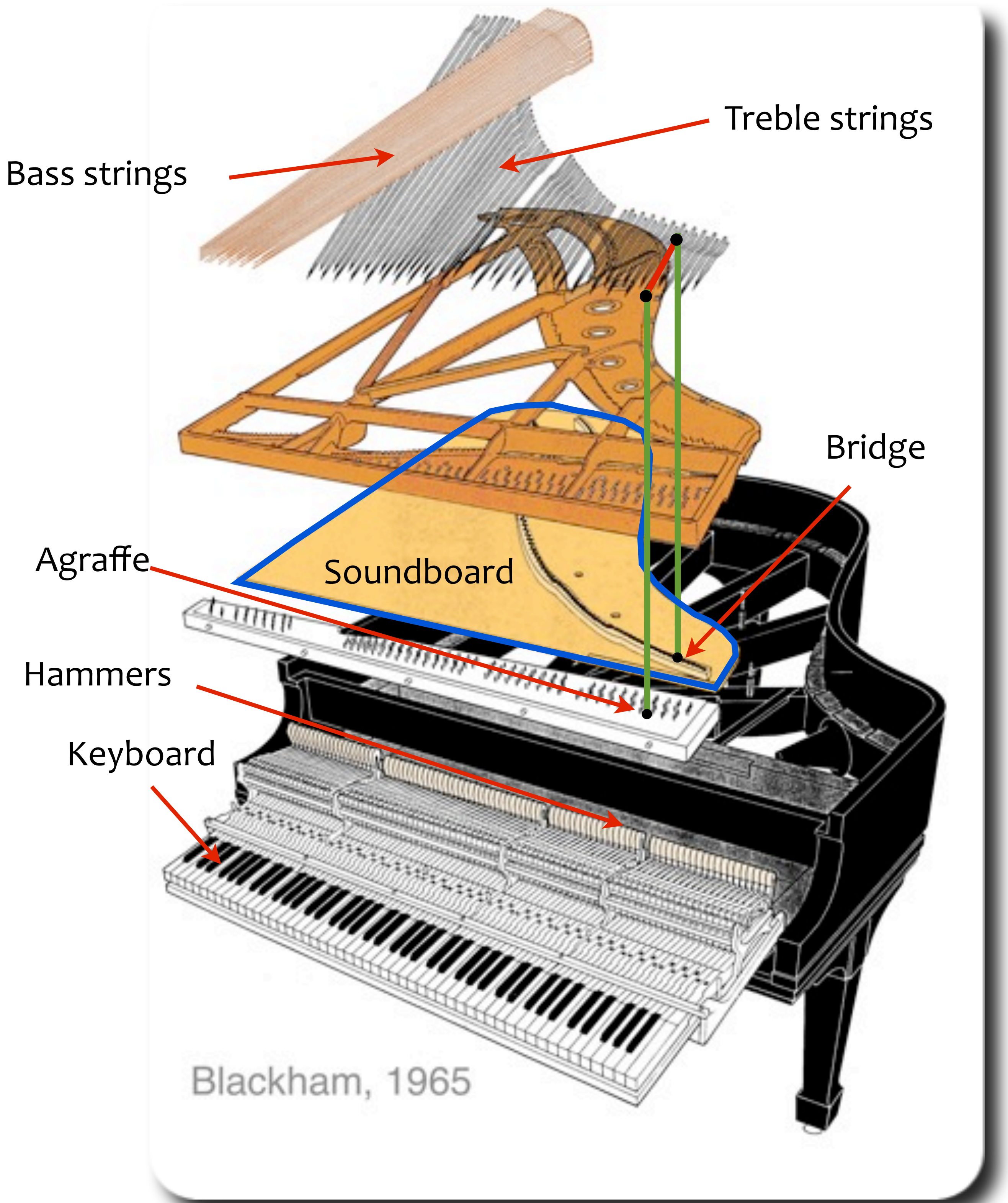
Stability and dispersion analysis of improved time discretization for prestressed Timoshenko systems.

Application to the stiff piano string.

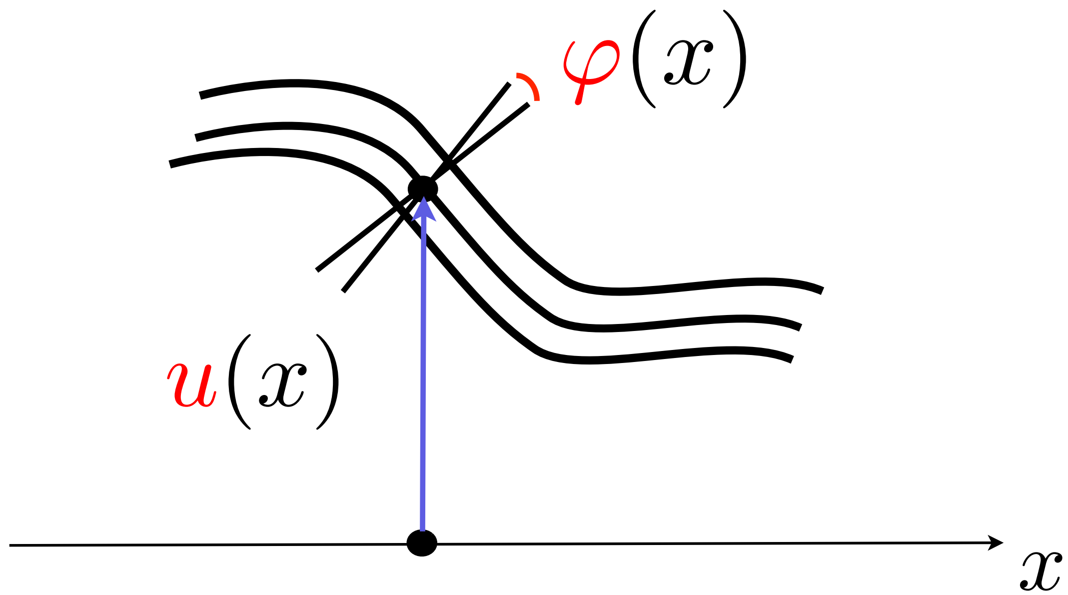
Juliette Chabassier
MAGIQUE 3D
Université de Pau et des pays de l'Adour

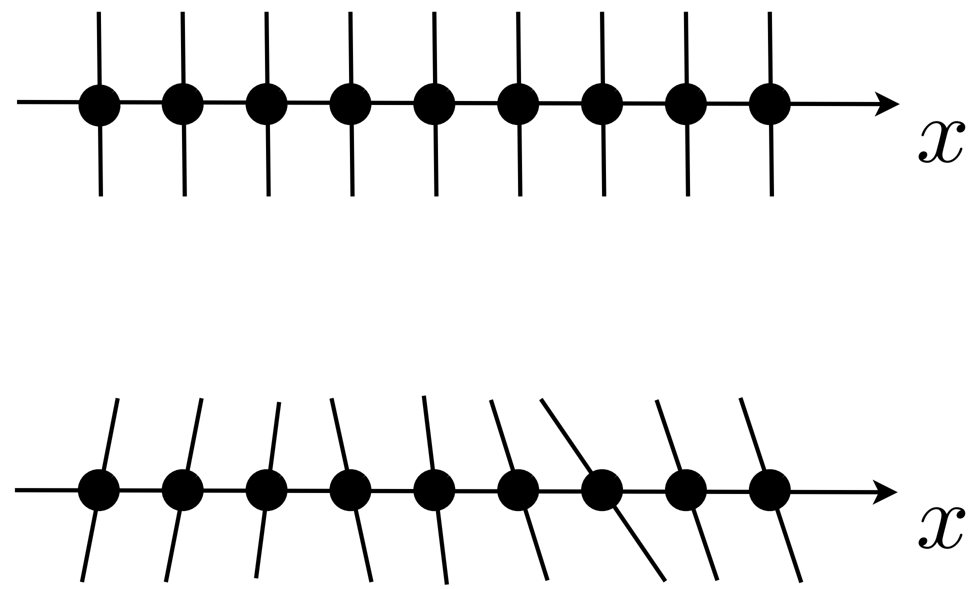
Sébastien Imperiale
Columbia University (New York City)
APAM (Applied Physics and Applied Mathematics department)

Motivations



Motivations





u : flexural wave

φ : shear wave

~~D'Alembert equation, harmonic spectrum~~

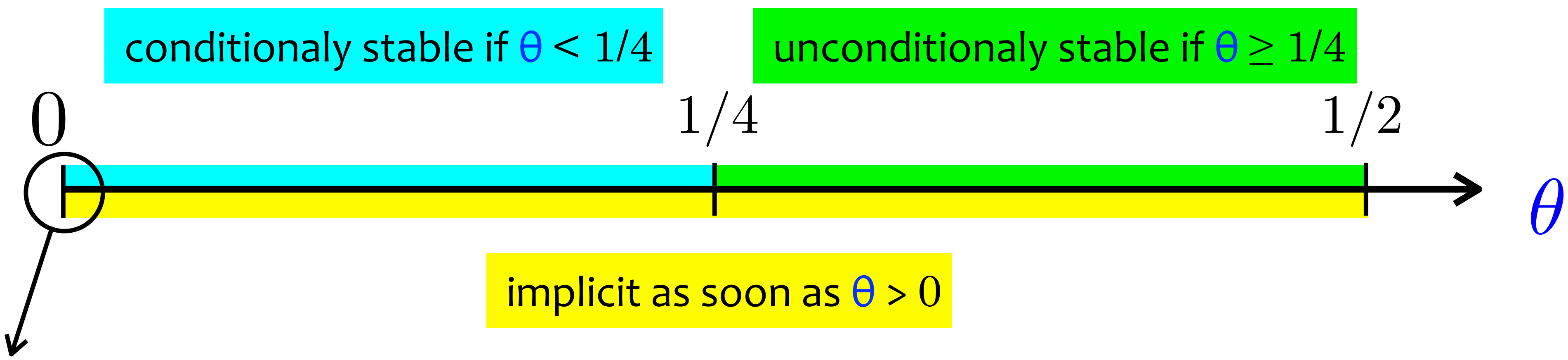
Timoshenko system, inharmonic spectrum

Motivations

After space semi discretization with a standard method as FEM

$$\frac{d^2}{dt^2} M_h U_h + K_h U_h = M_h \Sigma_h$$

$$M_h \frac{U_h^{n+1} - 2U_h^n + U_h^{n-1}}{\Delta t^2} + K_h \left(\theta\, U_h^{n+1} + (1-2\theta)\, U_h^n + \theta\, U_h^{n-1}\right) = M_h \Sigma_h^n$$



Explicit ($)
Time step restriction (D#1 string, reasonable spatial discretization) :
- Timoshenko : Δt ≤ 0.1 millisecond
- D'Alembert : Δt ≤ 1.4 milliseconds

Implicit ($$$)...

... But one solution of the 3D piano problem takes 86 ms per time step ! Δt = 1 millisecond

$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$

Motivations

Write a numerical scheme that

- will be used with a «large» time step ⇨ stability properties shown with energy techniques
- can be implicit
- limits numerical dispersion ⇨ dispersion in finite domains : eigenfrequencies

Improved time discretization for prestressed Timoshenko systems

I. Continuous equations
II. Classical θ-schemes
III. New θ-schemes

Improved time discretization for prestressed Timoshenko systems

I. Continuous equations

II. Classical θ-schemes

III. New θ-schemes

- Equations
- A priori estimates (energy, solution)
- Dispersion analysis (eigenfrequencies)

Prestressed Timoshenko system :

$$\begin{cases} \rho S \dfrac{\partial^2 u}{\partial t^2} - T_0 \dfrac{\partial^2 u}{\partial x^2} + SG\kappa \dfrac{\partial}{\partial x}\left(\varphi - \dfrac{\partial u}{\partial x}\right) = \rho S \sigma, \\ \rho I \dfrac{\partial^2 \varphi}{\partial t^2} - EI \dfrac{\partial^2 \varphi}{\partial x^2} + SG\kappa \left(\varphi - \dfrac{\partial u}{\partial x}\right) = 0, \end{cases}$$

Timoshenko : On the correction for shear of the differential equation for transverse vibrations of bars of uniform cross-section (Philosophical Magazine,1921)

Simply supported boundary conditions :

$$u(x=0,t)=0, \quad u(x=L,t)=0, \quad \partial_x \varphi(x=0,t)=0, \quad \partial_x \varphi(x=L,t)=0,$$

Initial conditions :

$$u(x,t=0)=u_0(x),\ \varphi(x,t=0)=\varphi_0(x),\ \partial_t u(x,t=0)=u_1(x),\ \partial_t \varphi(x,t=0)=\varphi_1(x).$$

Continuous energy identities

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \rho\, S\, |\partial_t u|^2 + \frac{1}{2}\int_0^L \rho\, I\, |\partial_t \varphi|^2$$

$$+ \frac{1}{2}\int_0^L T_0\, |\partial_x u|^2 + \frac{1}{2}\int_0^L EI\, |\partial_x \varphi|^2 + \frac{1}{2}\int_0^L SG\kappa\, |\varphi - \partial_x u|^2$$

$$\frac{d\mathcal{E}}{dt} = \int_0^L \rho S\, \sigma\, \partial_t u \leq \sqrt{\int_0^L \rho S\, |\sigma|^2}\ \underbrace{\sqrt{\int_0^L \rho S\, |\partial_t u|^2}}_{\leq \sqrt{2\mathcal{E}(t)}}$$

$$\frac{d\mathcal{E}}{dt} = 2\sqrt{\mathcal{E}}\frac{d\sqrt{\mathcal{E}}}{dt}$$

Continuous energy identities

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \rho\, S\, |\partial_t u|^2 + \frac{1}{2}\int_0^L \rho\, I\, |\partial_t \varphi|^2 \quad := \frac{1}{2}\,\|\partial_t U\|_M^2$$

$$U = {}^t(u, \varphi)$$

$$+ \frac{1}{2}\int_0^L T_0\, |\partial_x u|^2 + \frac{1}{2}\int_0^L EI\, |\partial_x \varphi|^2 + \frac{1}{2}\int_0^L SG\kappa\, |\varphi - \partial_x u|^2$$

$$\mathcal{E}(t) \leq \left[\sqrt{\mathcal{E}(0)} + \frac{1}{\sqrt{2}}\int_0^t \sqrt{\int_0^L \rho S\, |\sigma|^2}\,\right]^2$$

$$U(\cdot, t) = U_0(\cdot) + \int_0^t \partial_t U(\cdot, s)\, ds$$

$$\|U(\cdot, t)\|_M \leq \|U_0\|_M + t\,\sqrt{2\,\mathcal{E}(0)} + \int_0^t (t-s)\sqrt{\int_0^L \rho S\, |\sigma(\cdot, s)|^2}$$

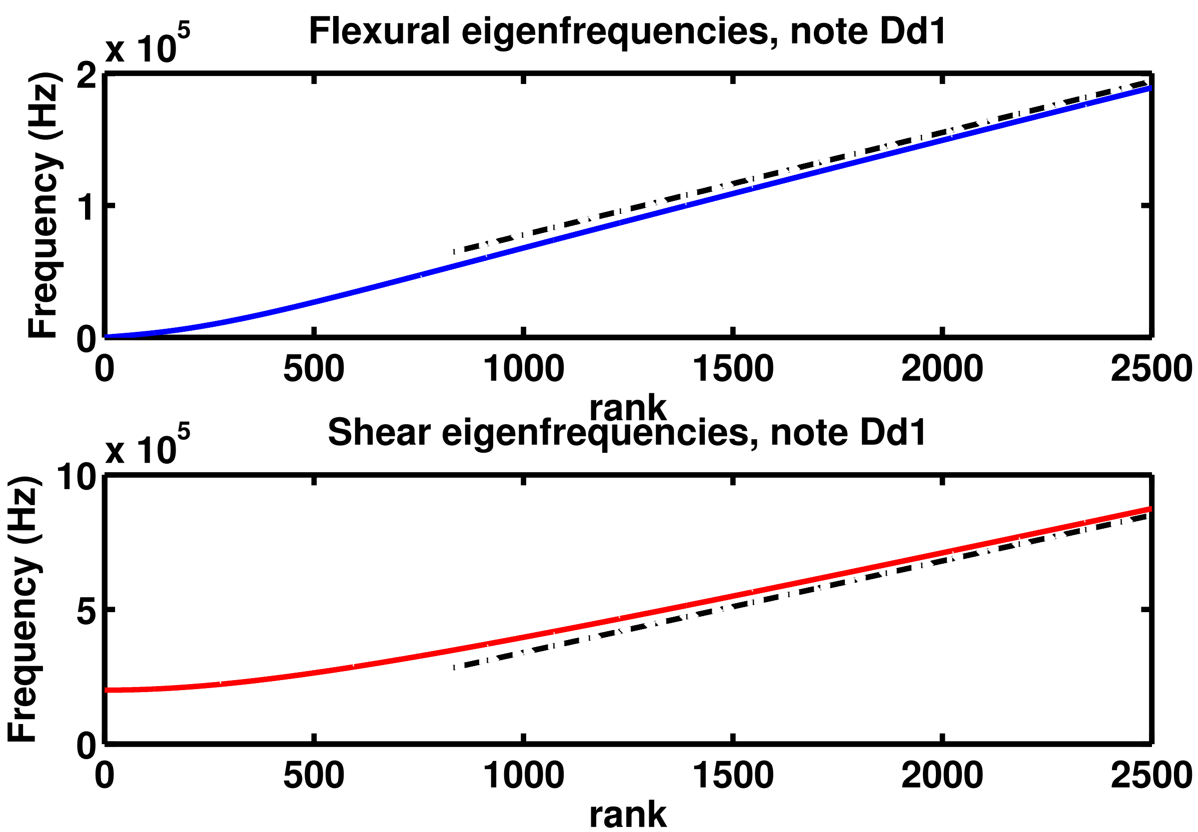
Dispersion in a finite domain

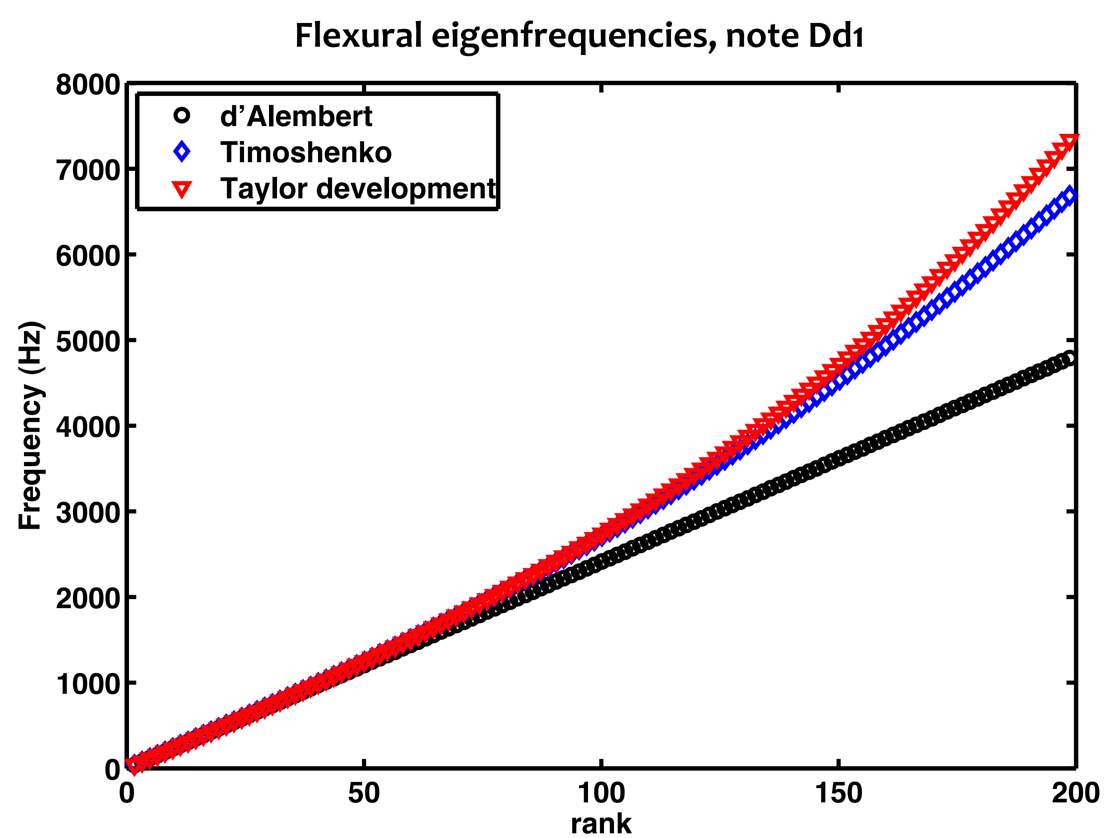
If we seek a solution of the form : $U(x,t) = e^{-i\,2\pi f\,t}\, V(x)$

then there exists ℓ such that $f = f_\ell^{\pm}$ = explicit but complicated formula

flexural : $f_\ell^- = \ell f_0^- \,(1 + \epsilon\,\ell^2) + \mathcal{O}(\ell^5)$, where $f_0^- = \dfrac{1}{2L}\sqrt{\dfrac{T_0}{\rho S}}$, $\epsilon = \dfrac{\pi^2}{2L^2}\dfrac{EI}{T_0}\left[1 - \dfrac{T_0}{ES}\right]$.

shear : $f_\ell^+ = f_0^+ \,(1 + \eta\,\ell^2) + \mathcal{O}(\ell^4)$, where $f_0^+ = \dfrac{1}{2\pi}\sqrt{\dfrac{SG\kappa}{\rho I}}$, $\eta = \dfrac{\pi^2}{2L^2}\dfrac{EI + IG\kappa}{SG\kappa}$.





Improved time discretization for prestressed Timoshenko systems

I. Continuous equations
II. Classical θ-schemes
III. New θ-schemes

- Numerical scheme
- Stability estimates (energy, solution)
- Dispersion analysis (eigenfrequencies)

After space semi discretization with a standard method as high order FEM

$$\frac{d^2}{dt^2} M_h \mathbf{U}_h + K_h \mathbf{U}_h = M_h \Sigma_h$$

Time discretisation with a θ-scheme :

$$M_h \frac{\mathbf{U}_h^{n+1} - 2\mathbf{U}_h^n + \mathbf{U}_h^{n-1}}{\Delta t^2} + K_h \left(\theta\, \mathbf{U}_h^{n+1} + (1-2\theta)\, \mathbf{U}_h^n + \theta\, \mathbf{U}_h^{n-1}\right) = M_h \Sigma_h^n$$

$$\boxed{\frac{\mathcal{E}_\theta^{n+1/2} - \mathcal{E}_\theta^{n-1/2}}{\Delta t} = M_h\, \mathbf{\Sigma}_h^n \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n-1}}{2\Delta t}}$$

$$\mathcal{E}_\theta^{n+1/2} = \frac{1}{2} \left\| \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \right\|^2_{\widetilde{M}_{h,\theta}} + \frac{1}{2} \left\| \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2} \right\|^2_{K_h} \qquad \widetilde{M}_{h,\theta} = M_h + (\theta - \frac{1}{4})\Delta t^2 K_h$$

$$\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n-1}}{2\Delta t} = \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{2\Delta t} + \frac{\mathbf{U}_h^n - \mathbf{U}_h^{n-1}}{2\Delta t}$$

Discrete estimates

$$\frac{\mathcal{E}_\theta^{n+1/2} - \mathcal{E}_\theta^{n-1/2}}{\Delta t} = M_h\, \mathbf{\Sigma}_h^n \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n-1}}{2\Delta t}$$

$$\mathcal{E}_\theta^{n+1/2} = \frac{1}{2} \left\| \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n}}{\Delta t} \right\|^2_{\widetilde{M}_{h,\theta}} + \frac{1}{2} \left\| \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^{n}}{2} \right\|^2_{K_h} \qquad \widetilde{M}_{h,\theta} = M_h + (\theta - \frac{1}{4})\Delta t^2 K_h$$

If $\theta \geq \frac{1}{4}$

$$\left\| \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n}}{\Delta t} \right\|_{M_h} \leq \left\| \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n}}{\Delta t} \right\|_{\widetilde{M}_{h,\theta}} \leq \sqrt{2\mathcal{E}_\theta^{n+1/2}}$$

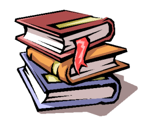
If $\theta < \frac{1}{4}$

Positive energy if $\Delta t^2 \rho \left(M_h^{-1} K_h \right) \leq \dfrac{1}{1-4\theta}$

$$\|\mathbf{X}\|_{M_h} \leq \frac{\Delta t^\theta}{\sqrt{(\Delta t^\theta)^2 - \Delta t^2}} \|\mathbf{X}\|_{\widetilde{M}_{h,\theta}}$$

blows up when $\Delta t \rightarrow \Delta t^\theta$!!!

- Other techniques not based on the natural energy



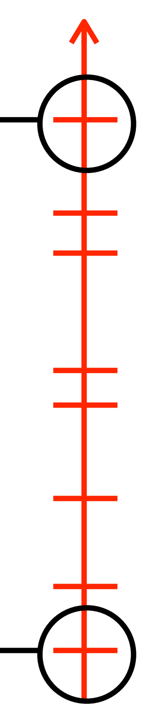
P Joly : *Variational Methods for Time Dependent Wave Propagation,* Topics in Computational Wave Propagation, 31:201264, 2003

- Not easy to use when couplings, nonlinear terms or dissipative terms are added

- New proof that avoids this difficulty : $\mathcal{E}_\theta^{n+1/2} = \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}\right\|^2_{\widetilde{M}_{h,\theta}} + \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1}+\mathbf{U}_h^n}{2}\right\|^2_{K_h}$

$$\frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^{n-1}}{2\Delta t} = \frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^{n}}{2\Delta t} + \frac{\mathbf{U}_h^{n}-\mathbf{U}_h^{n-1}}{2\Delta t}$$
$$= \frac{\mathbf{U}_h^{n+1}+\mathbf{U}_h^{n}}{2\Delta t} - \frac{\mathbf{U}_h^{n}+\mathbf{U}_h^{n-1}}{2\Delta t}$$

eigenvalues of $M_h^{-1}K_h$



$\frac{1}{\Delta t^2(1-4\theta)}$

If $\Delta t = \Delta t^\theta$ then $M_h + (\theta - \frac{1}{4})\Delta t^2 K_h$ is singular

$\|\mathbf{X}\|_{M_h} \leq C\!\!\!/\, \|\mathbf{X}\|_{\widetilde{M}_{h,\theta}}$

$\|\mathbf{X}\|_{M_h} \leq C\!\!\!/\, \|\mathbf{X}\|_{K_h}$

If K_h is singular, $\#\mathrm{Ker}(K_h) = s$

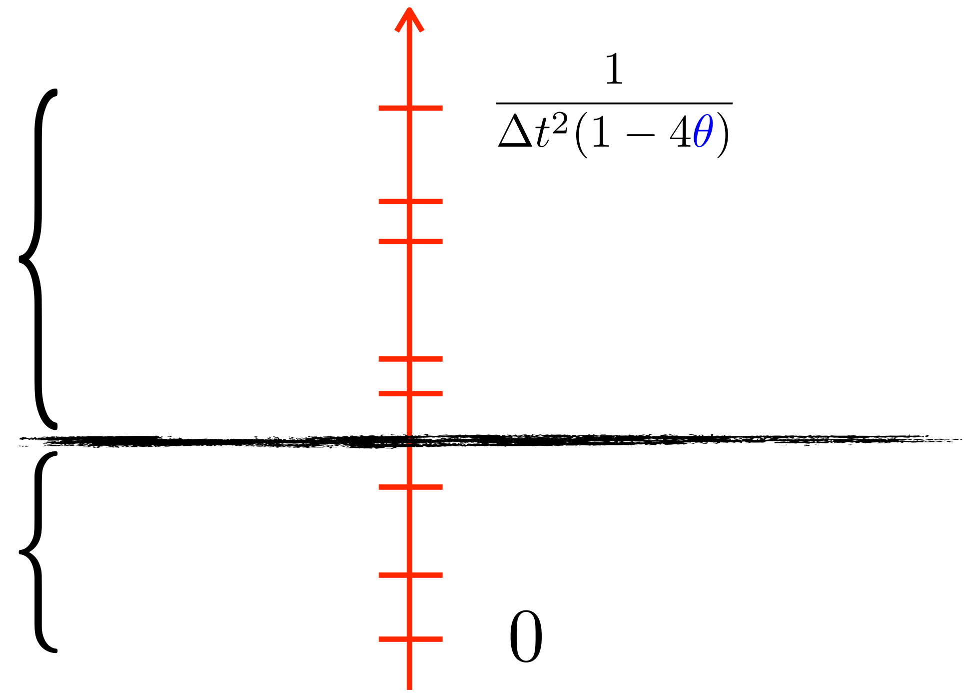
0

$$\mathcal{E}_\theta^{n+1/2} = \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\right\|^2_{\widetilde{M}_{h,\theta}} + \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}\right\|^2_{K_h}$$

$$\left\|\mathbf{X}^{HF}\right\|_{M_h} \leq C \left\|\mathbf{X}\right\|_{K_h}$$

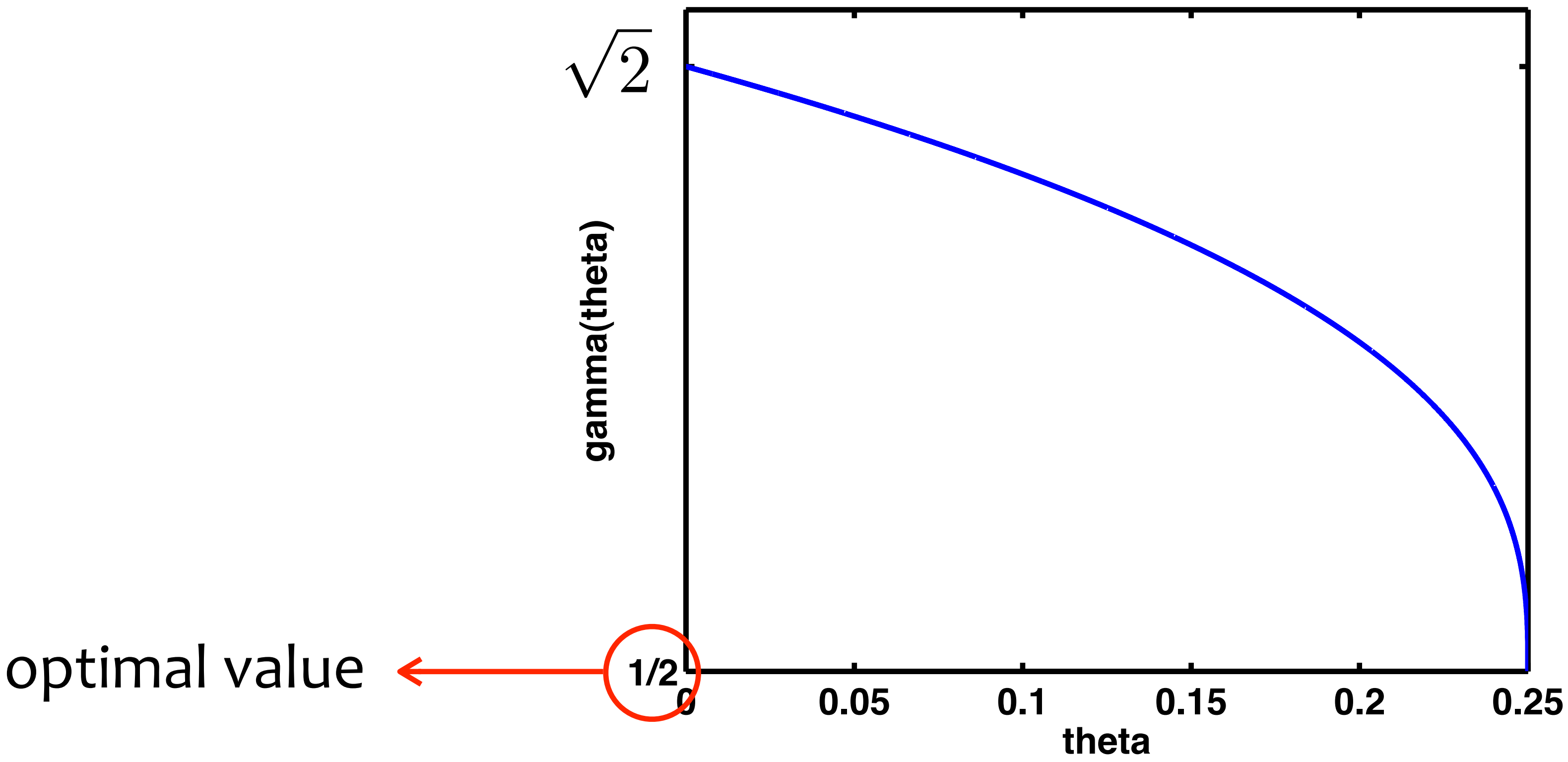
$$\left\|\mathbf{X}^{LF}\right\|_{M_h} \leq C \left\|\mathbf{X}\right\|_{\widetilde{M}_{h,\theta}}$$

eigenvalues of $M_h^{-1} K_h$



$$\sqrt{\mathcal{E}_\theta^{n+1/2}} \leq \sqrt{\mathcal{E}_\theta^{1/2}} + \sqrt{2}\,\gamma(\theta)\,\Delta t \sum_{\ell=1}^{n} \left\|\Sigma_h^\ell\right\|_{M_h}$$

Discrete estimates



$$\sqrt{\mathcal{E}_\theta^{n+1/2}} \leq \sqrt{\mathcal{E}_\theta^{1/2}} + \sqrt{2}\,\gamma(\theta)\,\Delta t \sum_{\ell=1}^{n} \left\| \Sigma_h^\ell \right\|_{M_h}$$

$$\left\| \mathbf{U}_h^{n+1} \right\|_{M_h} \leq \sqrt{2} \left\| \mathbf{U}_h^0 \right\|_{M_h} + 2\,\gamma(\theta)\, t^{n+1} \sqrt{2\mathcal{E}_\theta^{1/2}} + 4\,\gamma(\theta)^2\,\Delta t^2 \sum_{\ell=1}^{n} \sum_{k=1}^{\ell} \left\| \Sigma_h^k \right\|_{M_h}$$

Dispersion analysis

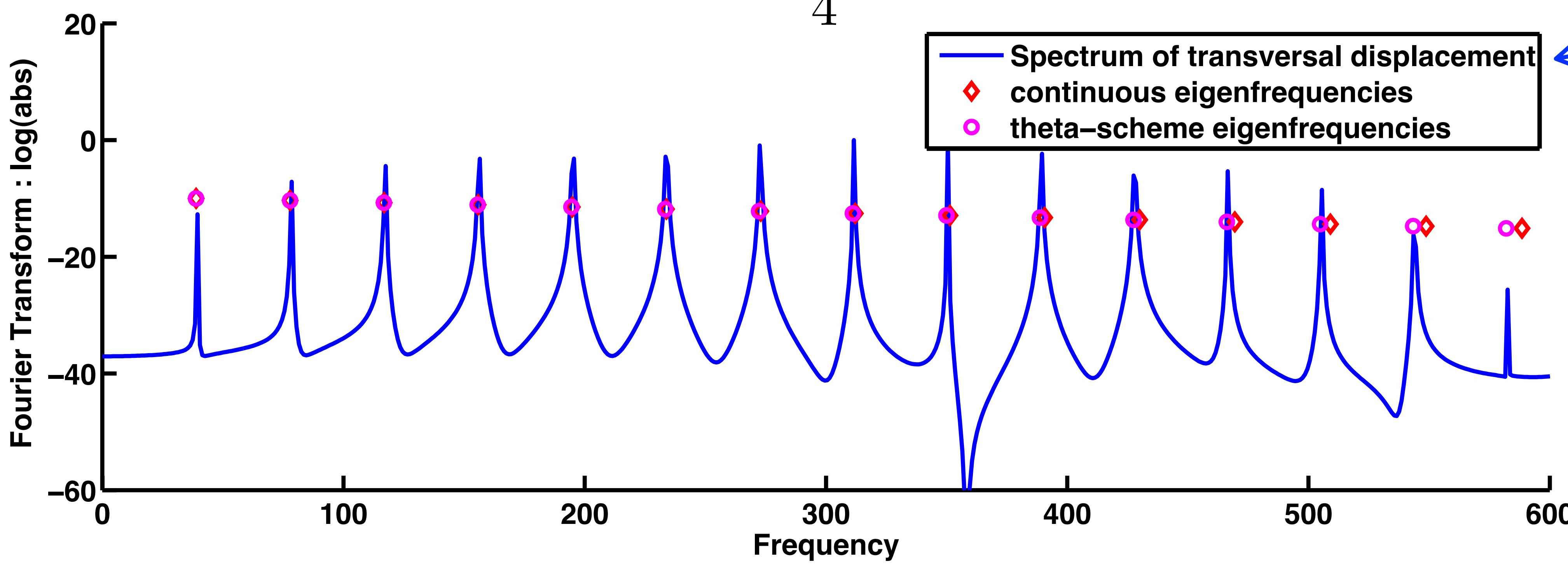
If we seek a solution of the form : $\mathbf{U}_h^n = e^{i\,2\pi f_h\,n\Delta t}\,\mathbf{V}_h^0$

then there exists ℓ such that $f_h = f_{h,\ell}$

$$f_{h,\ell} = f_\ell + \frac{f_\ell^3}{2}\left(\frac{1}{12} - \theta\right)\Delta t^2 + \mathcal{O}(\Delta t^4 + h^4)$$

explicit but complicated formula of the continuous system

Note D♯ 1 , $\theta = \frac{1}{4}$, $\Delta t = 10^{-4}$second



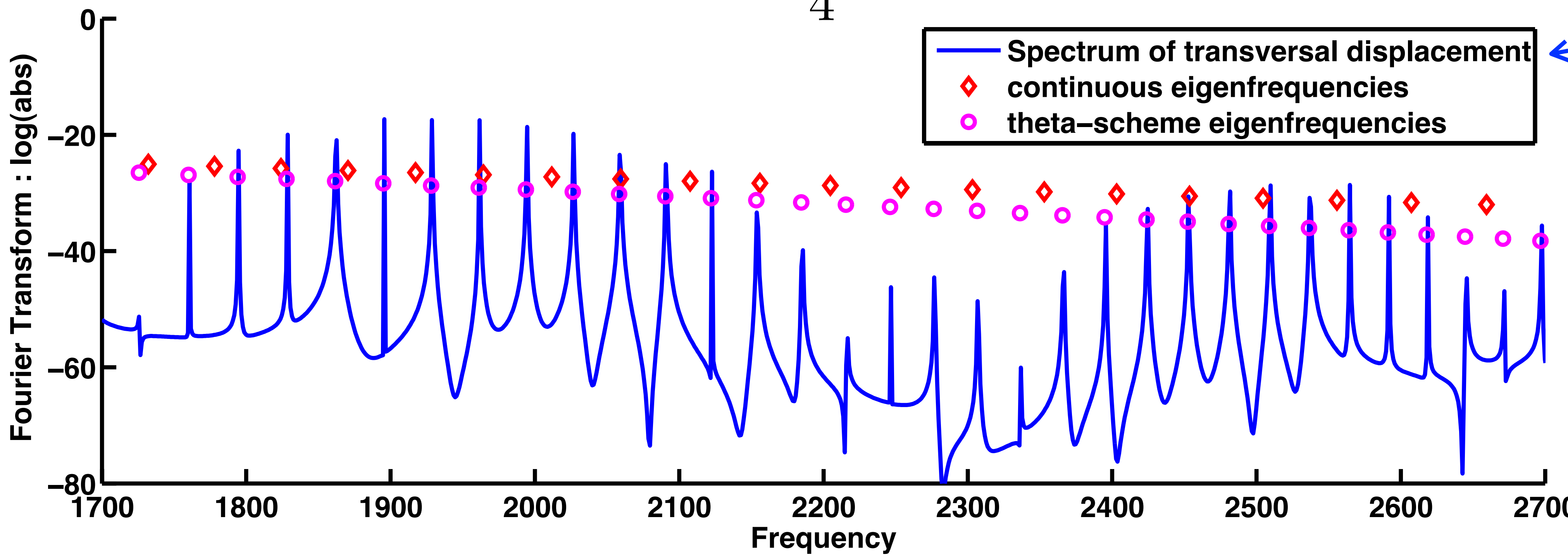
If we seek a solution of the form : $\mathbf{U}_h^n = e^{i\,2\pi f_h\, n\Delta t}\,\mathbf{V}_h^0$

then there exists ℓ such that $f_h = f_{h,\ell}$

$$f_{h,\ell} = f_\ell + \frac{f_\ell^3}{2}\left(\frac{1}{12} - \theta\right)\Delta t^2 + \mathcal{O}(\Delta t^4 + h^4)$$

explicit but complicated formula of the continuous system

Note D♯ 1 , $\theta = \frac{1}{4}$, $\Delta t = 10^{-4}$second



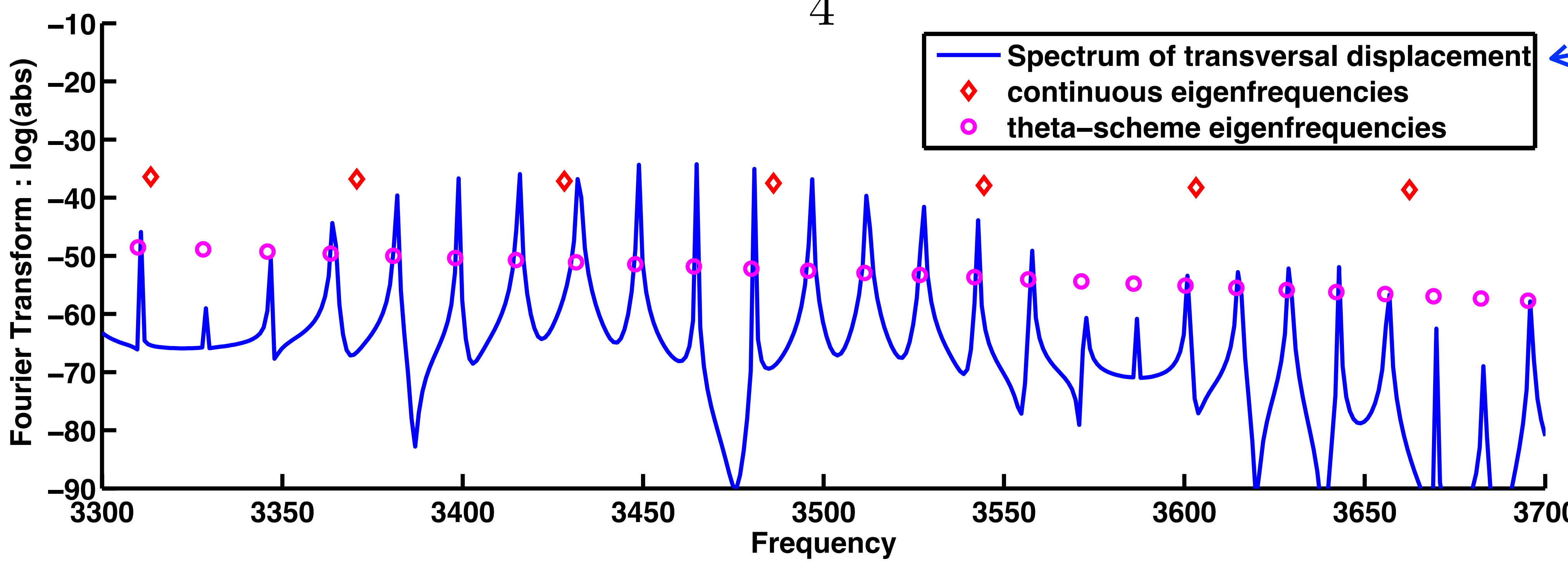
If we seek a solution of the form : $\mathbf{U}_h^n = e^{i\,2\pi f_h\, n\Delta t}\,\mathbf{V}_h^0$

then there exists ℓ such that $f_h = f_{h,\ell}$

$$f_{h,\ell} = f_\ell + \frac{f_\ell^3}{2}\left(\frac{1}{12} - \theta\right)\Delta t^2 + \mathcal{O}(\Delta t^4 + h^4)$$

explicit but complicated formula of the continuous system

$$\text{Note D}\sharp\ 1\ , \quad \theta = \frac{1}{4}, \quad \Delta t = 10^{-4}\text{second}$$



If we seek a solution of the form : $\mathbf{U}_h^n = e^{i\,2\pi f_h\, n\Delta t}\,\mathbf{V}_h^0$

then there exists ℓ such that $f_h = f_{h,\ell}$

$$f_{h,\ell} = f_\ell + \frac{f_\ell^3}{2}\left(\frac{1}{12} - \theta\right)\Delta t^2 + \mathcal{O}(\Delta t^4 + h^4)$$

The choice $\theta = \dfrac{1}{12}$ seems a good choice... but with this spatial discretisation,

$$\Delta t \leq 3,5 \times 10^{-7} \quad \text{second}$$

If we only had d'Alembert equation, the restriction would be : $\Delta t \leq 5 \times 10^{-6} \quad \text{second}$

Could we have a small numerical dispersion on the flexural wave,
without undergoing the CFL restriction coming from the shear wave ???

Improved time discretization for prestressed Timoshenko systems

I. Continuous equations

II. Classical θ-schemes

III. New θ-schemes

- Numerical scheme
- Stability estimates (energy, solution)
- Dispersion analysis (eigenfrequencies)

Juliette Chabassier

$$\begin{cases} \rho S \dfrac{\partial^2 u}{\partial t^2} - T_0 \dfrac{\partial^2 u}{\partial x^2} + SG\kappa \dfrac{\partial}{\partial x}\left(\varphi - \dfrac{\partial u}{\partial x}\right) = \rho S \sigma, \\ \rho I \dfrac{\partial^2 \varphi}{\partial t^2} - EI \dfrac{\partial^2 \varphi}{\partial x^2} + SG\kappa\left(\varphi - \dfrac{\partial u}{\partial x}\right) = 0, \end{cases}$$

$$\frac{d^2}{dt^2} M_h \mathbf{U}_h + \overline{K}_h \mathbf{U}_h + \underline{K}_h \mathbf{U}_h = M_h \Sigma_h$$

$$M_h \frac{\mathbf{U}_h^{n+1} - 2\mathbf{U}_h^n + \mathbf{U}_h^{n-1}}{\Delta t^2} + \underline{K}_h \{\mathbf{U}_h\}_\theta^n + \overline{K}_h \{\mathbf{U}_h\}_{\overline{\theta}}^n = M_h \mathbf{\Sigma}_h^n$$

$$\boxed{\frac{\mathcal{E}_{\theta,\overline{\theta}}^{n+1/2} - \mathcal{E}_{\theta,\overline{\theta}}^{n-1/2}}{\Delta t} = M_h \mathbf{\Sigma}_h^n \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^{n-1}}{2\Delta t}}$$

$$\mathcal{E}_\theta^{n+1/2} = \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\right\|^2_{\widetilde{M}_{h,\theta,\overline{\theta}}} + \frac{1}{2}\left\|\frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}\right\|^2_{K_h} \qquad \widetilde{M}_{h,\theta,\overline{\theta}} = M_h - \frac{\Delta t^2}{4}\left((1-4\,\theta)\underline{K}_h + (1-4\,\overline{\theta})\overline{K}_h\right)$$

Discrete estimates

$$\begin{cases} \text{If } \overline{\theta} \geq 1/4 \text{ and } \theta \geq 1/4 & \sqrt{\mathcal{E}^{n+1/2}_{\theta,\overline{\theta}}} \leq \sqrt{\mathcal{E}^{1/2}_{\theta,\overline{\theta}}} + \dfrac{\Delta t}{\sqrt{2}} \displaystyle\sum_{\ell=1}^{n} \|\boldsymbol{\Sigma}^{\ell}_h\|_{M_h} \\ \text{otherwise} & \sqrt{\mathcal{E}^{n+1/2}_{\theta,\overline{\theta}}} \leq \sqrt{\mathcal{E}^{1/2}_{\theta,\overline{\theta}}} + \Delta t\,\sqrt{2}\,\gamma\big(\min(\theta,\overline{\theta})\big) \displaystyle\sum_{\ell=1}^{n} \|\boldsymbol{\Sigma}^{\ell}_h\|_{M_h} \end{cases}$$

$$\begin{cases} \text{If } \overline{\theta} \geq 1/4 \text{ and } \theta \geq 1/4 \\ \qquad \|\mathbf{U}^{n+1}_h\|_{M_h} \leq \|\mathbf{U}^{0}_h\|_{M_h} + t^{n+1}\sqrt{2\mathcal{E}^{1/2}_{\theta,\overline{\theta}}} + \Delta t^2 \displaystyle\sum_{\ell=1}^{n}\sum_{k=1}^{\ell} \left\|\boldsymbol{\Sigma}^{k}_h\right\|_{M_h} \\ \text{otherwise} \\ \left\|\mathbf{U}^{n+1}_h\right\|_{M_h} \leq \sqrt{2}\left\|\mathbf{U}^{0}_h\right\|_{M_h} + 2\,\gamma\big(\min(\theta,\overline{\theta})\big)\, t^{n+1}\sqrt{2\mathcal{E}^{1/2}_{\theta,\overline{\theta}}} + 4\,\gamma\big(\min(\theta,\overline{\theta})\big)^2\,\Delta t^2 \displaystyle\sum_{\ell=1}^{n}\sum_{k=1}^{\ell} \left\|\boldsymbol{\Sigma}^{k}_h\right\|_{M_h} \end{cases}$$

If we seek a solution of the form : $\mathbf{U}_h^n = e^{i\,2\pi f_h\, n\Delta t}\,\mathbf{V}_h^0$

then there exists ℓ such that $f_h = f_{h,\ell}^{\pm}$

flexural : $f_{h,\ell}^{-} = \ell\, f_0^{-}\,(1 + \epsilon_{\Delta t}\,\ell^2) + \mathcal{O}(\ell^5 + \Delta t^4 + h^4)$

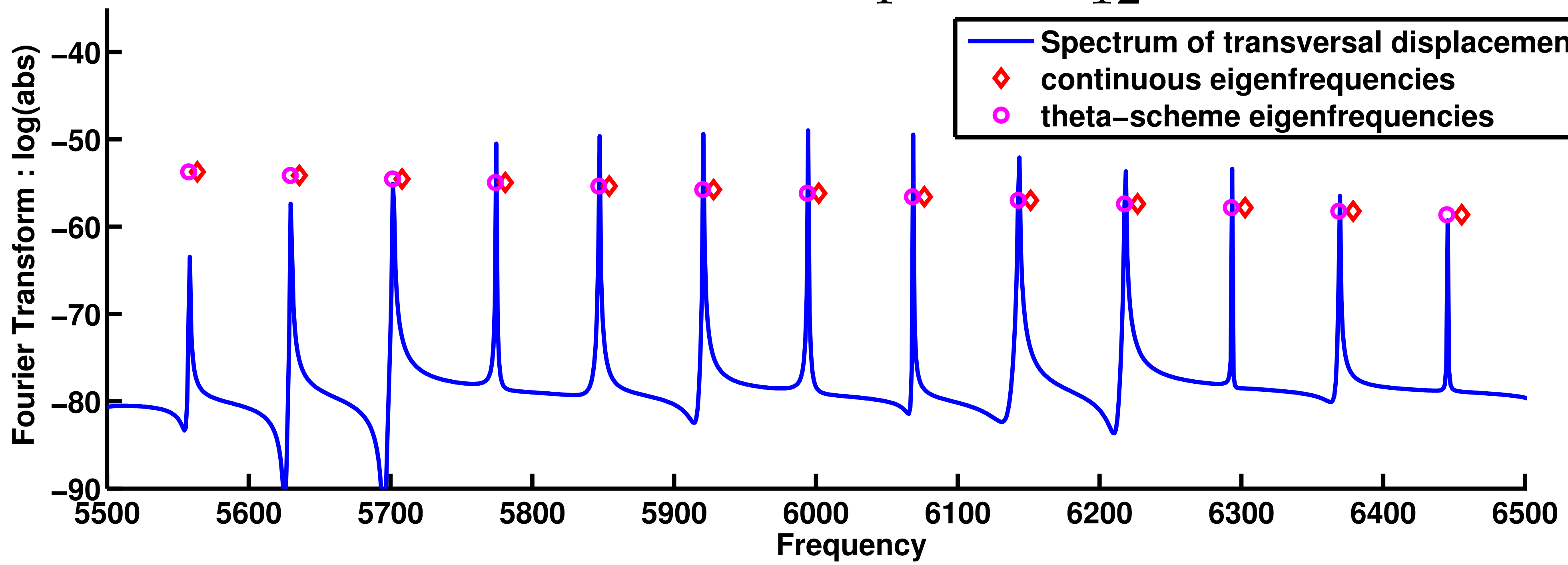
if $\bar{\theta}$ = 1/12, exact up to order 4 !

shear : $f_{h,\ell}^{+} = f_{0,\Delta t}^{+}\,(1 + \eta_{\Delta t}\,\ell^2) + \mathcal{O}(\ell^3 + \Delta t^4 + h^4)$

$$\begin{cases} \epsilon_{\Delta t} = \epsilon + 2\pi^2\,\Delta t^2 \left(\frac{1}{12} - \bar{\theta}\right) (f_0^-)^2 \\ f_{0,\Delta t}^{+} = f_0^{+}\left[1 + (2\pi f_0^{+})^2 \left(\frac{1}{12} - \theta\right) \Delta t^2\right] \\ \eta_{\Delta t} = \eta + \frac{\pi^2\,(E + G\kappa)}{2\rho L^2} \left(\theta - \frac{1}{12}\right) \Delta t^2 \end{cases}$$

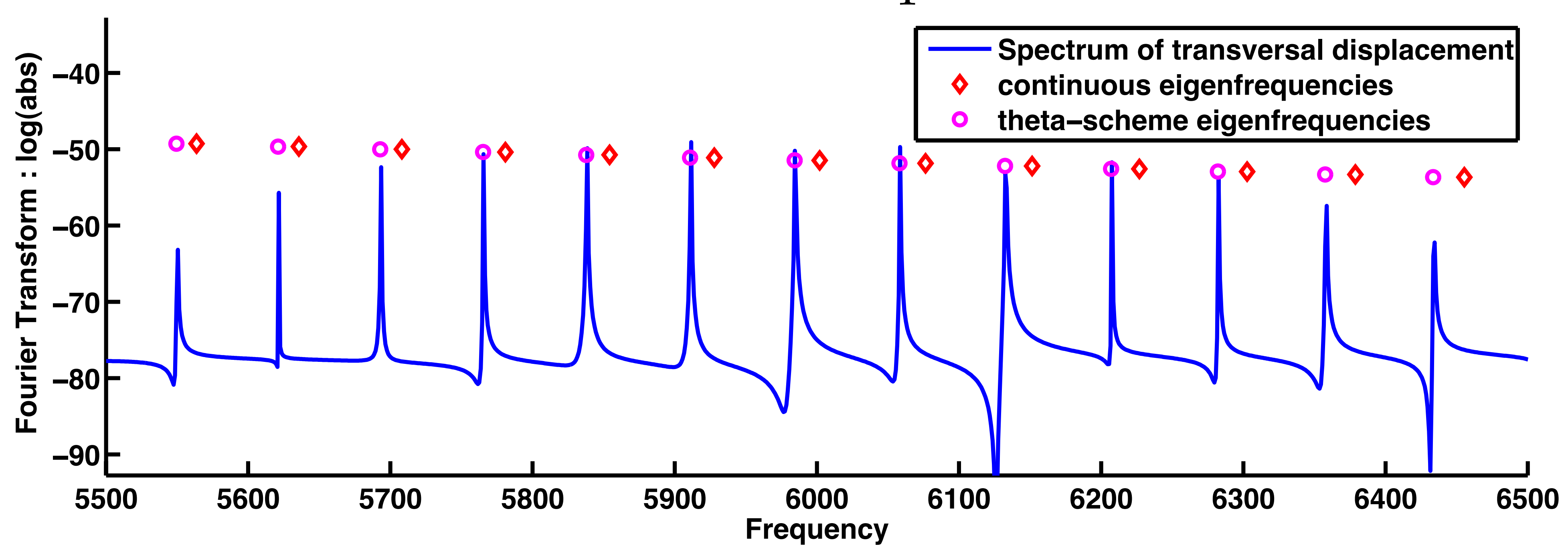
Dispersion analysis $\Delta t = 5 \times 10^{-6}$ second

Note D♯ 1 , $\theta = \frac{1}{4}, \quad \overline{\theta} = \frac{1}{12}$



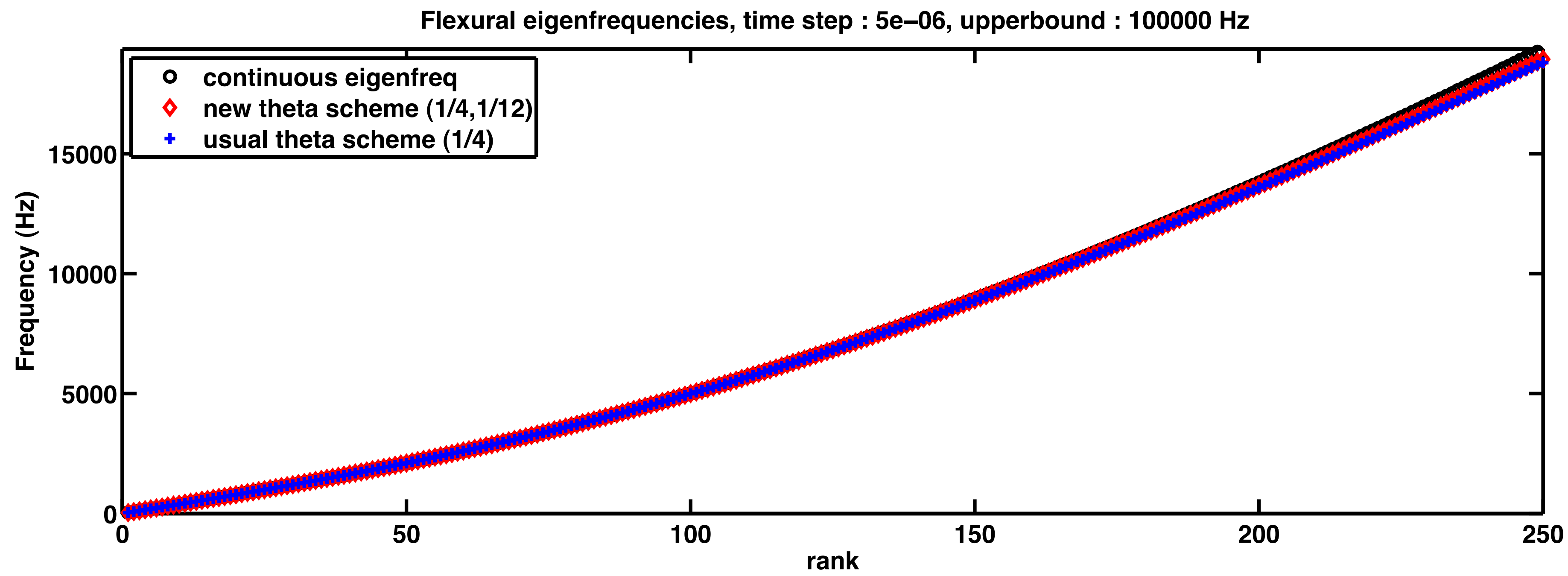
New scheme

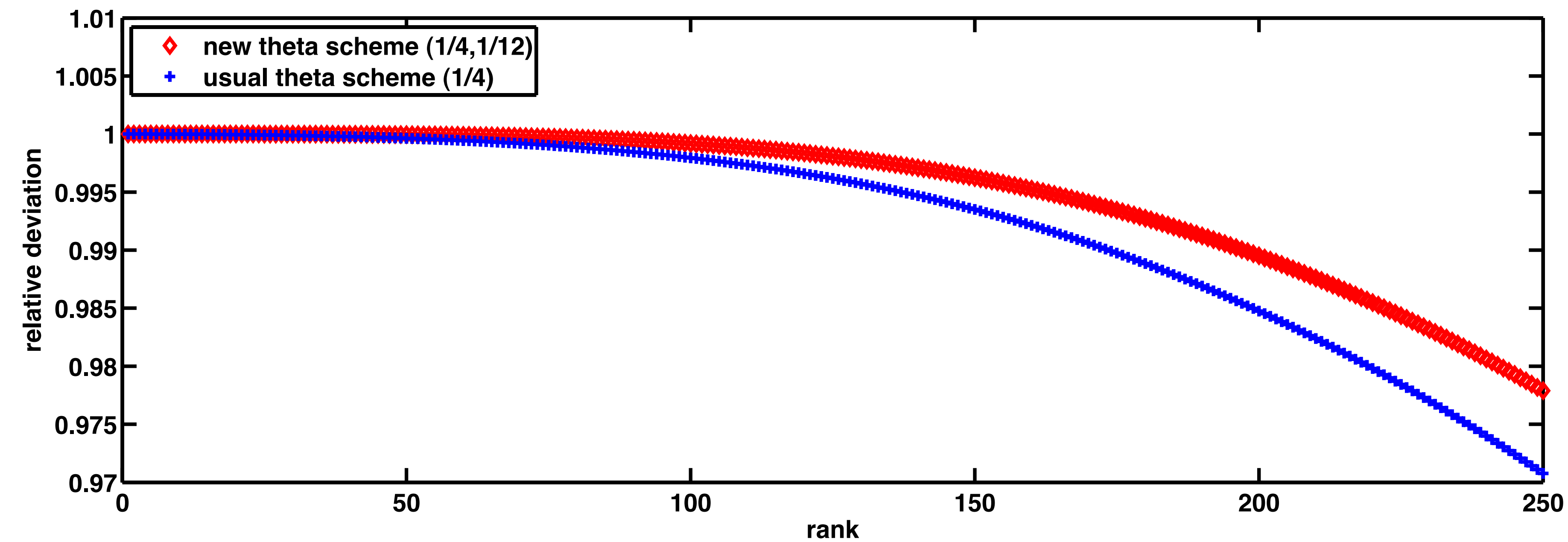
Note D♯ 1 , $\theta = \frac{1}{4}$



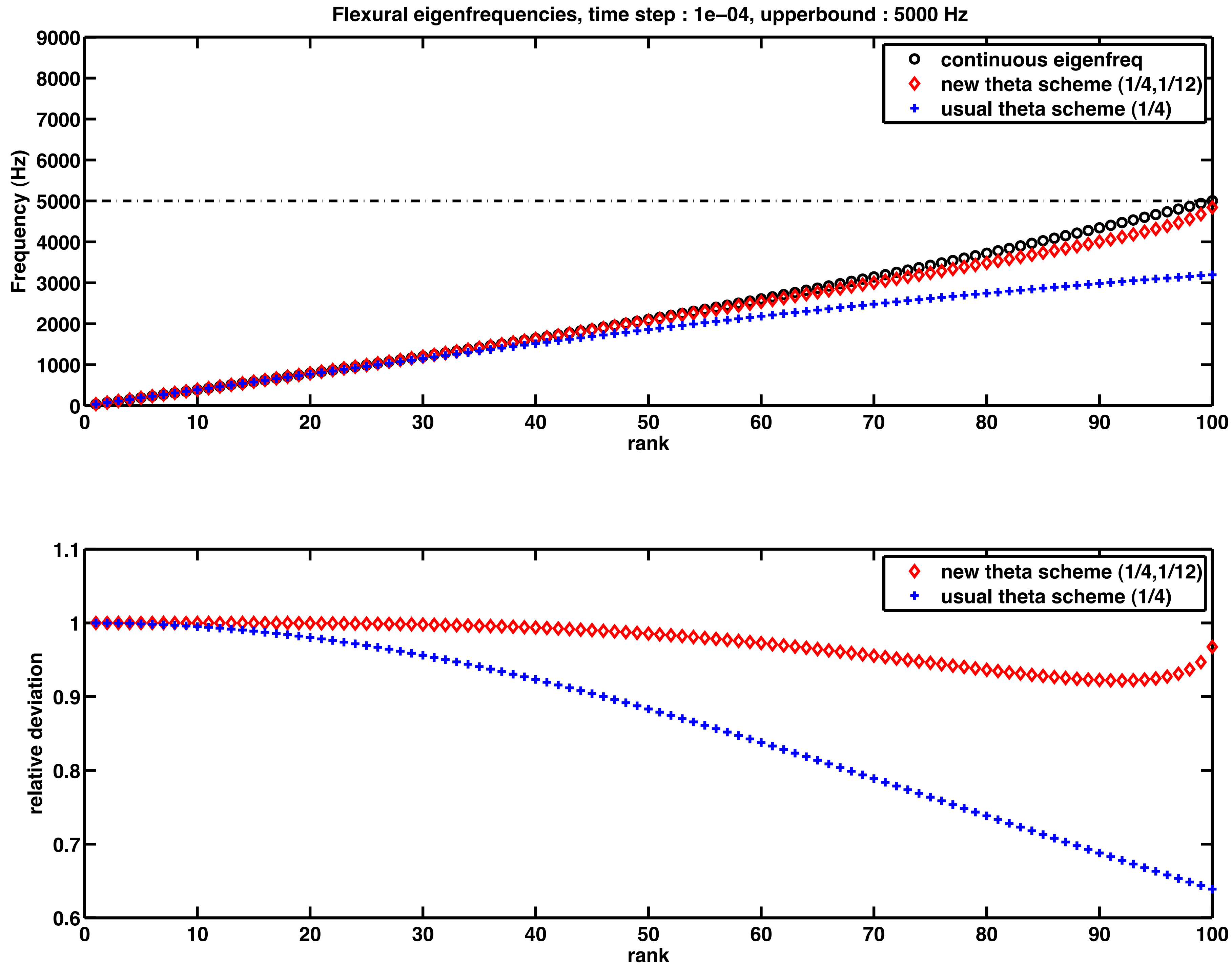
Classical scheme

Dispersion analysis

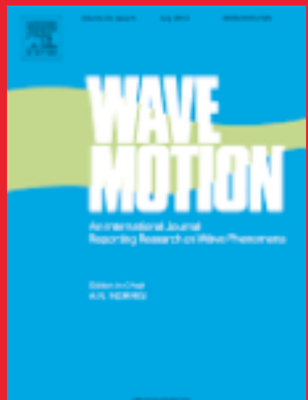




Dispersion analysis



Conclusions and Prospects



J Chabassier & S Imperiale : *Stability and dispersion analysis of improved time discretization for prestressed Timoshenko systems. Application to the stiff piano string.* Submitted to Wave Motion, 2012

Preprint available : people.bordeaux.inria.fr/chabassier

- Dissipative case : very easy improvement
- Other wave systems with contrasted velocities
  - P and S waves in soft material
  - acoustic & elastic waves in poro-elastic media
- Higher order in time ?

Thank you !