Discontinuous Galerkin Time-Domain method for solving Maxwell's equations on hybrid and non-conforming meshes

Clément Durochat

NACHOS project-team, INRIA Sophia Antipolis - Méditerranée, France

Clement.Durochat@inria.fr

Biot, Thursday 26th July 2012

Outline

Ω, bounded polyhedral domain of $\mathbb{R}^3$, boundary $\Gamma = \Gamma^a \cup \Gamma^m$;
the system of Maxwell's equation in three space dimensions is given by :

$$\boxed{\left\{\begin{array}{rcl} \varepsilon\dfrac{\partial \mathbf{E}}{\partial t} - \mathrm{curl}(\mathbf{H}) & = & 0, \\ \mu\dfrac{\partial \mathbf{H}}{\partial t} + \mathrm{curl}(\mathbf{E}) & = & 0, \end{array}\right.}$$

where :

- $\mathbf{E} \equiv {}^t(E_1(\mathbf{x},t), E_2(\mathbf{x},t), E_3(\mathbf{x},t))$ & $\mathbf{H} \equiv {}^t(H_1(\mathbf{x},t), H_2(\mathbf{x},t), H_3(\mathbf{x},t))$ are the electric field and the magnetic field
- $\varepsilon \equiv \varepsilon(\mathbf{x})$, $\mu \equiv \mu(\mathbf{x})$, are the electric permittivity and the magnetic permeability, respectively
- Metallic boundary condition on Γ^m : $\mathbf{n} \times \mathbf{E} = 0$ ($\mathbf{n}$ outwards normal to Γ)
  Silver-Mller boundary condition on Γ^a : $\mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\varepsilon}}\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = 0$
- Pseudo-conservative form : $Q(\partial_t \mathbf{W}) + \nabla \cdot F(\mathbf{W}) = 0$ $\quad (\mathbf{W} = {}^t(\mathbf{E}, \mathbf{H}) \in \mathbb{R}^6)$

Ω, bounded polyhedral domain of $\mathbb{R}^3$, boundary $\Gamma = \Gamma^a \cup \Gamma^m$;
the system of Maxwell's equation in three space dimensions is given by :

$$\boxed{\begin{cases} \varepsilon \dfrac{\partial \mathbf{E}}{\partial t} - \mathrm{curl}(\mathbf{H}) & = & 0, \\ \mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathrm{curl}(\mathbf{E}) & = & 0, \end{cases}}$$

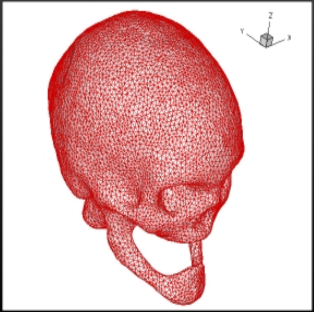
where :

- $\mathbf{E} \equiv {}^t(E_1(\mathbf{x},t), E_2(\mathbf{x},t), E_3(\mathbf{x},t))$ & $\mathbf{H} \equiv {}^t(H_1(\mathbf{x},t), H_2(\mathbf{x},t), H_3(\mathbf{x},t))$ are the electric field and the magnetic field
- $\varepsilon \equiv \varepsilon(\mathbf{x})$, $\mu \equiv \mu(\mathbf{x})$, are the electric permittivity and the magnetic permeability, respectively
- Metallic boundary condition on Γ^m : $\mathbf{n} \times \mathbf{E} = 0$ ($\mathbf{n}$ outwards normal to Γ)
  Silver-Mller boundary condition on Γ^a : $\mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\varepsilon}}\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = 0$
- Pseudo-conservative form : $Q(\partial_t \mathbf{W}) + \nabla \cdot F(\mathbf{W}) = 0 \quad (\mathbf{W} = {}^t(\mathbf{E},\mathbf{H}) \in \mathbb{R}^6)$

Outline

Objective : Formulate, study and validate a DGTD-$\mathbb{P}_p\mathbb{Q}_k$ method to solve Maxwell's equations :

- mesh objects with complex geometry by tetrahedra (triangles in 2D) for high precision
- mesh the surrounding space by square elements (large size) for simplicity and speed



- Ω is discretized by $\mathscr{C}_h = \bigcup_{i=1}^{N} c_i = \mathscr{T}_h \bigcup \mathscr{Q}_h$, where c_i are tetrahedra ($\in \mathscr{T}_h$) or hexahedra ($\in \mathscr{Q}_h$) in 3D (triangles or quadrangles in 2D)
- For theoritical aspects we consider only metallic boundaries
- $\mathbb{P}_p[c_i]$ the space of polynomial functions with degree at most p on $c_i \in \mathscr{T}_h$ (ex : form of polynomials $\mathbb{P}_1$ in 2D : $\xi_0 + \xi_1 x_1 + \xi_2 x_2$),
  $\mathbb{Q}_k[c_i]$ the space of polynomial functions with degree at most k with respect to each variable separately on $c_i \in \mathscr{Q}_h$ (ex : form of polynomials $\mathbb{Q}_1$ in 2D : $\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2$)
- $\phi_i = (\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{id_i})$ local basis of $\mathbb{P}_p[c_i]$
  $\theta_i = (\vartheta_{i1}, \vartheta_{i2}, \dots, \vartheta_{ib_i})$ local basis of $\mathbb{Q}_k[c_i]$
- The discrete solution vector $\mathbf{W}_h$ is searched for in the approximation space V_h^6 defined by :

$$V_h = \left\{ v_h \in L^2(\Omega) \;\middle|\; \begin{array}{l} \forall c_i \in \mathscr{T}_h,\ \ v_{h|c_i} \in \mathbb{P}_p[c_i] \\ \forall c_i \in \mathscr{Q}_h,\ \ v_{h|c_i} \in \mathbb{Q}_k[c_i] \end{array} \right\}$$

- Ω is discretized by $\mathscr{C}_h = \bigcup_{i=1}^{N} c_i = \mathscr{T}_h \bigcup \mathscr{Q}_h$, where c_i are tetrahedra ($\in \mathscr{T}_h$) or hexahedra ($\in \mathscr{Q}_h$) in 3D (triangles or quadrangles in 2D)

- For theoritical aspects we consider only metallic boundaries

- $\mathbb{P}_p[c_i]$ the space of polynomial functions with degree at most p on $c_i \in \mathscr{T}_h$ (ex : form of polynomials $\mathbb{P}_1$ in 2D : $\xi_0 + \xi_1 x_1 + \xi_2 x_2$),
  $\mathbb{Q}_k[c_i]$ the space of polynomial functions with degree at most k with respect to each variable separately on $c_i \in \mathscr{Q}_h$ (ex : form of polynomials $\mathbb{Q}_1$ in 2D : $\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2$)

- $\phi_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{id_i})$ local basis of $\mathbb{P}_p[c_i]$
  $\theta_i = (\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{ib_i})$ local basis of $\mathbb{Q}_k[c_i]$

- The discrete solution vector $\mathbf{W}_h$ is searched for in the approximation space V_h^6 defined by :

$$V_h = \left\{ v_h \in L^2(\Omega) \;\middle|\; \begin{array}{l} \forall c_i \in \mathscr{T}_h, \;\; v_{h\,|c_i} \in \mathbb{P}_p[c_i] \\ \forall c_i \in \mathscr{Q}_h, \;\; v_{h\,|c_i} \in \mathbb{Q}_k[c_i] \end{array} \right\}$$

- Local degrees of freedom denoted by $\mathbf{W}_{il} \in \mathbb{R}^6$
- $\mathbf{W}_i$ defines the restriction of the approximate solution to the cell c_i $(\mathbf{W}_h|_{c_i})$
- $$c_i \in \mathscr{T}_h \Longrightarrow \mathbf{W}_i \in \mathbb{P}_p[c_i] : \mathbf{W}_i(\mathbf{x}) = \sum_{l=1}^{d_i} \mathbf{W}_{il}\varphi_{il}(\mathbf{x}) \in \mathbb{R}^6$$

$$c_i \in \mathscr{Q}_h \Longrightarrow \mathbf{W}_i \in \mathbb{Q}_k[c_i] : \mathbf{W}_i(\mathbf{x}) = \sum_{l=1}^{b_i} \mathbf{W}_{il}\vartheta_{il}(\mathbf{x}) \in \mathbb{R}^6$$

- The local representation of $\mathbf{W}$ does not provide any form of continuity from one element to another. We use a centered numerical flux on $a_{ij} = c_i \cap c_j$

$$\mathbf{W}_h|_{a_{ij}} = \frac{\mathbf{W}_i|_{a_{ij}} + \mathbf{W}_j|_{a_{ij}}}{2}$$

If a_{ij} on the metallic boundary : ${}^t(\mathbf{E}_j, \mathbf{H}_j) = {}^t(-\mathbf{E}_i, \mathbf{H}_i)$

- Local degrees of freedom denoted by $\mathbf{W}_{il} \in \mathbb{R}^6$
- $\mathbf{W}_i$ defines the restriction of the approximate solution to the cell c_i $(\mathbf{W}_h|_{c_i})$
- $$c_i \in \mathscr{T}_h \Longrightarrow \mathbf{W}_i \in \mathbb{P}_p[c_i] : \mathbf{W}_i(\mathbf{x}) = \sum_{l=1}^{d_i} \mathbf{W}_{il}\varphi_{il}(\mathbf{x}) \in \mathbb{R}^6$$

  $$c_i \in \mathscr{Q}_h \Longrightarrow \mathbf{W}_i \in \mathbb{Q}_k[c_i] : \mathbf{W}_i(\mathbf{x}) = \sum_{l=1}^{b_i} \mathbf{W}_{il}\vartheta_{il}(\mathbf{x}) \in \mathbb{R}^6$$
- The local representation of $\mathbf{W}$ does not provide any form of continuity from one element to another. We use a centered numerical flux on $a_{ij} = c_i \cap c_j$

$$\mathbf{W}_h|_{a_{ij}} = \frac{\mathbf{W}_i|_{a_{ij}} + \mathbf{W}_j|_{a_{ij}}}{2}$$

If a_{ij} on the metallic boundary : ${}^t(\mathbf{E}_j, \mathbf{H}_j) = {}^t(-\mathbf{E}_i, \mathbf{H}_i)$

Case (A) :

c_i is a tetrahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another tetrahedron, **or** to a hexahedron (**hybrid**)

6d$_\mathbf{i}$ semi-discretized equations system :

$$\boxed{\begin{cases} 2\mathcal{X}_{\varepsilon,i}\dfrac{d\overline{\mathbf{E}}_i}{dt} + \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{H}}_j + \sum_{a_{ij}\in\mathcal{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{H}}_j = 0, \\ 2\mathcal{X}_{\mu,i}\dfrac{d\overline{\mathbf{H}}_i}{dt} - \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{E}}_j + \sum_{a_{ij}\in\mathcal{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{E}}_j = 0, \end{cases}}$$

with :

- $\overline{\mathbf{E}}_i = {}^t(\mathbf{E}_{i1}, \mathbf{E}_{i2}, \cdots, \mathbf{E}_{id_i})$ and $\overline{\mathbf{H}}_i = {}^t(\mathbf{H}_{i1}, \mathbf{H}_{i2}, \cdots, \mathbf{H}_{id_i}) \in \mathbb{R}^{3d_i}$
- $\widetilde{\mathbf{E}}_j = {}^t(\mathbf{E}_{j1}, \mathbf{E}_{j2}, \cdots, \mathbf{E}_{jb_j})$ and $\widetilde{\mathbf{H}}_j = {}^t(\mathbf{H}_{j1}, \mathbf{H}_{j2}, \cdots, \mathbf{H}_{jb_j}) \in \mathbb{R}^{3b_j}$
- $\mathcal{X}_{\varepsilon,i}$ and $\mathcal{X}_{\mu,i}$ are mass matrices, $\mathcal{X}_i^{x_k}$ gradient matrix, $\mathcal{X}_{ij}$ surface matrix $\Longrightarrow$ All have a $3d_i \times 3d_i$ size, **except** $\mathcal{A}_{ij}$, whose size is $3d_i \times 3b_j$

Case (A) :

c_i is a tetrahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another tetrahedron, **or** to a hexahedron (**hybrid**)

6d$_i$ semi-discretized equations system :

$$
\boxed{\left\{\begin{array}{l}
2\mathcal{X}_{\varepsilon,i}\dfrac{d\overline{\mathbf{E}}_i}{dt} + \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathscr{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{H}}_j + \sum_{a_{ij}\in\mathscr{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathscr{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{H}}_j = 0, \\
2\mathcal{X}_{\mu,i}\dfrac{d\overline{\mathbf{H}}_i}{dt} - \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathscr{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{E}}_j + \sum_{a_{ij}\in\mathscr{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathscr{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{E}}_j = 0,
\end{array}\right.}
$$

with :

- $\overline{\mathbf{E}}_i = {}^t(\mathbf{E}_{i1}, \mathbf{E}_{i2}, \cdots, \mathbf{E}_{id_i})$ and $\overline{\mathbf{H}}_i = {}^t(\mathbf{H}_{i1}, \mathbf{H}_{i2}, \cdots, \mathbf{H}_{id_i}) \in \mathbb{R}^{3d_i}$
- $\widetilde{\mathbf{E}}_j = {}^t(\mathbf{E}_{j1}, \mathbf{E}_{j2}, \cdots, \mathbf{E}_{jb_j})$ and $\widetilde{\mathbf{H}}_j = {}^t(\mathbf{H}_{j1}, \mathbf{H}_{j2}, \cdots, \mathbf{H}_{jb_j}) \in \mathbb{R}^{3b_j}$
- $\mathcal{X}_{\varepsilon,i}$ and $\mathcal{X}_{\mu,i}$ are mass matrices, $\mathcal{X}_i^{x_k}$ gradient matrix, $\mathcal{X}_{ij}$ surface matrix $\Longrightarrow$ All have a $3d_i \times 3d_i$ size, **except** $\mathcal{A}_{ij}$, whose size is $3d_i \times 3b_j$

Case (B) :

c_i is an hexahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another hexahedron, **or** to a tetrahedron (**hybrid**)

$6b_i$ semi-discretized equations system :

$$\begin{cases} 2\mathcal{W}_{\varepsilon,i}\dfrac{d\widetilde{\mathbf{E}}_i}{dt} + \displaystyle\sum_{k=1}^{3} \mathcal{W}_i^{x_k}\widetilde{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{Q}_d^i} \mathcal{W}_{ij}\widetilde{\mathbf{H}}_j + \sum_{a_{ij}\in\mathcal{Q}_m^i} \mathcal{W}_{im}\widetilde{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{H}_d^i} \mathcal{B}_{ij}\overline{\mathbf{H}}_j = 0, \\ 2\mathcal{W}_{\mu,i}\dfrac{d\widetilde{\mathbf{H}}_i}{dt} - \displaystyle\sum_{k=1}^{3} \mathcal{W}_i^{x_k}\widetilde{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{Q}_d^i} \mathcal{W}_{ij}\widetilde{\mathbf{E}}_j + \sum_{a_{ij}\in\mathcal{Q}_m^i} \mathcal{W}_{im}\widetilde{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{H}_d^i} \mathcal{B}_{ij}\overline{\mathbf{E}}_j = 0, \end{cases}$$

with :

- $\widetilde{\mathbf{E}}_i = {}^t(\mathbf{E}_{i1}, \mathbf{E}_{i2}, \cdots, \mathbf{E}_{ib_i})$ and $\widetilde{\mathbf{H}}_i = {}^t(\mathbf{H}_{i1}, \mathbf{H}_{i2}, \cdots, \mathbf{H}_{ib_i}) \in \mathbb{R}^{3b_i}$
- $\overline{\mathbf{E}}_j = {}^t(\mathbf{E}_{j1}, \mathbf{E}_{j2}, \cdots, \mathbf{E}_{jd_j})$ and $\overline{\mathbf{H}}_j = {}^t(\mathbf{H}_{j1}, \mathbf{H}_{j2}, \cdots, \mathbf{H}_{jd_j}) \in \mathbb{R}^{3d_j}$
- $\mathcal{W}_{\varepsilon,i}$ and $\mathcal{W}_{\mu,i}$ are mass matrices, $\mathcal{W}_i^{x_k}$ gradient matrix, $\mathcal{W}_{ij}$ surface matrix $\Longrightarrow$ All have a $3b_i \times 3b_i$ size, **except** $\mathcal{B}_{ij}$, whose size is $3b_i \times 3d_j$

Case (B) :

c_i is an hexahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another hexahedron, **or** to a tetrahedron (**hybrid**)

6b$_i$ semi-discretized equations system :

$$\boxed{\begin{cases} 2\mathcal{W}_{\varepsilon,i}\dfrac{d\widetilde{\mathbf{E}}_i}{dt} + \displaystyle\sum_{k=1}^{3}\mathcal{W}_i^{x_k}\widetilde{\mathbf{H}}_i + \sum_{a_{ij}\in\mathscr{Q}_d^i}\mathcal{W}_{ij}\widetilde{\mathbf{H}}_j + \sum_{a_{ij}\in\mathscr{Q}_m^i}\mathcal{W}_{im}\widetilde{\mathbf{H}}_i + \sum_{a_{ij}\in\mathscr{H}_d^i}\mathcal{B}_{ij}\overline{\mathbf{H}}_j = 0, \\ 2\mathcal{W}_{\mu,i}\dfrac{d\widetilde{\mathbf{H}}_i}{dt} - \displaystyle\sum_{k=1}^{3}\mathcal{W}_i^{x_k}\widetilde{\mathbf{E}}_i - \sum_{a_{ij}\in\mathscr{Q}_d^i}\mathcal{W}_{ij}\widetilde{\mathbf{E}}_j + \sum_{a_{ij}\in\mathscr{Q}_m^i}\mathcal{W}_{im}\widetilde{\mathbf{E}}_i - \sum_{a_{ij}\in\mathscr{H}_d^i}\mathcal{B}_{ij}\overline{\mathbf{E}}_j = 0, \end{cases}}$$

with :

- $\widetilde{\mathbf{E}}_i = {}^t(\mathbf{E}_{i1}, \mathbf{E}_{i2}, \cdots, \mathbf{E}_{ib_i})$ and $\widetilde{\mathbf{H}}_i = {}^t(\mathbf{H}_{i1}, \mathbf{H}_{i2}, \cdots, \mathbf{H}_{ib_i}) \in \mathbb{R}^{3b_i}$
- $\overline{\mathbf{E}}_j = {}^t(\mathbf{E}_{j1}, \mathbf{E}_{j2}, \cdots, \mathbf{E}_{jd_j})$ and $\overline{\mathbf{H}}_j = {}^t(\mathbf{H}_{j1}, \mathbf{H}_{j2}, \cdots, \mathbf{H}_{jd_j}) \in \mathbb{R}^{3d_j}$
- $\mathcal{W}_{\varepsilon,i}$ and $\mathcal{W}_{\mu,i}$ are mass matrices, $\mathcal{W}_i^{x_k}$ gradient matrix, $\mathcal{W}_{ij}$ surface matrix $\Longrightarrow$ All have a $3b_i \times 3b_i$ size, **except** $\mathcal{B}_{ij}$, whose size is $3b_i \times 3d_j$

Second order Leap-Frog scheme :

- **Case (A)** :

$$\begin{cases} \overline{\mathbf{H}}_i^{n+\frac{1}{2}} = \overline{\mathbf{H}}_i^{n-\frac{1}{2}} + \dfrac{\Delta t}{2} \left[\mathcal{X}_{\mu,i}\right]^{-1} \mathbf{A}_{\mathbf{E},i}^{n}, \\ \overline{\mathbf{E}}_i^{n+1} = \overline{\mathbf{E}}_i^{n} + \dfrac{\Delta t}{2} \left[\mathcal{X}_{\varepsilon,i}\right]^{-1} \mathbf{A}_{\mathbf{H},i}^{n+\frac{1}{2}} \end{cases}$$

- **Case (B)** :

$$\begin{cases} \widetilde{\mathbf{H}}_i^{n+\frac{1}{2}} = \widetilde{\mathbf{H}}_i^{n-\frac{1}{2}} + \dfrac{\Delta t}{2} \left[\mathcal{W}_{\mu,i}\right]^{-1} \mathbf{B}_{\mathbf{E},i}^{n}, \\ \widetilde{\mathbf{E}}_i^{n+1} = \widetilde{\mathbf{E}}_i^{n} + \dfrac{\Delta t}{2} \left[\mathcal{W}_{\varepsilon,i}\right]^{-1} \mathbf{B}_{\mathbf{H},i}^{n+\frac{1}{2}} \end{cases}$$

Outline

- We define a discrete energy $\mathfrak{E}^n$
- We assume that this is an energy and we check that it is exactly conserved, i.e. $\Delta\mathfrak{E} = \mathfrak{E}^{n+1} - \mathfrak{E}^n = 0$
- We make hypothesis for fields in to prove that $\mathfrak{E}^n$ is a positive definite quadratic form under a CFL condition :

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathrm{rot}(\mathbf{X})\|_{c_i} \leq \left(\alpha_i^{\mathcal{T}} p_i \|\mathbf{X}\|_{c_i}\right)/|c_i|,$$

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathbf{X}\|_{a_{ij}}^2 \leq \left(\beta_{ij}^{\mathcal{T}} \|\mathbf{n}_{ij}\| \|\mathbf{X}\|_{c_i}^2\right)/|c_i|$$

where $\alpha_i^{\mathcal{T}}$ and $\beta_{ij}^{\mathcal{T}}$ $(j \in \{j | c_i \cap c_j \neq \varnothing\})$ defining the constant parameters

- We also admit similar hypothesis $\forall \mathbf{X} \in \left(\mathbb{Q}_k[c_i]\right)^3$ with constants α_i^q and β_{ij}^q
- $\|.\|_{c_i}$ and $\|.\|_{a_{ij}}$ are L^2-norm. $\|\mathbf{n}_{ij}\| = \int_{a_{ij}} 1 d\sigma$ with $\mathbf{n}_{ij}$ non-unitary normal to a_{ij} oriented from c_i towards c_j. $|c_i| = \int_{c_i} 1 d\mathbf{x}$ and $p_i = \sum_{j \in \mathcal{V}_i} \|\mathbf{n}_{ij}\|$

- We define a discrete energy $\mathfrak{E}^n$
- We assume that this is an energy and we check that it is exactly conserved, i.e. $\Delta\mathfrak{E} = \mathfrak{E}^{n+1} - \mathfrak{E}^n = 0$
- We make hypothesis for fields in to prove that $\mathfrak{E}^n$ is a positive definite quadratic form under a CFL condition :

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathrm{rot}(\mathbf{X})\|_{c_i} \leq \left(\alpha_i^{\mathcal{T}} p_i \|\mathbf{X}\|_{c_i}\right) / |c_i|,$$

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathbf{X}\|_{a_{ij}}^2 \leq \left(\beta_{ij}^{\mathcal{T}} \|\mathbf{n}_{ij}\| \|\mathbf{X}\|_{c_i}^2\right) / |c_i|$$

  where $\alpha_i^{\mathcal{T}}$ and $\beta_{ij}^{\mathcal{T}}$ $(j \in \{j | c_i \cap c_j \neq \varnothing\})$ defining the constant parameters
- We also admit similar hypothesis $\forall \mathbf{X} \in \left(\mathbb{Q}_k[c_i]\right)^3$ with constants α_i^q and β_{ij}^q
- $\|.\|_{c_i}$ and $\|.\|_{a_{ij}}$ are L^2-norm. $\|\mathbf{n}_{ij}\| = \int_{a_{ij}} 1 d\sigma$ with $\mathbf{n}_{ij}$ non-unitary normal to a_{ij} oriented from c_i towards c_j. $|c_i| = \int_{c_i} 1 d\mathbf{x}$ and $p_i = \sum_{j \in \mathcal{V}_i} \|\mathbf{n}_{ij}\|$

- For the DGTD-$\mathbb{P}_p$ method, the sufficient condition on Δt_τ is [1] :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_\tau \left[2\alpha_i^\tau + \beta_{ij}^\tau \max\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\varepsilon_i \mu_i}}{p_i}$$

- For DGTD-$\mathbb{Q}_k$ method, the sufficient condition on Δt_q is :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_q \left[2\alpha_i^q + \beta_{ij}^q \max\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\varepsilon_i \mu_i}}{p_i}$$

Finally, noting Δt the global time step for the hybrid method, we have shown that the sufficient stability condition is defined by :

$$\boxed{\Delta t = \min(\Delta t_\tau, \Delta t_q)}$$

Under this condition and hypothesis, $\mathfrak{E}^n$ is a positive definite quadratic form

[1] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno
Convergence and stability of a discontinuous Galerkin time-domain method for the heterogeneous Maxwell equations on unstructured meshes
ESAIM : Math. Model. and Numer. Anal. 39, no. 6, p. 1149-1176 (2005)

- For the DGTD-$\mathbb{P}_p$ method, the sufficient condition on Δt_τ is [1] :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_\tau \left[2\alpha_i^\tau + \beta_{ij}^\tau \max\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\varepsilon_i \mu_i}}{p_i}$$

- For DGTD-$\mathbb{Q}_k$ method, the sufficient condition on Δt_q is :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_q \left[2\alpha_i^q + \beta_{ij}^q \max\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\varepsilon_i \mu_i}}{p_i}$$

Finally, noting Δt the global time step for the hybrid method, we have shown that the sufficient stability condition is defined by :

$$\boxed{\Delta t = \min(\Delta t_\tau, \Delta t_q)}$$

Under this condition and hypothesis, $\mathfrak{E}^n$ is a positive definite quadratic form

[1] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno
Convergence and stability of a discontinuous Galerkin time-domain method for the heterogeneous Maxwell equations on unstructured meshes
ESAIM : Math. Model. and Numer. Anal. 39, no. 6, p. 1149-1176 (2005)

- Two cases for weak formulations ($c_i \in \mathscr{T}_h$ / $c_i \in \mathscr{Q}_h$), vector test functions

- $$\begin{cases} m(\mathbf{T},\mathbf{T}') &= 2\displaystyle\int_\Omega \langle Q\mathbf{T} \ , \ \mathbf{T}' \rangle dx \\ a(\mathbf{T},\mathbf{T}') &= \displaystyle\int_\Omega \left(\left\langle \sum_{k=1}^{3} \partial^h_{x_k} \mathcal{O}^k \mathbf{T} \ , \ \mathbf{T}' \right\rangle - \sum_{k=1}^{3} \langle \partial^h_{x_k} \mathbf{T}' \ , \ \mathcal{O}^k \mathbf{T} \rangle \right) dx \\ b(\mathbf{T},\mathbf{T}') &= \displaystyle\int_{\mathscr{F}_d} \big(\langle \{\mathbf{V}\} \ , \ [\![\mathbf{U}']\!] \rangle - \langle \{\mathbf{U}\} \ , \ [\![\mathbf{V}']\!] \rangle - \\ & \qquad \langle \{\mathbf{V}'\} \ , \ [\![\mathbf{U}]\!] \rangle + \langle \{\mathbf{U}'\} \ , \ [\![\mathbf{V}]\!] \rangle \big) \ d\sigma \ + \\ & \quad \displaystyle\int_{\mathscr{F}_m} \big(\langle \mathbf{U} \ , \ \check{\mathbf{n}} \times \mathbf{V}' \rangle + \langle \mathbf{V} \ , \ \check{\mathbf{n}} \times \mathbf{U}' \rangle \big) \ d\sigma \end{cases}$$

  with :
  - $\mathbf{T} = {}^t(\mathbf{U},\mathbf{V}), \quad \mathbf{T}' = {}^t(\mathbf{U}',\mathbf{V}')$
  - $[\![\mathbf{U}_h]\!]_{ij} = \left(\mathbf{U}_j|_{a_{ij}} - \mathbf{U}_i|_{a_{ij}} \right) \times \check{\mathbf{n}}_{ij}, \ \ \{\mathbf{U}_h\}_{ij} = \dfrac{\mathbf{U}_i|_{a_{ij}} + \mathbf{U}_j|_{a_{ij}}}{2}$
  - $\mathscr{F}_d$ set of internal faces, $\mathscr{F}_m$ set of metallic boundary faces

- Two cases for weak formulations ($c_i \in \mathscr{T}_h$ / $c_i \in \mathscr{Q}_h$), vector test functions
- $$\left\{ \begin{aligned} m(\mathbf{T},\mathbf{T}') &= 2\int_\Omega \langle Q\mathbf{T} \,,\, \mathbf{T}'\rangle d\mathbf{x} \\ a(\mathbf{T},\mathbf{T}') &= \int_\Omega \left(\left\langle \sum_{k=1}^{3} \partial^h_{x_k} \mathcal{O}^k \mathbf{T} \,,\, \mathbf{T}' \right\rangle - \sum_{k=1}^{3} \langle \partial^h_{x_k} \mathbf{T}' \,,\, \mathcal{O}^k \mathbf{T} \rangle \right) d\mathbf{x} \\ b(\mathbf{T},\mathbf{T}') &= \int_{\mathscr{F}_d} \Big(\langle \{\mathbf{V}\} \,,\, [\![\mathbf{U}']\!] \rangle - \langle \{\mathbf{U}\} \,,\, [\![\mathbf{V}']\!] \rangle - \\ &\qquad \langle \{\mathbf{V}'\} \,,\, [\![\mathbf{U}]\!] \rangle + \langle \{\mathbf{U}'\} \,,\, [\![\mathbf{V}]\!] \rangle \Big) \, d\sigma \, + \\ &\quad \int_{\mathscr{F}_m} \Big(\langle \mathbf{U} \,,\, \check{\mathbf{n}} \times \mathbf{V}' \rangle + \langle \mathbf{V} \,,\, \check{\mathbf{n}} \times \mathbf{U}' \rangle \Big) \, d\sigma \end{aligned} \right.$$

with :

- $\mathbf{T} = {}^t(\mathbf{U},\mathbf{V})$, $\quad \mathbf{T}' = {}^t(\mathbf{U}',\mathbf{V}')$
- $[\![\mathbf{U}_h]\!]_{ij} = \left(\mathbf{U}_j|_{a_{ij}} - \mathbf{U}_i|_{a_{ij}} \right) \times \check{\mathbf{n}}_{ij}$, $\;\{\mathbf{U}_h\}_{ij} = \dfrac{\mathbf{U}_i|_{a_{ij}} + \mathbf{U}_j|_{a_{ij}}}{2}$
- $\mathscr{F}_d$ set of internal faces, $\quad \mathscr{F}_m$ set of metallic boundary faces

- Summing up weak formulations on each c_i, the discrete solution $\mathbf{W}_h = {}^t(\mathbf{E}_h, \mathbf{H}_h)$ satisfies :

$$m(\partial_t \mathbf{W}_h, \mathbf{T}') + a(\mathbf{W}_h, \mathbf{T}') + b(\mathbf{W}_h, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- We assume that the exact solution $\mathbf{W} = {}^t(\mathbf{E}, \mathbf{H}) \in (H(\text{curl}, \Omega))^6$. Using the continuity of the tangential traces of $\mathbf{E}$ and $\mathbf{H}$ accross $a_{ij} \in \mathscr{F}_d$, and the metallic boundary condition $\mathbf{E} \times \check{\mathbf{n}} = 0$ on $a_{ij} \in \mathscr{F}_m$, we prove :

$$m(\partial_t \mathbf{W}, \mathbf{T}') + a(\mathbf{W}, \mathbf{T}') + b(\mathbf{W}, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- Let $\mathbf{W}_h \in \mathcal{C}^1([0, t_f]; V_h^6)$ and let $\mathbf{W} \in \mathcal{C}^0([0, t_f]; (PH^{s+1}(\Omega))^6)$ for $s \leq 0$ with t_f the final time and :

$$PH^{s+1}(\Omega) = \{v \mid \forall j, v_{|\Omega_j} \in H^{s+1}(\Omega_j)\}$$

- Let $h_\tau = \max_{\tau_i \in \mathscr{T}_h}(h_{\tau_i}), \ h_q = \max_{q_i \in \mathscr{Q}_h}(h_{q_i})$ and :

$$\eta_h = \max\left\{h_\tau^{\min\{s,p\}}, h_q^{\min\{s,k\}}\right\}$$

- Summing up weak formulations on each c_i, the discrete solution $\mathbf{W}_h = {}^t(\mathbf{E}_h, \mathbf{H}_h)$ satisfies :

$$m(\partial_t \mathbf{W}_h, \mathbf{T}') + a(\mathbf{W}_h, \mathbf{T}') + b(\mathbf{W}_h, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- We assume that the exact solution $\mathbf{W} = {}^t(\mathbf{E}, \mathbf{H}) \in (H(\mathrm{curl}, \Omega))^6$. Using the continuity of the tangential traces of $\mathbf{E}$ and $\mathbf{H}$ accross $a_{ij} \in \mathscr{F}_d$, and the metallic boundary condition $\mathbf{E} \times \check{\mathbf{n}} = 0$ on $a_{ij} \in \mathscr{F}_m$, we prove :

$$m(\partial_t \mathbf{W}, \mathbf{T}') + a(\mathbf{W}, \mathbf{T}') + b(\mathbf{W}, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- Let $\mathbf{W}_h \in \mathcal{C}^1([0, t_f]; V_h^6)$ and let $\mathbf{W} \in \mathcal{C}^0([0, t_f]; (PH^{s+1}(\Omega))^6)$ for $s \leq 0$ with t_f the final time and :

$$PH^{s+1}(\Omega) = \{v \mid \forall j, v_{|\Omega_j} \in H^{s+1}(\Omega_j)\}$$

- Let $h_\tau = \max_{\tau_i \in \mathscr{T}_h}(h_{\tau_i}), \ h_q = \max_{q_i \in \mathscr{Q}_h}(h_{q_i})$ and :

$$\eta_h = \max\left\{ h_\tau^{\min\{s,p\}}, h_q^{\min\{s,k\}} \right\}$$

- Then there is a constant $C > 0$ independent of h such that :

$$\max_{t\in[0,t_f]} \left(\|P_h(\mathbf{W}(t)) - \mathbf{W}_h(t)\|_{0,\Omega}\right) \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}$$

- For the semi-discretized problem, the error $\mathbf{w} = \mathbf{W} - \mathbf{W}_h$ satisfies the estimate :

$$\boxed{\|\mathbf{w}\|_{\mathcal{C}^0([0,t_f],L^2(\Omega))} \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}}$$

- The fully discretized scheme may be seen as the discretization in time of a system of ODE
- Since the Leap-Frog scheme is second-order accurate, we found the consistency error altogether of order $\mathcal{O}(\Delta t^2)$
- Finally, together with the stability result we thus get an error of order (if the exact solution is regular enough) :

$$\boxed{\mathcal{O}(\Delta t^2) + \mathcal{O}(t_f \eta_h)}$$

- Then there is a constant $C > 0$ independent of h such that :

$$\max_{t\in[0,t_f]} \left(\|P_h(\mathbf{W}(t)) - \mathbf{W}_h(t)\|_{0,\Omega}\right) \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}$$

- For the semi-discretized problem, the error $\mathbf{w} = \mathbf{W} - \mathbf{W}_h$ satisfies the estimate :

$$\boxed{\|\mathbf{w}\|_{\mathcal{C}^0([0,t_f],L^2(\Omega))} \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}}$$

- The fully discretized scheme may be seen as the discretization in time of a system of ODE
- Since the Leap-Frog scheme is second-order accurate, we found the consistency error altogether of order $\mathcal{O}(\Delta t^2)$

- Finally, together with the stability result we thus get an error of order (if the exact solution is regular enough) :

$$\boxed{\mathcal{O}(\Delta t^2) + \mathcal{O}(t_f \eta_h)}$$

- Then there is a constant $C > 0$ independent of h such that :

$$\max_{t\in[0,t_f]} (\|P_h(\mathbf{W}(t)) - \mathbf{W}_h(t)\|_{0,\Omega}) \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}$$

- For the semi-discretized problem, the error $\mathbf{w} = \mathbf{W} - \mathbf{W}_h$ satisfies the estimate :

$$\boxed{\|\mathbf{w}\|_{\mathcal{C}^0([0,t_f],L^2(\Omega))} \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}}$$

- The fully discretized scheme may be seen as the discretization in time of a system of ODE
- Since the Leap-Frog scheme is second-order accurate, we found the consistency error altogether of order $\mathcal{O}(\Delta t^2)$
- Finally, together with the stability result we thus get an error of order (if the exact solution is regular enough) :

$$\boxed{\mathcal{O}(\Delta t^2) + \mathcal{O}(t_f \eta_h)}$$

Outline

- 2D transverse magnetic waves (TM_z) : $\mathbf{H} \equiv {}^t(H_x, H_y, 0)$ et $\mathbf{E} \equiv {}^t(0, 0, E_z)$
- 2D Maxwell's equations are given by :

$$\left\{ \begin{aligned} \epsilon \frac{\partial E_z}{\partial t} - \frac{\partial H_y}{\partial x_1} + \frac{\partial H_x}{\partial x_2} &= 0, \\ \mu \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial x_2} &= 0, \\ \mu \frac{\partial H_y}{\partial t} - \frac{\partial E_z}{\partial x_1} &= 0. \end{aligned} \right.$$

- We compute the evolution of the (1,1) mode in a PEC square cavity, the exact solution is :

$$\left\{ \begin{aligned} H_x(x_1, x_2, t) &= -(\pi/\omega)\sin(\pi x_1)\cos(\pi x_2)\sin(\omega t), \\ H_y(x_1, x_2, t) &= (\pi/\omega)\cos(\pi x_1)\sin(\pi x_2)\sin(\omega t), \\ E_z(x_1, x_2, t) &= \sin(\pi x_1)\sin(\pi x_2)\cos(\omega t), \end{aligned} \right.$$

where $\omega = 2\pi f$, with f the frequency

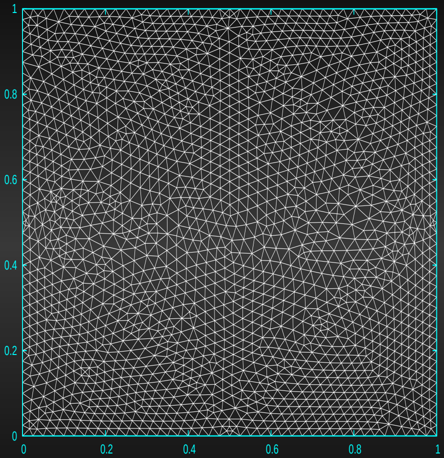
- 2D transverse magnetic waves (TM_z) : $\mathbf{H} \equiv {}^t(H_x, H_y, 0)$ et $\mathbf{E} \equiv {}^t(0, 0, E_z)$
- 2D Maxwell's equations are given by :

$$\begin{cases} \epsilon \dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_y}{\partial x_1} + \dfrac{\partial H_x}{\partial x_2} & = & 0, \\ \mu \dfrac{\partial H_x}{\partial t} + \dfrac{\partial E_z}{\partial x_2} & = & 0, \\ \mu \dfrac{\partial H_y}{\partial t} - \dfrac{\partial E_z}{\partial x_1} & = & 0. \end{cases}$$

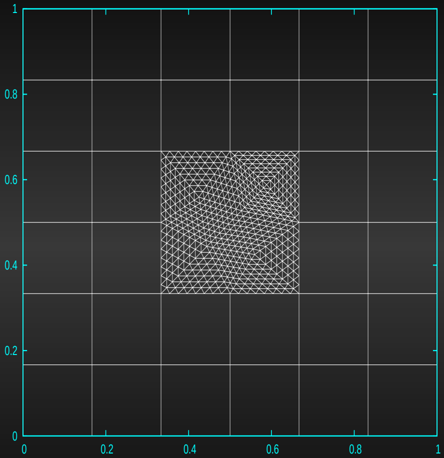
- We compute the evolution of the (1,1) mode in a PEC square cavity, the exact solution is :

$$\begin{cases} H_x(x_1, x_2, t) & = & -(\pi/\omega)\sin(\pi x_1)\cos(\pi x_2)\sin(\omega t), \\ H_y(x_1, x_2, t) & = & (\pi/\omega)\cos(\pi x_1)\sin(\pi x_2)\sin(\omega t), \\ E_z(x_1, x_2, t) & = & \sin(\pi x_1)\sin(\pi x_2)\cos(\omega t), \end{cases}$$

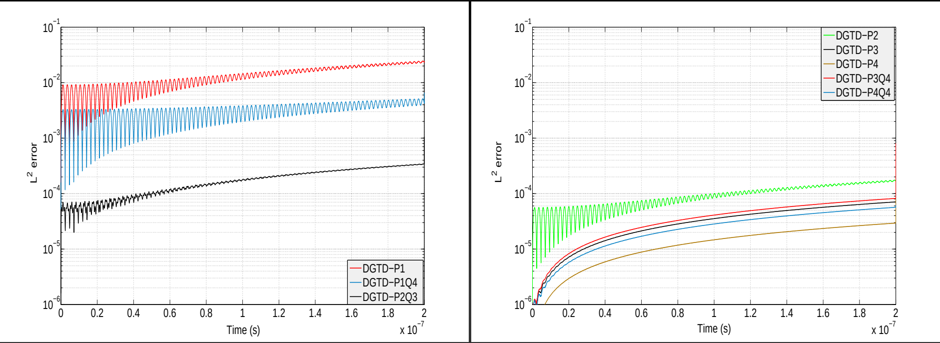
where $\omega = 2\pi f$, with f the frequency



Triangles : 3778
Quadrangles : 0

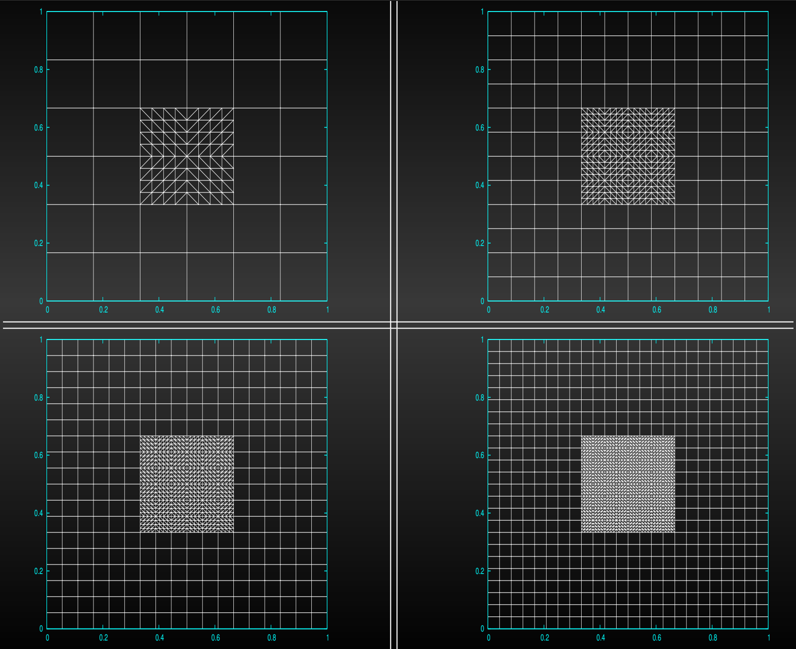


Triangles : 1024
Quadrangles : 32

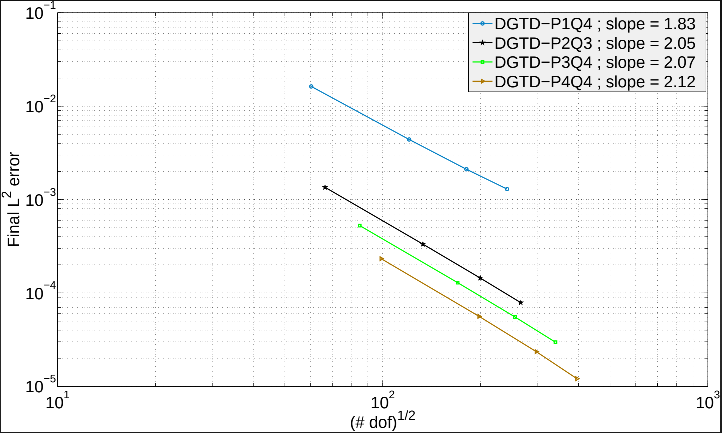


	CPU time	# *dof*	Final L^2-error
DGTD-$\mathbb{P}_1$	45 s	11334	2.33×10^{-2}
DGTD-$\mathbb{P}_2$	206 s	22668	1.68×10^{-4}
DGTD-$\mathbb{P}_3$	530 s	37780	7.09×10^{-5}
DGTD-$\mathbb{P}_4$	1511 s	56670	2.94×10^{-5}
DGTD-$\mathbb{P}_1\mathbb{Q}_4$	12 s	3872	4.24×10^{-3}
DGTD-$\mathbb{P}_2\mathbb{Q}_3$	45 s	6656	3.41×10^{-4}
DGTD-$\mathbb{P}_3\mathbb{Q}_4$	160 s	11040	8.21×10^{-5}
DGTD-$\mathbb{P}_4\mathbb{Q}_4$	346 s	16160	5.67×10^{-5}

Eigenmode in a unitary PEC square cavity

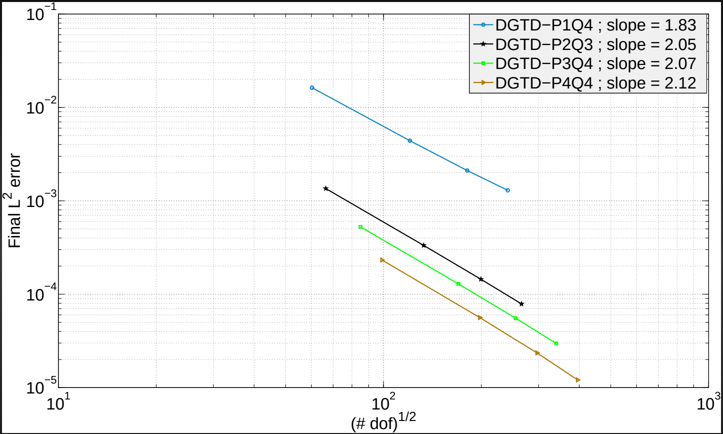


Numerical *h*-wise convergence for the second-order Leap-Frog scheme :



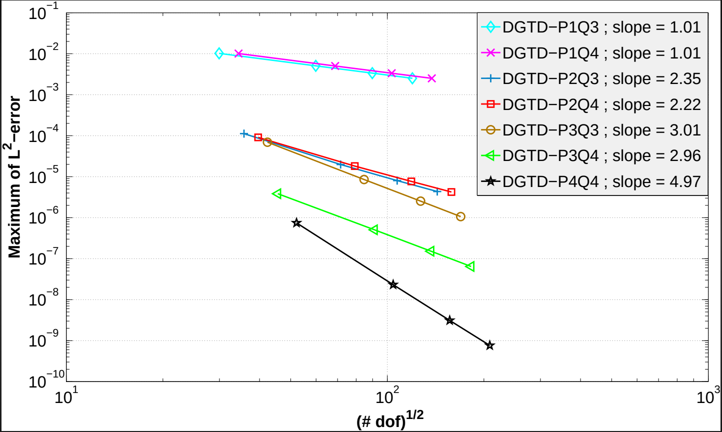
- Numerical validation of convergence in h. Stable method.
- Each time step used in the DGTD-$\mathbb{P}_p\mathbb{Q}_k$ is the minimum between the limit time step for DGTD-$\mathbb{P}_p$ and the one for DGTD-$\mathbb{Q}_k$ $\Longrightarrow$ first numerical validation of the stability analysis

Numerical h-wise convergence for the second-order Leap-Frog scheme :



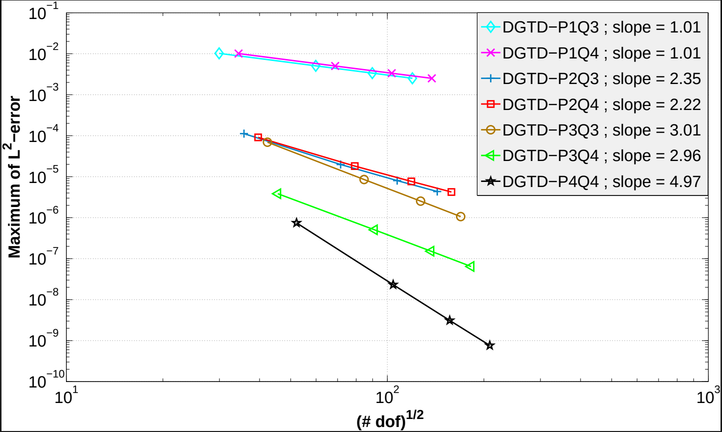
- Numerical validation of convergence in h. Stable method.
- Each time step used in the DGTD-$\mathbb{P}_p\mathbb{Q}_k$ is the minimum between the limit time step for DGTD-$\mathbb{P}_p$ and the one for DGTD-$\mathbb{Q}_k$ $\Longrightarrow$ first numerical validation of the stability analysis

Numerical *h*-wise convergence for the fourth-order Leap-Frog scheme :

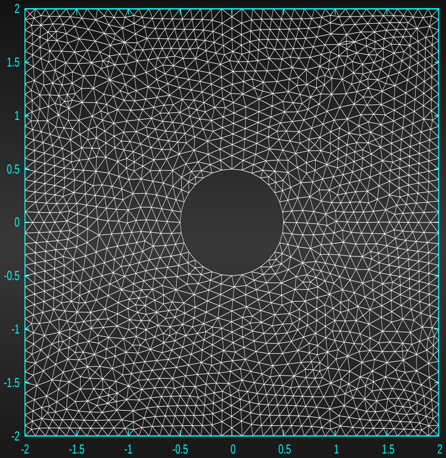


- Numerical validation of convergence in *h*. Stable method.
- LF4 more efficient and more accurate than LF2 for this test problem

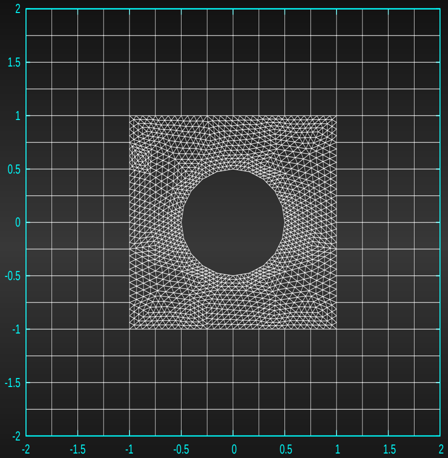
Numerical h-wise convergence for the fourth-order Leap-Frog scheme :



- Numerical validation of convergence in h. Stable method.
- LF4 more efficient and more accurate than LF2 for this test problem



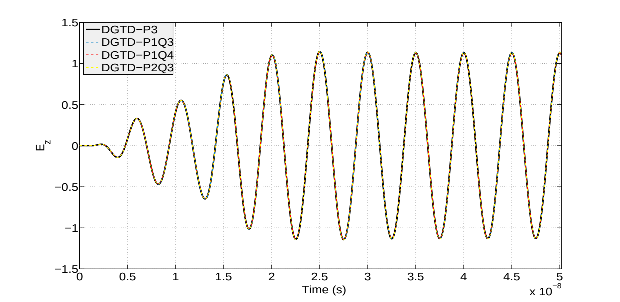
Triangles : 3276
Quadrangles : 0

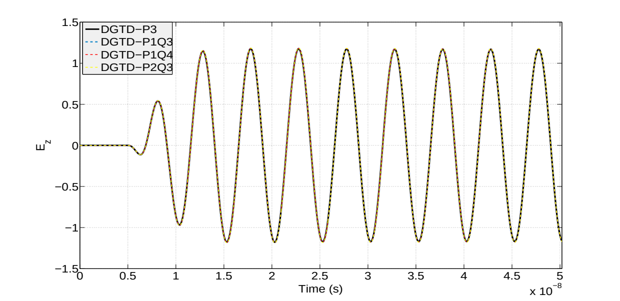


Triangles : 2656
Quadrangles : 192

Time evolution of component E_z at points $(-1.6; 1.6)$ and $(0.5; -0.5)$:

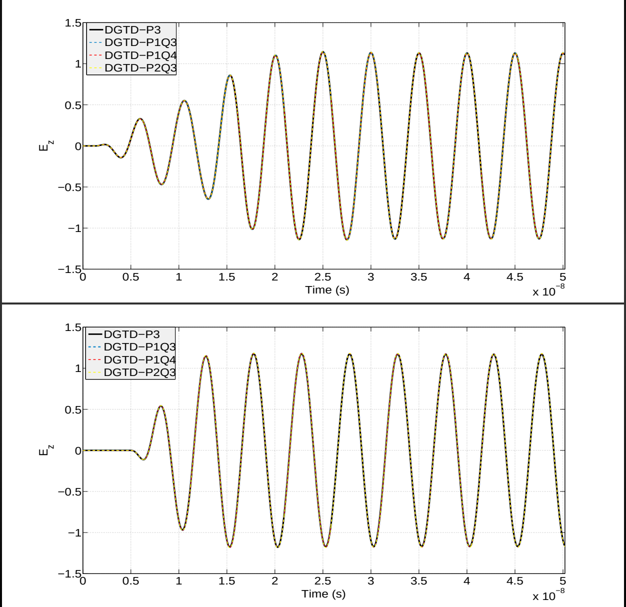
Frequency = 200 MHz		
	# *dof*	CPU time
$\mathbb{P}_3$	32760	20.3 s
$\mathbb{P}_1\mathbb{Q}_3$	11040	1.4 s
$\mathbb{P}_1\mathbb{Q}_4$	12495	2.6 s
$\mathbb{P}_2\mathbb{Q}_3$	19008	6.5 s



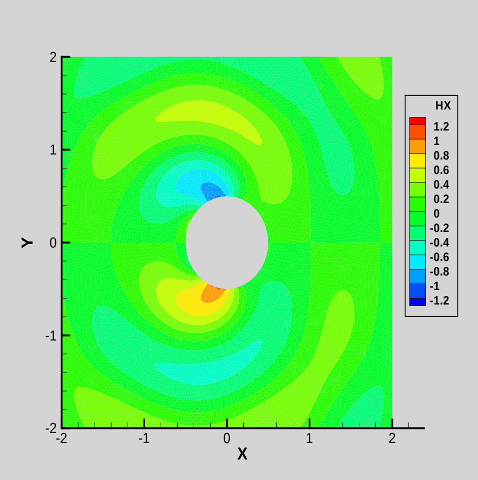


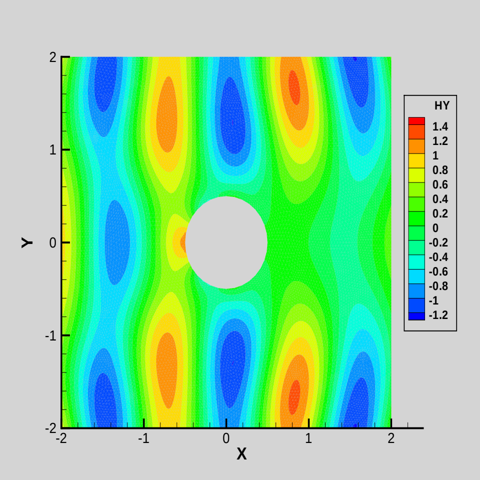
Time evolution of component E_z at points $(-1.6; 1.6)$ and $(0.5; -0.5)$:

Frequency = 200 MHz		
	# *dof*	CPU time
$\mathbb{P}_3$	32760	20.3 s
$\mathbb{P}_1\mathbb{Q}_3$	11040	1.4 s
$\mathbb{P}_1\mathbb{Q}_4$	12495	2.6 s
$\mathbb{P}_2\mathbb{Q}_3$	19008	6.5 s

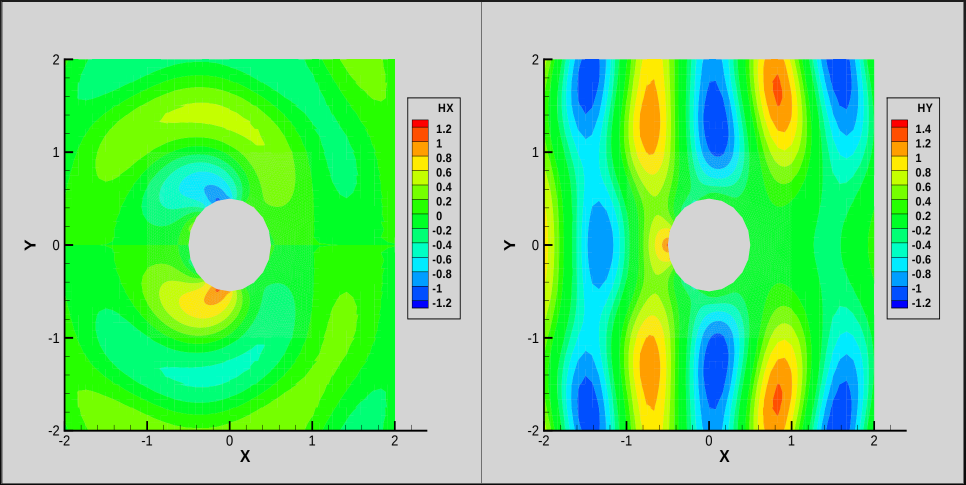


Contour lines of discret Fourier transform of H_x and H_y components (during the last period of the simulation) for calculations with DGTD-$\mathbb{P}_3$ method :

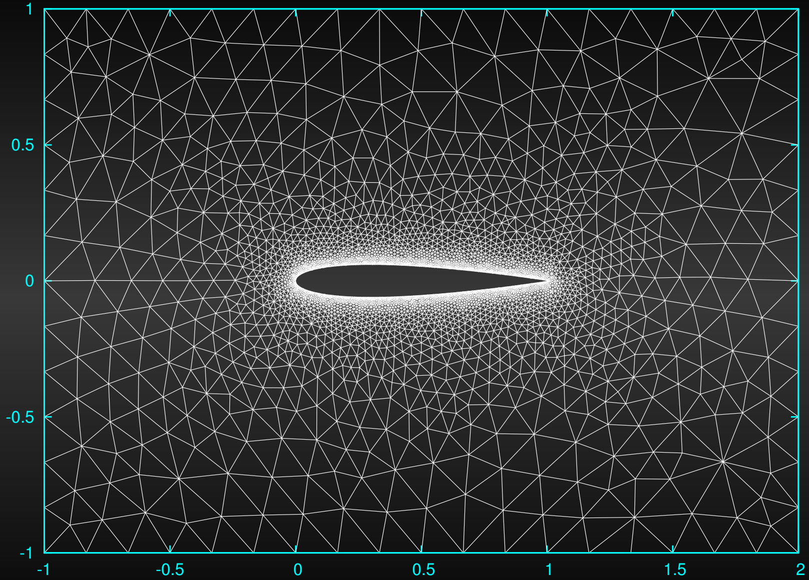




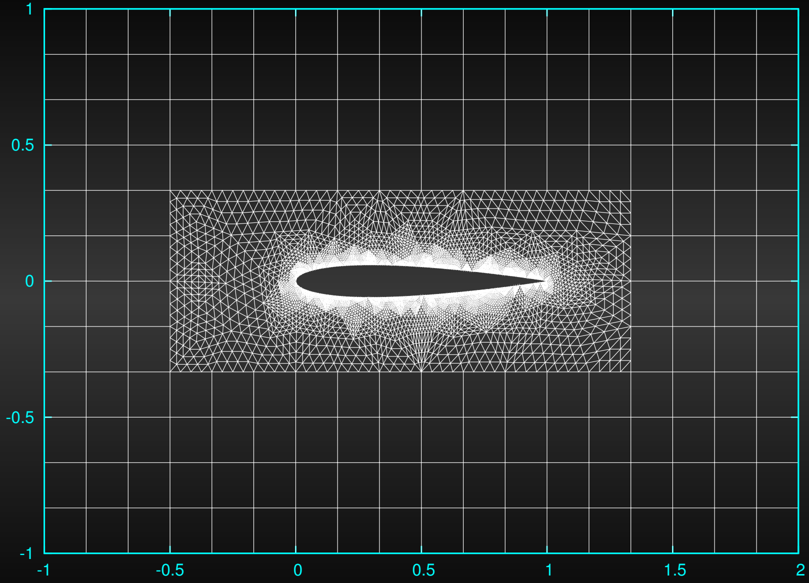
Contour lines of discret Fourier transform of H_x and H_y components (during the last period of the simulation) for calculations with DGTD-$\mathbb{P}_2\mathbb{Q}_3$ method :



Scattering of a plane wave

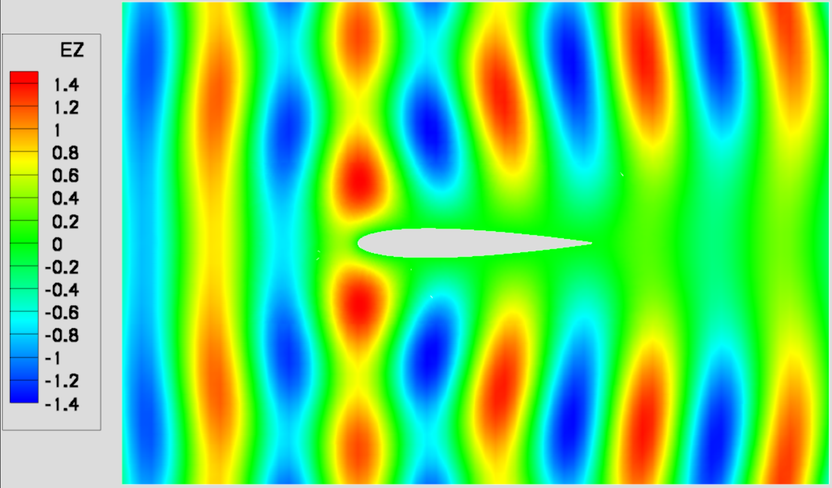


Triangles : 8444 || # Quadrangles : 0

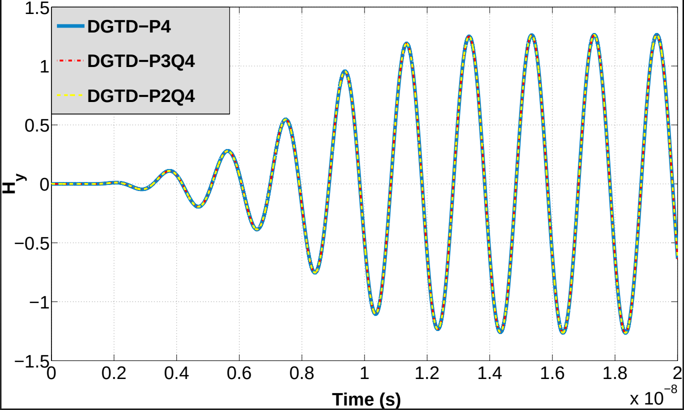


Triangles : 11584 || # Quadrangles : 172

Contour lines of discret Fourier transform of E_z component (calculated during the last period of the simulation) for calculations with DGTD-$\mathbb{P}_2\mathbb{Q}_4$ method :

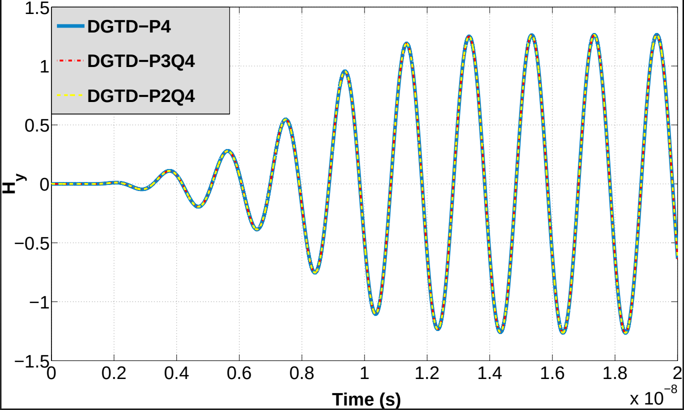


Time evolution of component H_y at point $(0.7; -0.7)$:



Frequency = 500 MHz		
	# *dof*	CPU time
$\mathbb{P}_4$	126660	1339 s
$\mathbb{P}_2\mathbb{Q}_4$	73804	200 s
$\mathbb{P}_3\mathbb{Q}_4$	120140	503 s

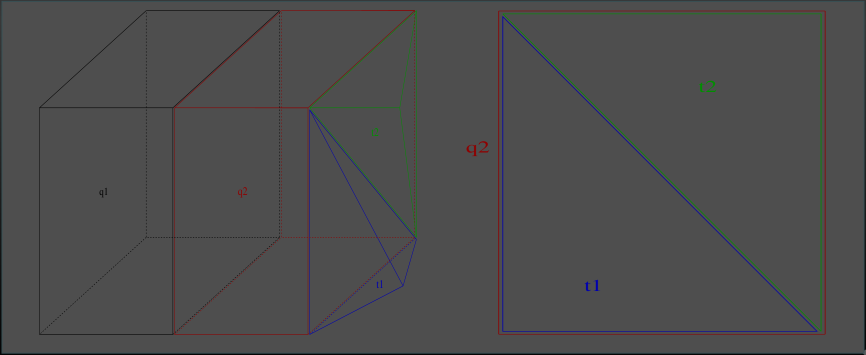
Time evolution of component H_y at point $(0.7; -0.7)$:



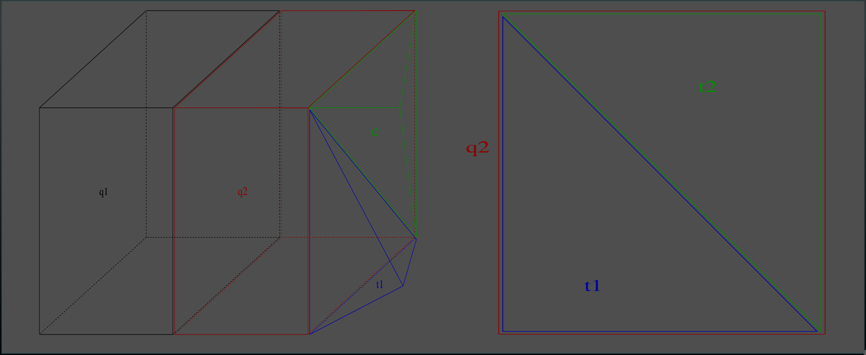
Frequency = 500 MHz		
	$\#$ *dof*	CPU time
$\mathbb{P}_4$	126660	1339 s
$\mathbb{P}_2\mathbb{Q}_4$	73804	200 s
$\mathbb{P}_3\mathbb{Q}_4$	120140	503 s

Outline

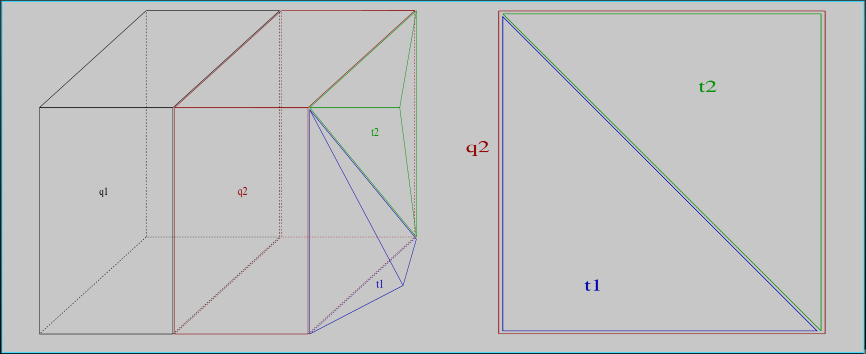
- Validation of this method
- Interesting compromises between accuracy and CPU time
- Work in progress :
  - Fourth order Leap-Frog scheme
  - Large number of new test cases
  - Using others basis function (orthogonal basis for $\mathbb{Q}_k$ on hexahedra)
  - Local time-stepping strategy
  - Transition to 3D

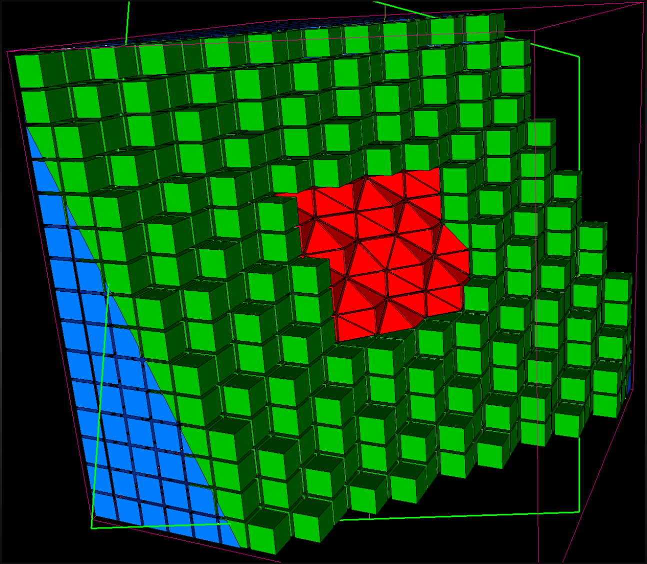


- Validation of this method
- Interesting compromises between accuracy and CPU time
- Work in progress :
  - Fourth order Leap-Frog scheme
  - Large number of new test cases
  - Using others basis function (orthogonal basis for $\mathbb{Q}_k$ on hexahedra)
  - Local time-stepping strategy
  - Transition to 3D

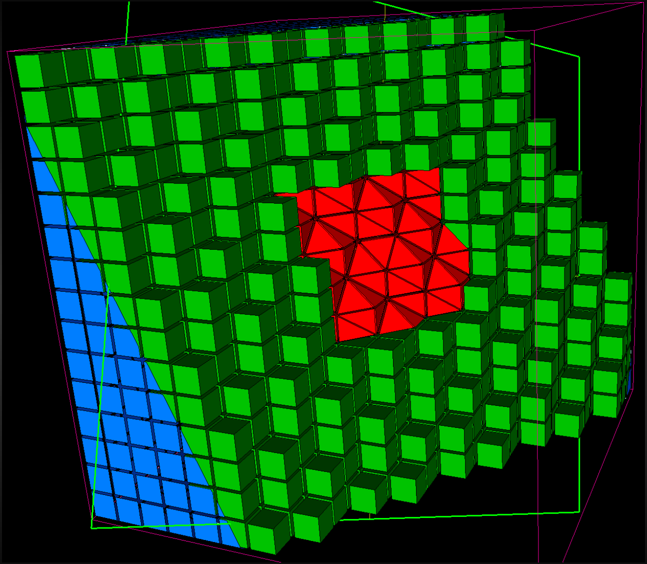


- Validation of this method
- Interesting compromises between accuracy and CPU time
- Work in progress :
  - Fourth order Leap-Frog scheme
  - Large number of new test cases
  - Using others basis function (orthogonal basis for $\mathbb{Q}_k$ on hexahedra)
  - Local time-stepping strategy
  - Transition to 3D





Thank you for your attention



Thank you for your attention