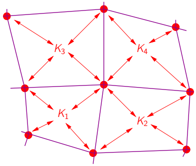
Application to two phase Darcy flows ($\alpha = o, w$).

The Darcy fluxes between κ and $\mathbf{s}$ are discretized by:

$$V^{\alpha}_{\kappa,\mathbf{s}} = F_{\kappa,\mathbf{s}}(P^{\alpha,n}_{\mathcal{T}}) + \rho^{\alpha}_{\kappa,\mathbf{s}}\; g\; F_{\kappa,\mathbf{s}}(Z_{\mathcal{T}}).$$

with the conservativity property $V^{\alpha}_{\mathbf{s},\kappa} = -V^{\alpha}_{\kappa,\mathbf{s}}$.

$$\left\{\begin{array}{l} m_\kappa \phi_\kappa \dfrac{S^{\alpha,n}_\kappa - S^{\alpha,n-1}_\kappa}{\Delta t} + \displaystyle\sum_{\mathbf{s}\in\mathcal{V}_\kappa} \dfrac{k^{\alpha}_r(S^{\alpha,n}_{up^\alpha})}{\mu^\alpha} V^{\alpha}_{\kappa,\mathbf{s}} = 0, \kappa \in \mathcal{M}, \\ m_\mathbf{s} \phi_\mathbf{s} \dfrac{S^{\alpha,n}_\mathbf{s} - S^{\alpha,n-1}_\mathbf{s}}{\Delta t} - \displaystyle\sum_{\kappa\in\mathcal{M}_\mathbf{s}} \dfrac{k^{\alpha}_r(S^{\alpha,n}_{up^\alpha})}{\mu^\alpha} V^{\alpha}_{\kappa,\mathbf{s}} = 0, \mathbf{s} \in \mathcal{V}\setminus\mathcal{V}_D, \\ up^\alpha = \left\{\begin{array}{l} \kappa \text{ if } V^{\alpha}_{\kappa,\mathbf{s}} \geq 0, \\ \mathbf{s} \text{ if } V^{\alpha}_{\kappa,\mathbf{s}} < 0. \end{array}\right. \end{array}\right.$$

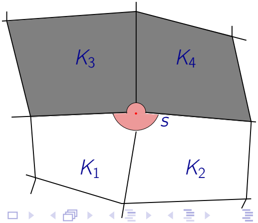


Definition of a volume and a porous volume to each cell and vertex

Volumes:
$$\begin{cases} m_{\kappa,\mathbf{s}} = \alpha_{\kappa,\mathbf{s}} \displaystyle\int_\kappa d\boldsymbol{x} & \text{for all } \kappa \in \mathcal{M}, \mathbf{s} \in \mathcal{V}_\kappa \setminus \mathcal{V}_D, \\ m_{\mathbf{s}} = \displaystyle\sum_{\kappa \in \mathcal{M}_{\mathbf{s}}} m_{\kappa,\mathbf{s}} & \text{for all } \mathbf{s} \in \mathcal{V} \setminus \mathcal{V}_D, \\ m_\kappa = \displaystyle\int_\kappa d\boldsymbol{x} - \sum_{\mathbf{s} \in \mathcal{V}_\kappa \setminus \mathcal{V}_D} m_{\kappa,\mathbf{s}} & \text{for all } \kappa \in \mathcal{M}, \end{cases}$$

which are such that
$$\sum_{\kappa \in \mathcal{M}} m_\kappa + \sum_{\mathbf{s} \in \mathcal{V} \setminus \mathcal{V}_D} m_{\mathbf{s}} = \int_\Omega d\boldsymbol{x}.$$

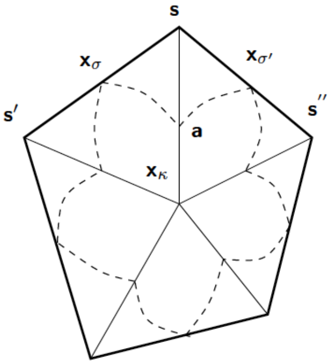
Porosities:
$$\begin{cases} \phi_\kappa = \dfrac{1}{\int_k d\boldsymbol{x}} \displaystyle\int_\kappa \phi(\boldsymbol{x}) d\boldsymbol{x}, \\ \phi_{\mathbf{s}} = \dfrac{\sum_{\kappa \in \mathcal{M}_{\mathbf{s}}} \phi_\kappa m_{\kappa,\mathbf{s}}}{m_{\mathbf{s}}}. \end{cases}$$



Control Volume Finite Element (CVFE) interpretation of the VAG Fluxes in 2D

$$F_{\kappa,\mathbf{s}}(u_{\mathcal{T}}) = \int_{\kappa} -K_{\kappa}\nabla u_{\mathcal{T}}(\boldsymbol{x}) \cdot \nabla \eta_{\mathbf{s}}(\boldsymbol{x})\ d\boldsymbol{x},$$

$$= \int_{\widehat{x_{\sigma}\mathbf{a}} \cup \widehat{x_{\sigma'}\mathbf{a}}} -K_{\kappa}\nabla u_{\mathcal{T}}(\boldsymbol{x}) \cdot \mathbf{n}_{\kappa} d\sigma.$$



Questions

- How does the choice of the volumes $m_{\kappa,\mathbf{s}}$ affect the convergence of the scheme for two phase Darcy flows?

- How to deal with different rocktypes ?

Convergence analysis of a two phase immiscible incompressible Darcy flow model

$$\begin{cases} \mathrm{div}\Big(-\lambda(S)\ K\nabla p\Big) = k^o + k^w & \text{on } \Omega \times (0, t_f), \\ \phi\partial_t S + \mathrm{div}\Big(-f(S)\lambda(S)\ K\nabla p\Big) + \mathrm{div}\Big(-K\nabla\varphi(S)\Big) = k^o & \text{on } \Omega \times (0, t_f), \\ S = 0,\ p = 0 & \text{on } \partial\Omega \times (0, t_f), \\ S|_{t=0} = S_0 & \text{on } \Omega. \end{cases}$$

- Capillary diffusion: $\varphi(S)$ strictly increasing with $\varphi'(0) = \varphi'(1) = 0$ (degenerate parabolic equation for S), φ^{-1} Holder continuous.
- Total Mobility: $\underline{\lambda} < \lambda(S) \leq \overline{\lambda}$
- Fractional flow: $f(S)$ nondecreasing with $f(0) = 0$, $f(1) = 1$.
- Source term: $k^o + k^w \geq 0$.

Convergence analysis for TPFA [Eymard et al 2006], and for SUSHI-Mimetic schemes [Brenner 2011].

Why it will work independently of the choice of the $m_{\kappa,\mathbf{s}}$?

Two representations of a discrete function u (saturation S):

$$\text{Finite Element interpolation: } u_{\mathcal{T}}(\boldsymbol{x}) = \sum_{\kappa\in\mathcal{M}} u_\kappa \eta_\kappa(\boldsymbol{x}) + \sum_{\mathbf{s}\in\mathcal{V}} u_{\mathbf{s}} \eta_{\mathbf{s}}(\boldsymbol{x})$$

and

$$\text{Finite Volume interpolation: } \left\{ \begin{array}{l} u_{\mathcal{D}}(\boldsymbol{x}) = u_\kappa \text{ on } \omega_\kappa, \;\; \mathrm{Vol}(\omega_\kappa) = m_\kappa, \\ u_{\mathcal{D}}(\boldsymbol{x}) = u_{\mathbf{s}} \text{ on } \omega_{\mathbf{s}}, \;\; \mathrm{Vol}(\omega_{\mathbf{s}}) = m_{\mathbf{s}}. \end{array} \right.$$

Under shape regularity assumptions we can prove the estimates:

$$\|u_{\mathcal{D}}\|_{L^2(\Omega)} \lesssim \|u_{\mathcal{T}}\|_{L^2(\Omega)},$$

and

$$\|u_{\mathcal{D}} - u_{\mathcal{T}}\|_{L^2(\Omega)} \lesssim h_{\mathcal{T}} \|\nabla u_{\mathcal{T}}\|_{L^2(\Omega)^d}.$$

Discrete a priori estimates and convergence theorem

For any choice of the weights $\alpha^m_{\kappa,\mathbf{s}} \in [0,1)$ such that $1 - \sum_{\mathbf{s}\in\mathcal{V}_\kappa\setminus\mathcal{V}_{\boldsymbol{D}}} \alpha_{\kappa,\mathbf{s}} \geq 0$, one has the a priori estimates:

$$\|\varphi(S)_{\mathcal{D},\Delta t}\|_{L^\infty(0,t_f;L^2(\Omega))} + \|\nabla\varphi(S)_{\mathcal{T},\Delta t}\|_{L^2((0,t_f)\times\Omega)} + \|\nabla p_{\mathcal{T},\Delta t}\|_{L^\infty(0,t_f;L^2(\Omega))} \leq C.$$

with a constant C depending on the shape regularity constant of the submesh $\mathcal{T}$ and on $\max_{\mathbf{s}\in\mathcal{V}} \#\mathcal{M}_{\mathbf{s}}$.

Proof: use basically the equivalence

$$\sum_{\mathbf{s}\in\mathcal{V}_\kappa} (u_\kappa - u_{\mathbf{s}}) F_{\kappa,\mathbf{s}}(u) \sim \|\nabla u_{\mathcal{T}}\|^2_{L^2(\kappa)},$$

and the discrete Poincaré inequality

$$\|u_{\mathcal{D}}\|_{L^2(\Omega)} \lesssim \|u_{\mathcal{T}}\|_{L^2(\Omega)} \lesssim \|\nabla u_{\mathcal{T}}\|_{L^2(\Omega)^d}.$$

Convergence theorem

Let $\mathcal{D}^m, m \in \mathbb{N}$ be a family of discretizations such that the family of simpletic submeshes $\mathcal{T}^m, m \in \mathbb{N}$ is shape regular, $\max_{\mathbf{s}\in\mathcal{V}^m} \#\mathcal{M}_\mathbf{s}$ is bounded, and $h_{\mathcal{T}^m} \to 0$. Let $\Delta t^m \to 0$.
Let the weights $\alpha^m_{\kappa,\mathbf{s}} \in [0,1)$ be chosen such that $1 - \sum_{\mathbf{s}\in\mathcal{V}_\kappa\setminus\mathcal{V}_D} \alpha^m_{\kappa,\mathbf{s}} \geq 0$.

Then, up to a subsequence, one has

$$S_{\mathcal{T}^m,\Delta t^m} \to S \text{ strongly in } L^2(0,t_f;L^2(\Omega))$$

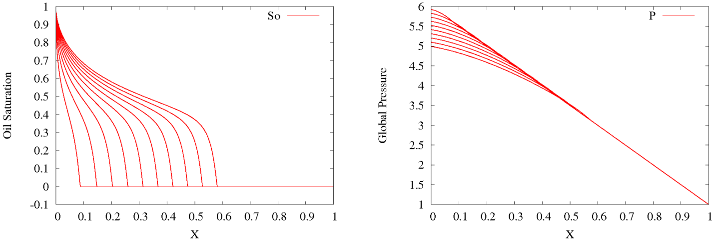
$$P_{\mathcal{T}^m,\Delta t^m} \to P \text{ weakly in } L^2(0,t_f;L^2(\Omega))$$

where (P,S) is a weak solution.

Numerical experiment on the Buckley Leverett 1D solution

$$\begin{cases} \partial_t S + \partial_x f(S) - \partial_{x^2}\varphi(S) = 0 & \text{on } (0,1)\times(0,t_f), \\ S = 1 & \text{on } \{0\}\times(0,t_f), \\ S = 0 & \text{on } \{1\}\times(0,t_f), \\ S|_{t=0} = 0 & \text{on } (0,1), \end{cases} \qquad P(x,t) = 1 + \int_x^1 \frac{du}{\lambda(S(u,t))}$$

$$\lambda(S) = \frac{S^2}{5} + (1-S)^2, \quad f(S) = \frac{\frac{S^2}{5}}{\frac{S^2}{5} + (1-S)^2}, \quad \varphi(S) = \frac{P_{c,1}}{5}\int_0^S \frac{u^2(1-u)}{\frac{u^2}{5} + (1-u)^2}\,du.$$

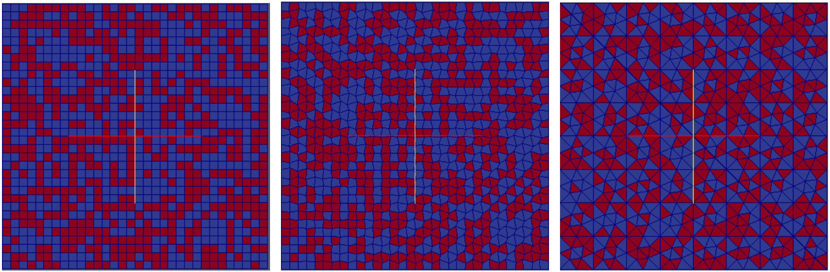


Oil saturation and Global Pressure for $P_{c,1} = 0.1$ ($Pe \sim 150$)

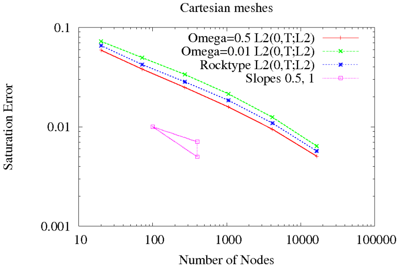
Three choices of the $m_{\kappa,\mathbf{s}}$. Meshes from the 2D FVCA5 and 3D FVCA6 Benchmarks

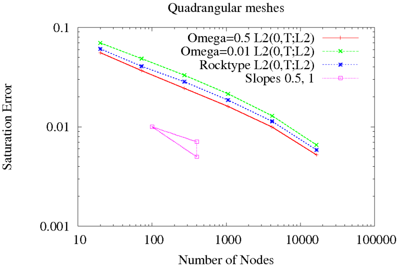
$$\text{Choices 1 and 2: } \alpha_{\kappa,\mathbf{s}} = \omega \frac{1}{\#\mathcal{M}_\mathbf{s}}, \text{ for } \omega = 0.5 \text{ or } \omega = 0.01.$$

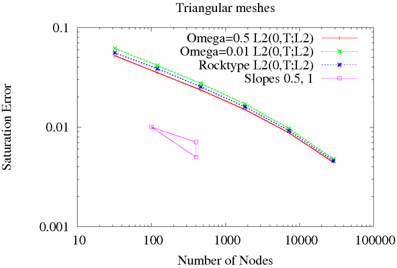
$$\text{Choice 3 (rocktype): } \alpha_{\kappa,\mathbf{s}} = \begin{cases} 0.5\frac{1}{\#\mathcal{M}_\mathbf{s}} & \text{if } \{\kappa \,|\, \text{rocktype}_\kappa = 2\} = \mathcal{M}_\mathbf{s}, \\ 0.5\frac{1}{\#\mathcal{M}_\mathbf{s}} & \text{else if } \text{rocktype}_\kappa = 1, \\ 0 & \text{else if } \text{rocktype}_\kappa = 2. \end{cases}$$



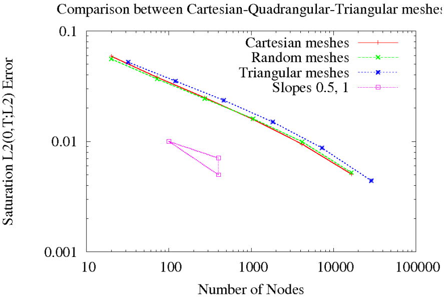
Numerical results in 2D: cartesian, random quadrangular and triangular meshes

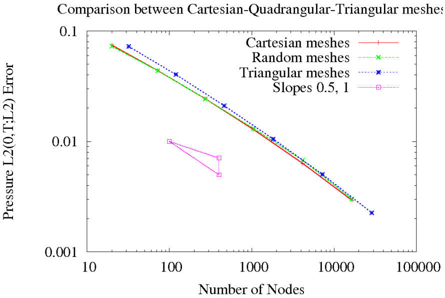




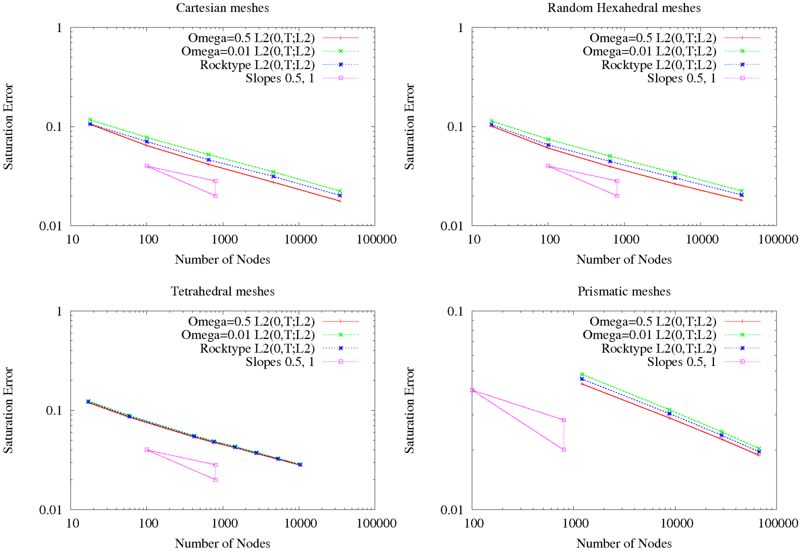


Numerical results in 2D: cartesian, random quadrangular and triangular meshes





Numerical results in 3D: cartesian, random hexahedral, tetrahedral and prismatic meshes



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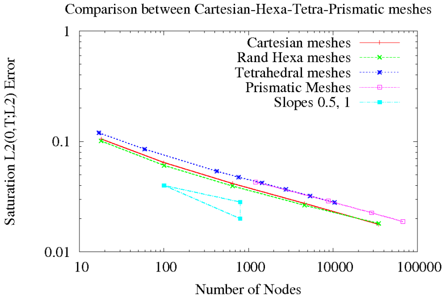
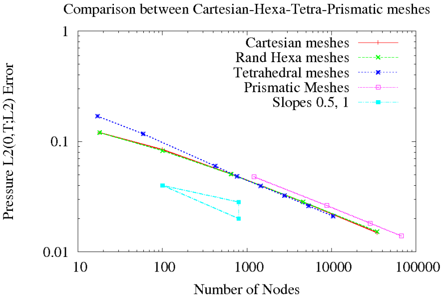




Conclusions (first part)

- Convergence slightly dependent on $m_{\kappa,\mathbf{s}}$
- Better accuracy for balanced volumes at cells and vertices

- Conclusion: choose the volumes
  - to match the heterogeneities
  - to balance the cell/vertex volumes as much as possible (improve the Newton convergence for more complex models with phase transitions)

Immiscible two phase Darcy flows with discontinuous capillary pressures

$$\left\{\begin{array}{l}\partial_t \rho^\alpha S^\alpha + \mathrm{div}\Big(\mathbf{Q}^\alpha\Big) = 0,\ \alpha = w, o,\\ \mathbf{Q}^\alpha = -\rho^\alpha \ \dfrac{k_r^\alpha(S^\alpha, \boldsymbol{x})}{\mu^\alpha} K(\boldsymbol{x})(\nabla P^\alpha - \rho^\alpha \mathbf{g}),\ \alpha = w, o,\\ S^w + S^o = 1,\\ P^o - P^w = P_c(S^o, \boldsymbol{x}).\end{array}\right.$$

Rocktypes 1 and 2:

$$\left\{\begin{array}{l}P_c(S^o, \boldsymbol{x}) = P_{c,i}(S^o),\\ k_r^\alpha(S^\alpha, \boldsymbol{x}) = k_{r,i}^\alpha(S^\alpha) \text{ if } \boldsymbol{x} \in \Omega_i,\ i = 1, 2.\end{array}\right.$$

Matching conditions at the interface $\Gamma = \Omega_1 \cap \Omega_2$ between the two rocktypes [Enchery et al 2008], [Cances et al 2011], [Brenner et al 2011]:

$$\left\{\begin{array}{l}P_{c,1}(S^o) \cap P_{c,2}(S^o) \neq \emptyset,\\ P_1^w = P_2^w (\text{if mobile phase}),\\ \mathbf{Q}_1^\alpha \cdot \mathbf{n}_1 + \mathbf{Q}_2^\alpha \cdot \mathbf{n}_1 = 0,\ \alpha = w, o.\end{array}\right.$$

VAG discretization: main ideas

- Allow for discontinuous saturations at the interfaces between two different rocktypes: $S_{\kappa,\mathbf{s}}$, $\kappa \in \mathcal{M}_{\mathbf{s}}$
- Fluxes continuity at a given interface **s** given by the conservation equations at **s**
- Reduce the number of unknowns by choosing the phase pressures as primary unknowns. The saturations are given for each cell κ by

$$\begin{cases} S^o_{\kappa,\mathbf{s}} = P^{-1}_{c,\kappa}(p^o_{\mathbf{s}} - p^w_{\mathbf{s}}), \ \kappa \in \mathcal{M}_{\mathbf{s}}, \\ S^o_{\kappa} = P^{-1}_{c,\kappa}(p^o_{\kappa} - p^w_{\kappa}). \end{cases}$$

  which accounts for the phase pressure continuity interface conditions
- Extension of the scheme [Brenner et al 2011], [Brenner 2011] to the VAG discretization on general meshes.

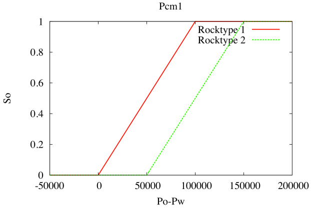
VAG discretization

$$\begin{cases} m_\kappa \phi_\kappa \dfrac{S_\kappa^{\alpha,n} - S_\kappa^{\alpha,n-1}}{\Delta t} + \displaystyle\sum_{\mathbf{s}\in\mathcal{V}_\kappa} \dfrac{k_{r,\kappa}^\alpha(S_\kappa^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^+ + \dfrac{k_{r,\kappa}^\alpha(S_{\kappa,\mathbf{s}}^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^- = 0, \\ \qquad k \in \mathcal{M},\ \alpha = w, o, \\ \displaystyle\sum_{\kappa\in\mathcal{M}_\mathbf{s}} m_{\kappa,\mathbf{s}}\phi_\kappa \dfrac{S_{\kappa,\mathbf{s}}^{\alpha,n} - S_{\kappa,s}^{\alpha,n-1}}{\Delta t} - \displaystyle\sum_{\kappa\in\mathcal{M}_\mathbf{s}} \dfrac{k_{r,\kappa}^\alpha(S_\kappa^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^+ + \dfrac{k_{r,\kappa}^\alpha(S_{\kappa,\mathbf{s}}^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^- = 0, \\ \qquad \mathbf{s} \in \mathcal{V}\setminus\mathcal{V}_D,\ \alpha = w, o. \end{cases}$$

$$\begin{cases} S_{\kappa,\mathbf{s}}^{o,n} = P_{c,\kappa}^{-1}(p_\mathbf{s}^{o,n} - p_\mathbf{s}^{w,n}),\ \kappa \in \mathcal{M}_\mathbf{s}, \mathbf{s} \in \mathcal{V}\setminus\mathcal{V}_D, \\ S_\kappa^{o,n} = P_{c,\kappa}^{-1}(p_\kappa^{o,n} - p_\kappa^{w,n}),\ \kappa \in \mathcal{M}. \end{cases}$$

Problem of non uniqueness of the solution P^w, P^o

Example: initial state P^w, $S^w = 1$:
P^o is clearly not unique !



To avoid this singularity when solving the discrete nonlinear system:

Projections of $P^o_\kappa - P^w_\kappa$ on the interval:

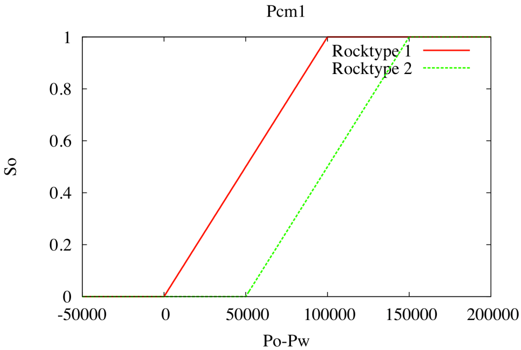
$$\left[\min_{\{p\,|\,(P_{\mathbf{c},\kappa}^{-1})'(p)>0\}} P_{c,\kappa}^{-1}(p), \max_{\{p\,|\,(P_{\mathbf{c},\kappa}^{-1})'(p)>0\}} P_{c,\kappa}^{-1}(p)\right]$$

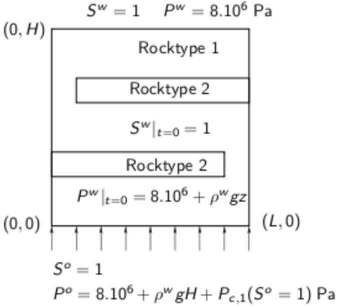
and of $P^o_{\mathbf{s}} - P^w_{\mathbf{s}}$ on

$$\left[\min_{\kappa\in\mathcal{M}_{\mathbf{s}}} \min_{\{p\,|\,(P_{\mathbf{c},\kappa}^{-1})'(p)>0\}} P_{c,\kappa}^{-1}(p), \max_{\kappa\in\mathcal{M}_{\mathbf{s}}} \max_{\{p\,|\,(P_{\mathbf{c},\kappa}^{-1})'(p)>0\}} P_{c,\kappa}^{-1}(p)\right].$$

Test Case: two barriers

Porous media with two rocktypes: $K_1 = K_2 = 1.10^{-12}\ m^2,\ \phi_1 = \phi_2 = 0.1$, $k^{\alpha}_{r,1} = k^{\alpha}_{r,2},\ \alpha = w, o$, and the following $P^{-1}_{c,1},\ P^{-1}_{c,2}$:



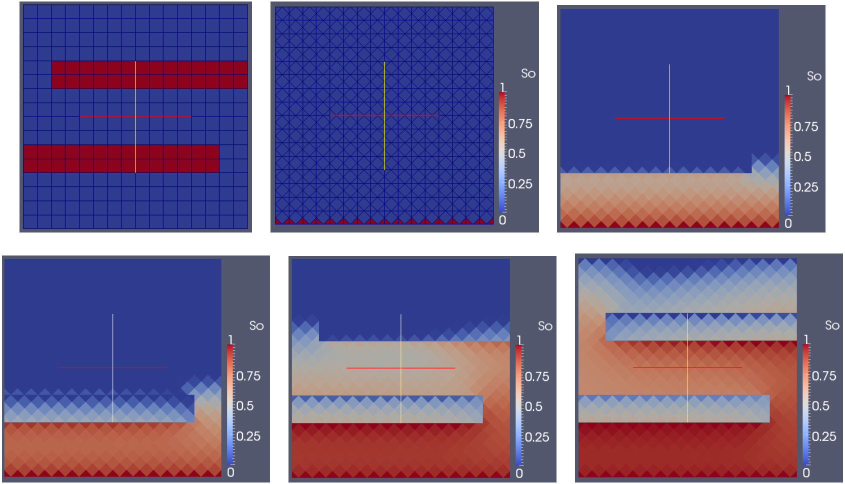


Density driven flow: $\rho^o = 800,\ \rho^w = 1000\ kg/m^3$,

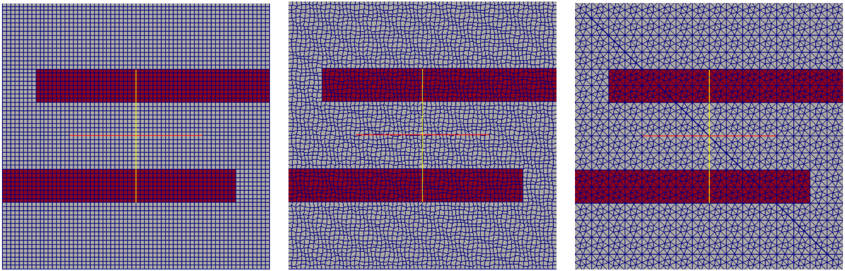
$$k^o_r(S^o) = (S^o)^2,\ \mu^o = 5.10^{-3},$$

$$k^w_r(S^w) = (S^w)^2,\ \mu^w = 1.10^{-3}.$$

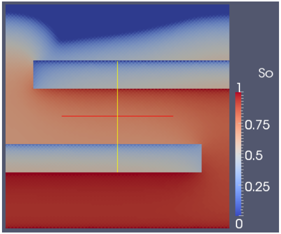
Barriers test case: numerical result on a Cartesian grid 16×16

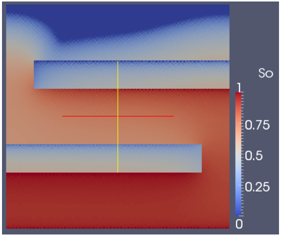


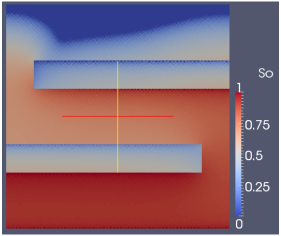
Comparison of the solution at final time on cartesian, random quadrangular and triangular meshes



Comparison of the solution on cartesian 64×64, random quadrangular 64×64, and triangular (1900 nodes) meshes







CPR-AMG preconditioner

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} R^w + R^o \\ R^o \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} P^w \\ P^o - P^w \end{pmatrix}.$$

$$A = \frac{\partial R}{\partial X} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

CPR-AMG Preconditioner: multiplicative combination of ILU0 on A and AMG on the elliptic bloc A_{11}.

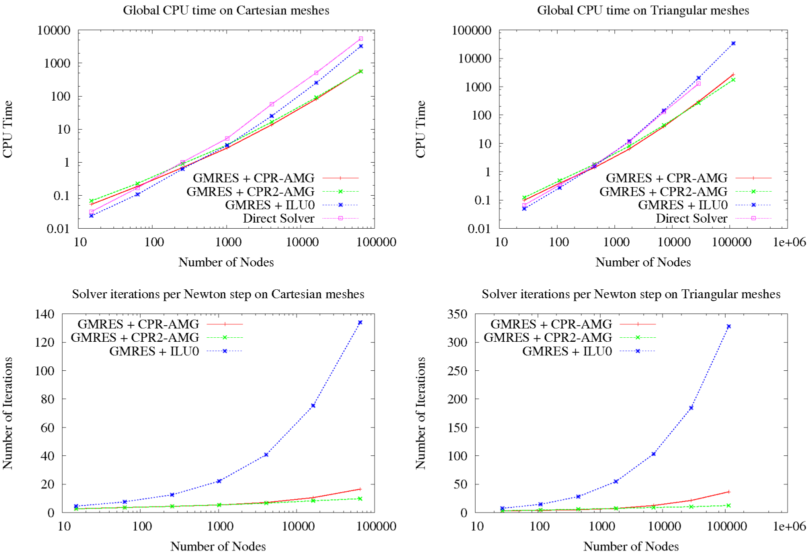
- $X^{(1)} = \mathrm{ILU0}(A)^{-1} R$
- $X_1^{(2)} = \mathrm{AMG}(A_{11})^{-1} \left(R_1 - A_{11} X_1^{(1)} - A_{12} X_2^{(1)} \right)$
- $X = \begin{pmatrix} X_1^{(1)} + X_1^{(2)} \\ X_2^{(1)} \end{pmatrix}$

CPR-AMG preconditioner: variant

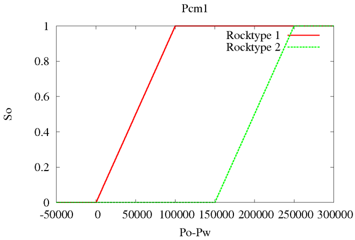
Apply AMG also on the A_{22} bloc (degenerate Parabolic equation).

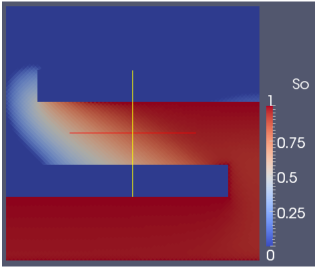
- $X^{(1)} = \mathrm{ILU0}(A)^{-1} R$
- $X^{(2)} = \begin{pmatrix} \mathrm{AMG}(A_{11})^{-1} & 0 \\ 0 & \mathrm{AMG}(A_{22})^{-1} \end{pmatrix} \left(R - AX^{(1)}\right)$
- $X = X^{(1)} + X^{(2)}$

CPU time with different solvers/preconditioners: barriers test case with 137 time steps.

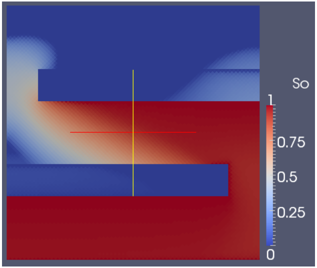


Barriers test case 2



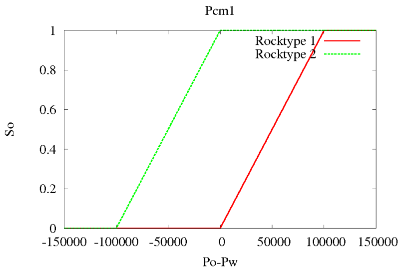


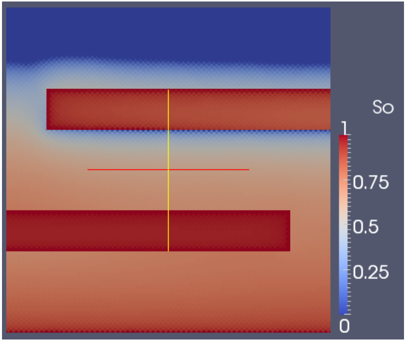
$\rho^o = 800$



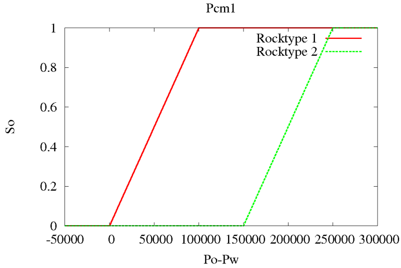
$\rho^o = 750$

Test case with change of wettability: imbibition in the barriers

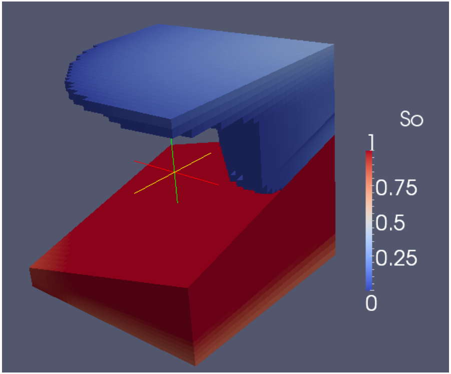




3D test case



Pcm1
Rocktype 1
Rocktype 2
So
Po-Pw
1
0.8
0.6
0.4
0.2
0
-50000
0
50000
100000
150000
200000
250000
300000



So
1
0.75
0.5
0.25
0

Conclusion and perspectives

- Vertex centred discretization of Darcy flows:
  - adapted to general meshes,
  - very efficient on simpletic meshes compared with cell centred schemes,
  - can be adapted to highly heterogeneous media and different rocktypes.

- Extension to compositional models with discontinuous capillary pressures (following the pressure-pressure formulation of [Angelini 2010] for Black Oil models)

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