Vertex centred finite volume scheme for compositional multiphase flows in porous media

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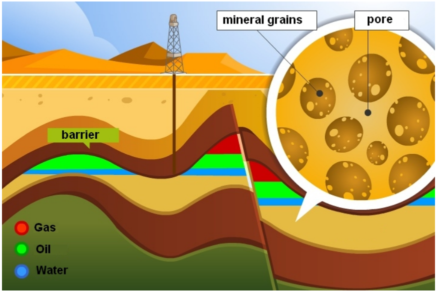
Context

- **Applications :** Petroleum reservoir and sedimentary basin simulation



- **Method :** Formulation and discretization of multiphase flow in porous media
- **Constraint :**
  - ▷ Anisotropy of the permeability tensor
  - ▷ Geological layers $\Longrightarrow$ heterogeneities on coarse grid
  - ▷ Simulation time long $\Longrightarrow$ efficient method in CPU time

Multiphase flow framework – Discrete balance laws

Simulation of flow with N_c miscibles components and N_α phases

On each control volume K of the domain :

$$\frac{\Phi_K}{\delta t}(A_{K,i}^{(n+1)} - A_{K,i}^{(n)}) + \sum_{\alpha=1}^{N_\alpha} \sum_{L \in \mathcal{N}_K} M_{K,L,i}^{(n+1),\alpha} F_{K,L}^{(n+1),\alpha} = 0, \ \forall i = 1, \ldots, N_c$$

with

Φ_K	:	porous volume of the control volume K
$A_{K,i}$	:	accumulation of i in K per unit pore volume
$M_{K,L,i}^{\alpha}$	:	upstream mobility of i in α from K to L

and $F_{K,L}^{\alpha}$ the Darcy fluxes

$$F_{KL}^{\alpha} = -F_{LK}^{\alpha} \approx \int_{KL} \mathbf{V}^{\alpha} \cdot \mathbf{n}_{KL} d\sigma$$

où $\mathbf{V}^{\alpha} = -\Lambda \left(\nabla \left[P + P_{c,\alpha}(S) \right] - \rho_\alpha \mathbf{g} \right)$

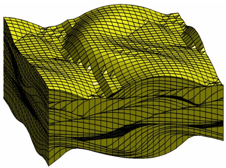
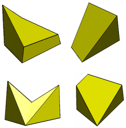
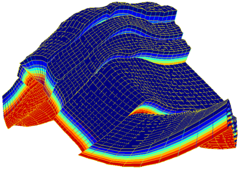
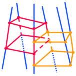
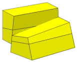
+ *closure equations* + *thermodynamical equilibrium* (phases can appear and disappear)

Which method for the approximation of F_{KL}^{α}?

Aim: the method should be ...

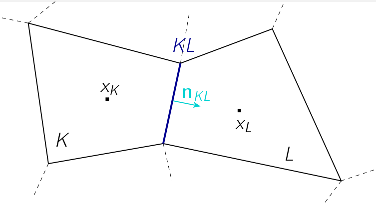
$\star$	conservative	$\star$
$\star$	accurate	$\star$
$\star$	compact	$\star$
$\star$	consistent	$\star$
$\star$	convergent	$\star$

... for **heterogenous anisotropic media** and **realistic meshes** ...

... *generalized polyhedral, non planar faces, faults, local grid refinement* ...

Industrial discretization of the Darcy fluxes
Cell Centered, Linear and Conservative



$$F_{KL} = -F_{LK} \approx \int_{KL} -\Lambda\, \nabla P \cdot \mathbf{n}_{KL} d\sigma$$

Two Point Flux Approximation (TPFA):

$$F_{KL} = \frac{\Lambda_{KL}|KL|}{d_{KL}} \, (P_K - P_L)\,,$$

conservative, cheap **but**
not accurate and convergent on realistic meshes

MultiPoint Flux Approximation schemes (MPFA):

$$F_{KL} = \sum_{M \in \mathcal{S}_{KL}} a_{KL}^M P_M,$$

conservative, consistent **but**
not always coercive and convergent

transmissivities $\left(a_{KL}^M\right)$ fonction of Λ and the mesh $\mathcal{M}$

$\Rightarrow$ transmissivities are computed by pre-processing

Today choices for the approximation of diffusive fluxes
An overview of the state of art...

- cell-centered, compact and linear schemes **but** not symmetric: conditionally coercive and convergent:

  MPFA O [Aavatsmark et al.] [Edwards et al.]
  MPFA L [Aavatsmark et al.] and G [Agélas et al.]
  Gradcell [Agélas et al.]

- symmetric, centered schemes **but** not compact (neighbours of the neighbours stencil)

  SUSHI [Eymard, Gallouet, Herbin]
  VFSym [Agélas, Di Pietro, Masson]

- symmetric, compact schemes **but** with additional unknowns at faces or vertices:

  VFH [Eymard, Gallouet, Herbin,...]
  MFD [Brezzi, Lipnikov,..]
  DDFV [Hermeline, Omnes, Boyer, Hubert, Coudière, ...]
  VAG - Vertex Approximate Gradient [Eymard, Guichard, Herbin]

Define a symmetric compact cell centered scheme is still an open problem

Continuous model problem of linear diffusion

$$\begin{cases} -\mathrm{div}(\Lambda\nabla\bar{u}) = f \text{ sur } \Omega \\ \bar{u} = 0 \text{ sur } \partial\Omega \end{cases}$$

Variational formulation :

$$\bar{u} \in H_0^1(\Omega),\ \forall \bar{v} \in H_0^1(\Omega),\ \int_\Omega \Lambda(\boldsymbol{x})\nabla\bar{u}(\boldsymbol{x}) \cdot \nabla\bar{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\bar{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

Nonconforming approximation :

$$u \in X_{\mathcal{D},0},\ \forall v \in X_{\mathcal{D},0},\ \int_\Omega \Lambda(\boldsymbol{x})\nabla_\mathcal{D} u(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\Pi_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

Definition of a gradient scheme : $\mathcal{D} = (X_{\mathcal{D},0},\ \Pi_\mathcal{D},\ \nabla_\mathcal{D})$

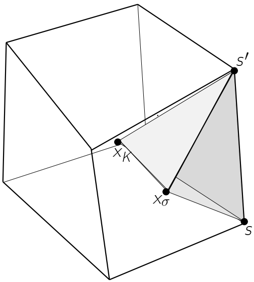
- ★ discrete space $X_{\mathcal{D},0} = \mathbb{R}^{\{d.o.f.\}}$ (suited to boundary conditions)
- ★ reconstruction of function $\Pi_\mathcal{D} \ :\ X_{\mathcal{D},0} \to L^2(\Omega)$ linear mapping
- ★ reconstruction of gradient $\nabla_\mathcal{D} \ :\ X_{\mathcal{D},0} \to L^2(\Omega)^d$ linear mapping

VAG scheme – definition of $\mathcal{D} = (X_{\mathcal{D},0},\ \Pi_{\mathcal{D}},\ \nabla_{\mathcal{D}})$

$X_{\mathcal{D},0} = \{$ discrete value u_K at the cell center $\boldsymbol{x}_K$ and u_s at the vertex $\boldsymbol{x}_s$ $\}$

- Barycentric cutting of a cell in tetrahedra

$$\mathbf{x}_\sigma = \sum_{s\in\mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma}\mathbf{x}_s, \quad u_\sigma = \sum_{s\in\mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma} u_s$$



- Local discrete gradient on each tetrahedra

$$\nabla_{K,\sigma,s,s'} u = \sum_{s\in\mathcal{V}_\sigma} (u_s - u_K) g^s_{K,\sigma,s,s'}$$

- Piecewise constant gradient in $L^2(\Omega)^d$

$$\nabla_{\mathcal{D}} u = \nabla_{K,\sigma,s,s'} u \ \text{ on each tetrahedra } x_K, x_\sigma, s, s'.$$

- Reconstruction operator

$$\Pi_{\mathcal{D}} u(\boldsymbol{x}) = u_K \text{ on } V_K\ ,\ u_s \text{ on } V_s$$

Define the volumes V_s, V_K at each vertex s and each cell K s.t. $\sum_{K\in\mathcal{M}} V_K + \sum_{s\in\mathcal{V}} V_s = \Omega$

VAG for multiphase flow – definition of the fluxes

$$\int_\Omega \Lambda\, \nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) \Pi_{\mathcal{D}}(v) \mathrm{d}\boldsymbol{x} \quad \forall v \in X_{\mathcal{D},0}$$

$\Longleftrightarrow$

$$\sum_{K\in\mathcal{M}} \sum_{s\in\mathcal{V}_K} F_{K,s}(u)\ (v_K - v_s) = \sum_{K\in\mathcal{M}} \int_{V_K} f(\boldsymbol{x}) v_K \mathrm{d}\boldsymbol{x} + \sum_{s\in\mathcal{V}} \int_{V_s} f(\boldsymbol{x}) v_s \mathrm{d}\boldsymbol{x} \qquad \forall v \in X_{\mathcal{D},0}$$

with the following linear and conservative fluxes

$$F_{K,s}(u) = -F_{s,K}(u) = \sum_{s'\in\mathcal{V}_K} A_K^{s,s'}\ (u_K - u_{s'})$$

which lead to the local mass balance equations

$$\sum_{s\in\mathcal{V}_K} F_{K,s}(u) = \int_{V_K} f(x)\ dx \text{ for all } K \in \mathcal{M},$$

$$\sum_{K\in\mathcal{M}_s} F_{K,s}(u) = \int_{V_s} f(x)\ dx \text{ for all } s \in \mathcal{V}$$

Advantages

- ⋆ similar to MPFA scheme in the new system of control volumes $\mathcal{M} \cup \mathcal{V}$
  $\Rightarrow$ easily usable in standard industrial codes without any change
- ⋆ cell-unknowns are eliminated in the linear system using a Schur complement
  $\Rightarrow$ compact vertex-centered scheme with a 27-points stencil on hexahedral grids
- ⋆ really efficient on tetrahedral grids $\boxed{\text{CPU O-scheme} = 15 \times \text{CPU VAG}}$ (3D)
  $\Rightarrow$ new perspectives to use tetrahedral/pyramidal grids for reservoir simulation

Difficulties

How treat the vertices which are at the interface of heterogeneities ?

- **Repartition of the volume between vertices and cells :**

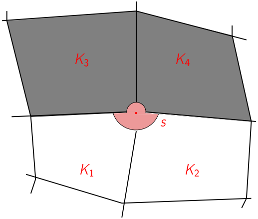
$\longrightarrow$ *Conservative redistribution of volume from the cells to the vertices*

$\longrightarrow$ *Volume has to be taken from the highest permeability cells around the vertex*

$\star$ Initial volume of K : $\widetilde{V}_K$

$\star$ Indicators of the transmissivity :

$$B_{K,s} = \sum_{s' \in \mathcal{V}_K} A_K^{s,s'} \rightsquigarrow \widetilde{B}_{K,s} = \frac{B_{K,s}}{\sum_{L \in \mathcal{M}_s} B_{L,s}}$$



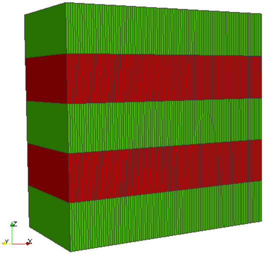
$\star$

$$\begin{cases} V_s(\omega) = \omega \sum_{K \in \mathcal{M}_s} \widetilde{B}_{K,s} \widetilde{V}_K & \text{for } s \in \mathcal{V} \\ V_K(\omega) = \widetilde{V}_K (1 - \omega \sum_{s \in \mathcal{V}_K} \widetilde{B}_{K,s}) & \text{for } K \in \mathcal{M}. \end{cases}$$

for a small $\omega > 0$ discussed in the numerical results

High heterogeneity and coarse grid

- Geometry : $[0, 100] \times [0, 50] \times [0, 100]\ m^3$
- Injection of immiscible gas in the water at $x = 0$
- Porous media heterogenous and isotropic
- Ratio of permeability between drains and barriers : 10^4
- Cartesian grid : $100 \times 1 \times 5$

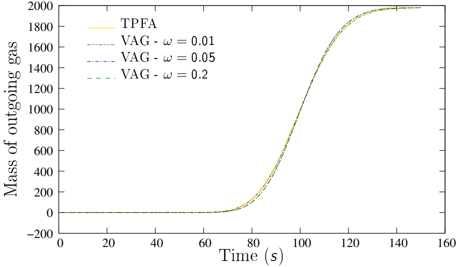


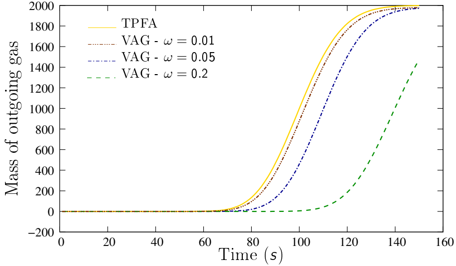
mesh and layers
drains
barriers

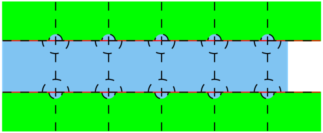
Study of the repartition of the pore volume – $V_s(\omega) = \omega \ \widetilde{V}_K \sum_{K \in \mathcal{M}_s} \widetilde{B}_{K,s} \quad s \in \mathcal{V}$

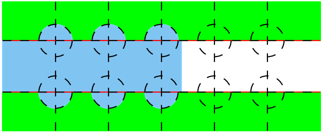
$\widetilde{B}_{K,s}$ relative weight of the transmissivity

$\widetilde{B}_{K,s} = \dfrac{1}{\mathrm{Card}\mathcal{M}_s}$ equidistributed

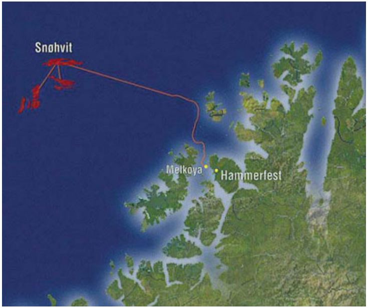


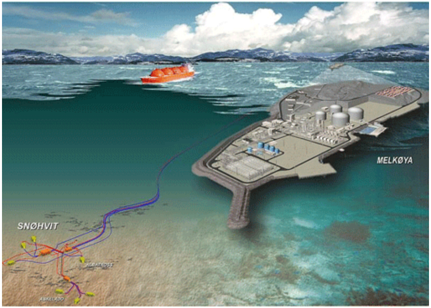






- Snohvit gas field contains 5 to 8 % of CO_2
- Reinjection of CO_2 in the saline aquifer Tubaen
- 700 000 tons reinjected since 2008
- Unexplained periodic loss of injectivity
- Assumption : near well drying and mineral salt precipitation
  $\Rightarrow$ alteration of the porosity





3 phases : water (w), gas (g), mineral (m) – **3 components :** $H2O$, $CO2$ and $salt$

$$\mathcal{C}^w = \{H2O, CO2, salt\}, \quad \mathcal{C}^g = \{H2O, CO2\}, \quad \mathcal{C}^m = \{salt\}$$

$$\left\{\begin{array}{l}
\partial_t \phi \; (\rho^w \; S^w \; C^w_{H2O} + \rho^g \; S^g \; C^g_{H20}) + \mathrm{div}\,(C^w_{H2O} \; \rho^w \; \mathbf{U}^w + C^g_{H2O} \; \rho^g \; \mathbf{U}^g) = 0, \\[2ex]
\partial_t \phi \; (\rho^w \; S^w \; C^w_{salt} + \rho^m \; S^m) + \mathrm{div}\,(C^w_{salt} \; \rho^w \; \mathbf{U}^w) = 0, \\[2ex]
\partial_t \phi \; (\rho^g \; S^g \; C^g_{CO2} + \rho^w \; S^w \; C^w_{CO2}) + \mathrm{div}\,(C^g_{CO2} \; \rho^g \; \mathbf{U}^g + C^w_{CO2} \; \rho^w \; \mathbf{U}^w) = 0, \\[2ex]
S^w + S^g + S^m = 1, \\
C^w_{H2O} + C^w_{CO2} + C^w_{salt} = 1 \;\; \text{if } w \text{ present}, \\
C^g_{H20} + C^g_{CO2} = 1 \;\; \text{if } g \text{ present} \\[2ex]
\mathbf{U}^g = -\frac{k_{r_g}(S)}{\mu^g} \Lambda \left(\nabla P^g - \rho^g \mathbf{g}\right), \\
\mathbf{U}^w = -\frac{k_{r_w}(S)}{\mu^w} \Lambda \left(\nabla \left[P^g + P_{c,w}(S)\right] - \rho^w \mathbf{g}\right).
\end{array}\right.$$

$$\{ \text{ phases in presence } \} = \text{Flash}(P, Z)$$

$$\begin{cases} C^w_{CO2} = K_{CO2}\ C^g_{CO2} & \text{if } w, g \text{ present} \\ C^g_{H2O} = K_{H20}\ C^w_{H2O} & \text{if } w, g \text{ present} \\ C^w_{salt} = K_{salt} & \text{if } w, m \text{ present} \end{cases}$$

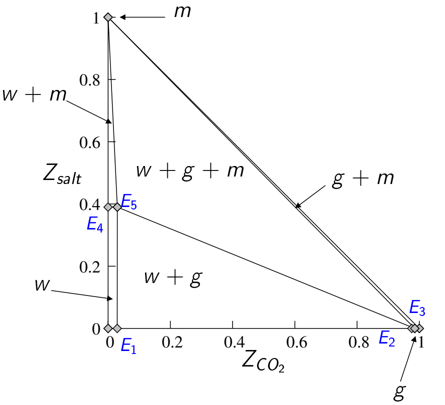
Phase diagram in the space Z at P fixed

Z_i : total molar fraction of the component i

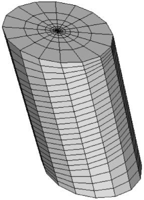
$i = CO2, H2O, salt$

$$(Z_{CO2}, Z_{salt})$$

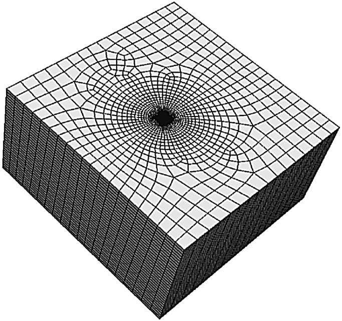
$$Z_{H2O} = 1 - Z_{CO2} - Z_{salt}$$



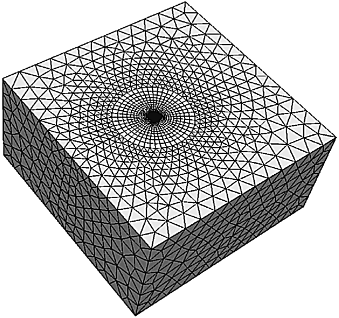
3D near well grids – deviated well



Radial mesh



Hexahedral grid



Hybrid grid

Influence of the parameter ω ?

Observations :

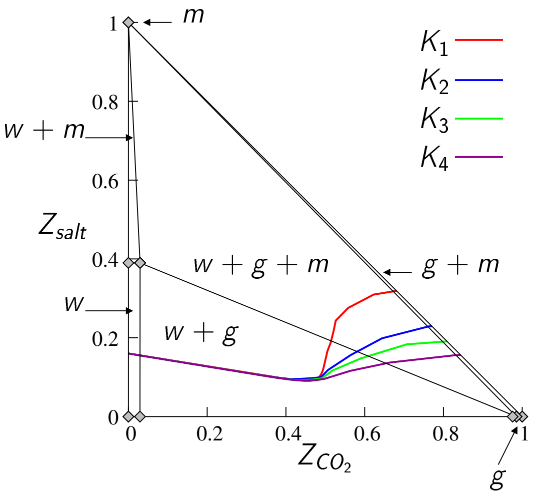
- Small influence on the accuracy of the numerical solution
- **But** very small pore volumes $\Rightarrow$ very small time steps
- Thus if ω is too small, the CPU time can increase
- Strategy: homogenization of the minimum pore volumes at vertices and cells

Compute ω_0 such that $\min_{K\in\mathcal{M}}\{\ V_K(\omega_0)\ \} = \min_{s\in\mathcal{V}}\{\ V_s(\omega_0)\ \}$

Data test case

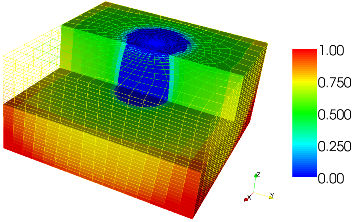
- Homogenous and isotropic porous media
- Injection of miscible gaseous $CO2$ such that $S^g = 1,\ C^g_{CO2} = 1$
- ingoing pressure boundary condition at the well bore P_{well} is imposed
- Aquifer initially composed of water $S_w = 1,\ C^w_{H2O} = 0.84,\ C^w_{sel} = 0.16$
- homogeneous Neumann boundary condition at the north and south faces, otherwise hydrostatic pressure boundary condition

Data are based on a comparison between
laboratory experiment vs. numerical simulation
(joint work of Roland Masson and Yannick Peysson)

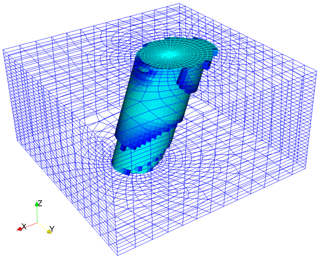


Trajectories in the space (Z_{CO_2}, Z_{sel}) of the Z_{K_i} for 4 cells K_i, $i = 1...4$, ordered according to their increasing distance to the well axis.

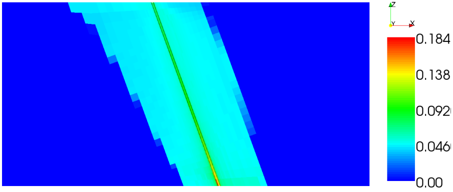
Results at the end of the simulation



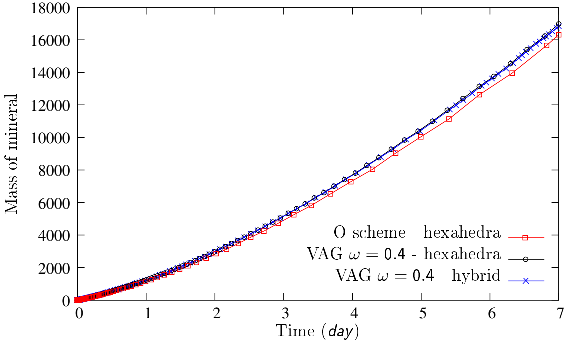
Water saturation S^w



Mineral saturation $|S^m > 0.1\%$



2D xz – Mineral saturation S^m



Variation of the mineral mass
inside the reservoir in function of time

- VAG scheme is an original **vertex centred finite volume** approach for compositional multiphase flows in porous media

- accurate on **coarse grid and high heterogeneities** thanks to
  – the conservation of the cell unknowns in the discretization
  – the original distribution of the porous volume at the vertices

- elimination of the cell unknowns in the linear system
  $\Rightarrow$ particularly efficient on tetrahedral and pyramidal grids
  $\Rightarrow$ open new perspectives for reservoir simulation

On going work... - Roland's talk

- **theoretical framework**
  - flux interpretation
  - convergence of the scheme whatever the value of ω

- improved treatment of the heterogeneities
  for example : **discontinuous capillary pressure field**