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A convergent finite volume scheme for two-phase flows in porous media with discontinuous capillary pressure field

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Two-phase flow problem

Continuity of phase $\alpha = \{o, w\}$

$$\phi \partial_t s + \nabla \cdot \mathbf{q}_\alpha = 0$$

Darcy law

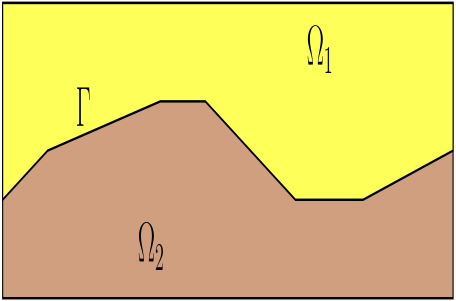
$$\mathbf{q}_\alpha = -\eta_{\alpha,i}(s_\alpha)(\nabla p_\alpha - \rho_\alpha \mathbf{g})$$

Immiscibility

$$s := s_o = 1 - s_w.$$

Capillary pressure law

$$p_o - p_w = \pi_i(s)$$



$\eta_i(s), \pi_i(s)$ depends on rocktype.

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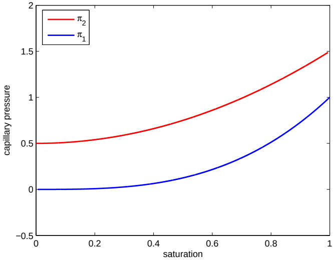

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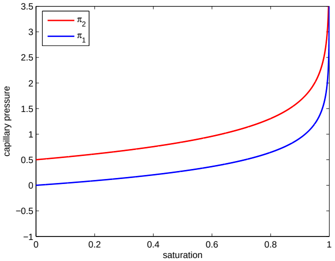

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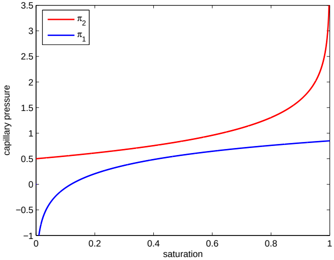

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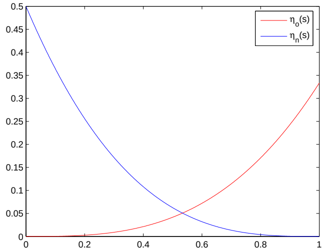


The natural topology for p_α is given by

$$\sum_{i=\{1,2\}} \int_0^T \int_{\Omega_i} \eta_i(s) |\nabla p_\alpha|^2,$$

if $s_\alpha = 0$, no control on p_α.

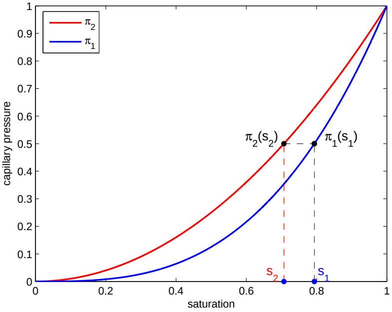
Simplified case

- $\pi_1(0) = \pi_2(0)$ and $\pi_1(1) = \pi_2(1)$

On Γ :

$$\mathbf{q}_{\alpha,1} = \mathbf{q}_{\alpha,2} \qquad \alpha \in \{o, w\},$$

$$p_{\alpha,1} = p_{\alpha,2} \qquad \alpha \in \{o, w\}.$$



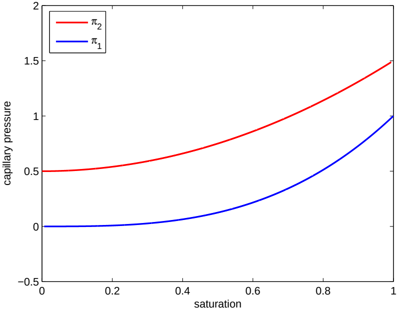
If $s_{\alpha,1}, s_{\alpha,2} \neq 0 \Rightarrow p_{\alpha,1} = p_{\alpha,2}$ for $\alpha \in \{o, w\}$.

$$\begin{aligned} (s_1, s_2) \in (0,1)^2 \quad &\Rightarrow \quad p_{\alpha,1} = p_{\alpha,2} \text{ for all } \alpha \in \{o, w\} \qquad (*) \\ &\Rightarrow \quad \pi_1(s_1) = \pi_2(s_2). \end{aligned}$$

By continuity $(*)$ is also true for $s_1 = s_2 = 0$ and $s_1 = s_2 = 1$.

General settings

What if $\pi_1(0) \neq \pi_2(0)$ and $\pi_1(1) \neq \pi_2(1)$



General settings

Conceder a sequence of regularized problems $\mathcal{P}^\varepsilon$ associated with $\pi_i^\varepsilon \in C^1([0,1])$

For each phase $\alpha \in \{o, w\}$:

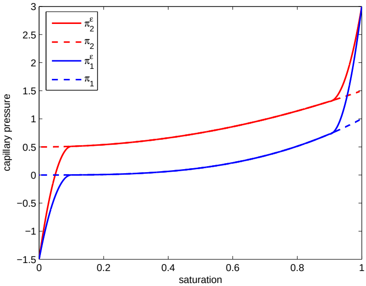
In Ω_i : $\quad \phi \partial_t s_\alpha^\varepsilon + \nabla \cdot \mathbf{q}_\alpha^\varepsilon = 0,$

$$\mathbf{q}_\alpha^\varepsilon = -\eta_{\alpha,i}(s_\alpha^\varepsilon)(\nabla p_\alpha^\varepsilon - \rho_\alpha \mathbf{g}$$

On Γ : $\quad \mathbf{q}_{\alpha,1}^\varepsilon = \mathbf{q}_{\alpha,2}^\varepsilon \quad \alpha \in \{o, w\},$

$$p_{\alpha,1}^\varepsilon = p_{\alpha,2}^\varepsilon \quad \alpha \in \{o, w\}.$$

Closure laws : $\quad s_o^\varepsilon = 1 - s_w^\varepsilon,$
$\quad p_o^\varepsilon - p_w^\varepsilon = \pi_i^\varepsilon(s_o^\varepsilon).$



$\pi_i^\varepsilon \longrightarrow \pi_i$ in $L^1((0,1))$.

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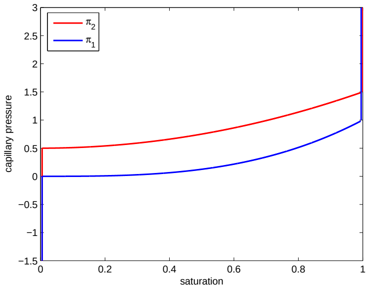

$$\pi_i^\varepsilon \longrightarrow \pi_i \text{ in } L^1((0,1)).$$

Transmission condition

The limit solution [Cancès *et al.*] satisfies :

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$$\tilde{\pi}_i(s) = \begin{cases} [-\infty, \pi_i(0)] & \text{if } s = 0, \\ \pi_i(s) & \text{if } s \in (0,1), \\ [\pi_i(1), +\infty] & \text{if } s = 1. \end{cases}$$

On Γ : $\mathbf{q}_{\alpha,1} = \mathbf{q}_{\alpha,2},$

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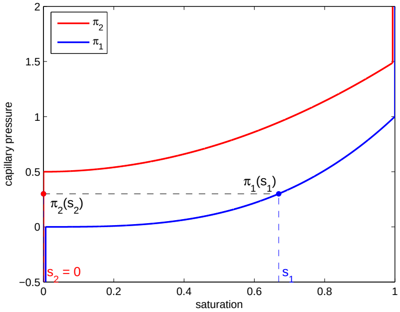
At the interface we have

$$\tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset.$$



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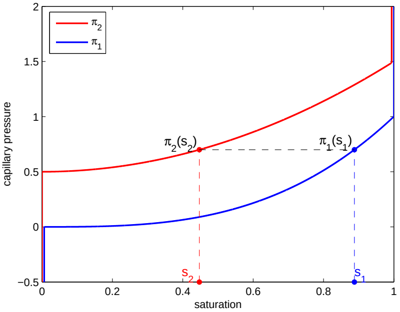

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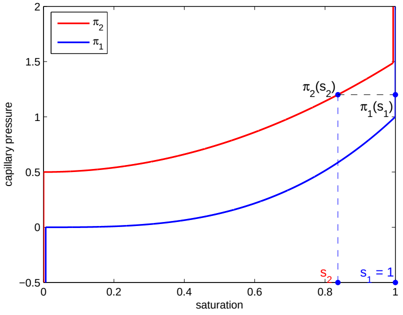


Global pressure formulation

In order to avoid the problems with degeneracy we introduce the global pressure [Chavent *et al.* '86]

$$\begin{aligned} P &= p_o - \int_0^{\pi_i(s)} \frac{\eta_{w,i}}{\eta_{o,i}+\eta_{w,i}} \left(\pi_i^{-1}(u)\right) du \\ &= p_w + \int_0^{\pi_i(s)} \frac{\eta_{o,i}}{\eta_{o,i}+\eta_{w,i}} \left(\pi_i^{-1}(u)\right) du = p_w + \lambda_{w,i}(\pi_i(s)) \end{aligned}$$

and the Kirchhoff transform

$$\varphi_i(s) = \int_0^{\pi_i(s)} \frac{\eta_{o,i}\eta_{w,i}}{\eta_{o,i}+\eta_{w,i}} \left(\pi_i^{-1}(u)\right) du.$$

The governing equations become

$$\nabla \cdot \mathbf{q} = 0, \text{ with } \mathbf{q} = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g},$$

$$\phi_i \partial_t s + \nabla \cdot \left(\mathbf{q} f_i(s) + \gamma_i(s)\mathbf{g} - \nabla \varphi_i(s)\right) = 0.$$

Interface condition

At the interface we prescribe :

The mass conservation

$$\sum_{i\in\{1,2\}} \mathbf{q}_i \cdot \mathbf{n}_i = 0,$$
$$\sum_{i\in\{1,2\}} (f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s)) \cdot \mathbf{n}_i = 0.$$

The continuity of phase pressures

$$\exists \pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \text{ s.t. } P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi),$$

Mesh and discrete unknowns

- We consider the admissible mesh s.t.

$$\sigma = K|L \perp [x_K, x_L]$$

- Mesh resolve the interface Γ:

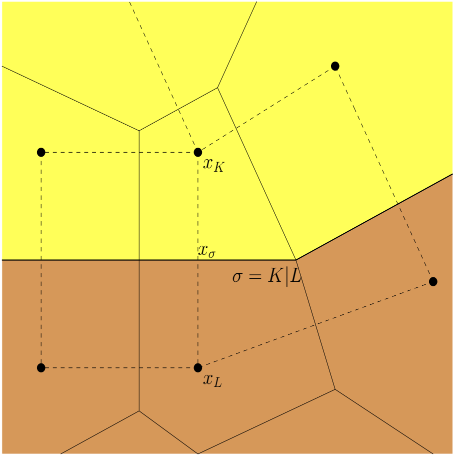
$$\forall K \subset \Omega_i$$

- Two unknowns per cell:

$$(s_K^n, P_K^n)$$

- Five unknowns per interface $\sigma \subset \Gamma$:

$$(s_{K,\sigma}^n, s_{L,\sigma}^n, P_{K,\sigma}^n, P_{L,\sigma}^n, \pi_\sigma^n).$$



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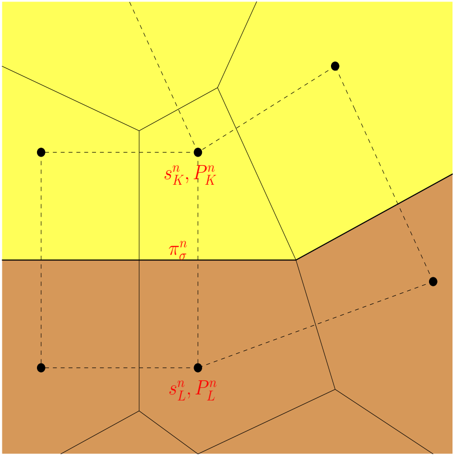

Discrete fluxes

We first discretize

$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q} = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g},$$

by

$$\sum_{\sigma \in \mathcal{E}_K} m(\sigma) Q_{K,\sigma}^{n+1} = 0, \qquad \forall n \in \{0, \dots, N\}, \forall K \in \mathcal{T},$$

where

$$Q_{K,\sigma}^n = \begin{cases} \frac{M_{K,L}(s_K^n, s_L^n)}{d_{K,L}} (P_K^n - P_L^n) + \mathcal{R}(Z_{K,\sigma}; s_K^n, s_L^n) & \text{if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ \frac{M_K(s_K^n)}{d_{K,\sigma}} \left(P_K^n - P_{K,\sigma}^n\right) + \mathcal{R}\left(Z_{K,\sigma}; s_K^n, s_{K,\sigma}^n\right) & \text{if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 & \text{if } \sigma \in \mathcal{E}_{K,\text{ext}}, \end{cases}$$

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$M_{K,L}(s^{n+1}_K, s^{n+1}_L)$ is a mean value between $M_K(s^{n+1}_K)$ and $M_L(s^{n+1}_L)$

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The function $Z_{K,\sigma}$ is defined by $Z_{K,\sigma}(s) = \zeta_K(s)\mathbf{g}\cdot\mathbf{n}_{K,\sigma}$ and $\mathcal{R}(f;a,b)$ is the Riemann solver

$$\mathcal{R}(f;a,b) = \begin{cases} \min_{c\in[a,b]} f(c) & \text{if } a \le b, \\ \max_{c\in[b,a]} f(c) & \text{if } b \le a. \end{cases}$$

Discrete fluxes

The discrete version of

$$\phi_i \partial_t s + \nabla \cdot (\mathbf{q} f_i(s) + \gamma_i(s)\mathbf{g} - \nabla \varphi_i(s)) = 0.$$

reads

$$\phi_K \frac{s_K^{n+1} - s_K^n}{\delta t} m(K) + \sum_{\sigma \in \mathcal{E}_K} m(\sigma) F_{K,\sigma}^{n+1} = 0,$$

where

$$F_{K,\sigma}^n = \begin{cases} Q_{K,\sigma}^n f_K(\overline{s}_{K,\sigma}^n) + \mathcal{R}(G_{K,\sigma}; s_K^n, s_L^n) + \frac{\varphi_K(s_K^n) - \varphi_K(s_L^n)}{d_{K,L}} \text{ if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ Q_{K,\sigma}^n f_K(\overline{s}_{K,\sigma}^n) + \mathcal{R}(G_{K,\sigma}; s_K^n, s_{K,\sigma}^n) + \frac{\varphi_K(s_K^n) - \varphi_K(s_{K,\sigma}^n)}{d_{K,\sigma}} \text{ if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 \text{ if } \sigma \in \mathcal{E}_{K,\text{ext}}, \end{cases}$$

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$\overline{s}_{K,\sigma}^{n+1}$ is the upwind value defined by

$$\overline{s}_{K,\sigma}^{n+1} = \begin{cases} s_K^{n+1} & \text{if } Q_{K,\sigma}^{n+1} \geq 0, \\ s_L^{n+1} & \text{if } Q_{K,\sigma}^{n+1} < 0 \text{ and } \sigma = K|L \in \mathcal{E}_{K,i}, \\ s_{K,\sigma}^{n+1} & \text{if } Q_{K,\sigma}^{n+1} < 0 \text{ and } \sigma \in \mathcal{E}_{K,\Gamma}. \end{cases}$$

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Two-point flux approximation.

Interface condition system

At the interface Γ we impose

$$Q_{K,\sigma}^{n+1} + Q_{L,\sigma}^{n+1} = 0,$$

$$F_{K,\sigma}^{n+1} + F_{L,\sigma}^{n+1} = 0,$$

and

$$\exists \pi_\sigma^{n+1} \in \tilde{\pi}_K(s_{K,\sigma}^{n+1}) \cap \tilde{\pi}_L(s_{L,\sigma}^{n+1}) \text{ s.t.}$$

$$P_{K,\sigma}^{n+1} - \lambda_{w,1}(\pi_\sigma^{n+1}) = P_{L,\sigma}^{n+1} - \lambda_{w,2}(\pi_\sigma^{n+1}).$$

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$$\exists \pi_\sigma^{n+1} \in \tilde{\pi}_K(s_{K,\sigma}^{n+1}) \cap \tilde{\pi}_L(s_{L,\sigma}^{n+1}) \text{ s.t.}$$

$$P_{K,\sigma}^{n+1} - \lambda_{w,1}(\pi_\sigma^{n+1}) = P_{L,\sigma}^{n+1} - \lambda_{w,2}(\pi_\sigma^{n+1}).$$

Remark that $s_{K,\sigma}^{n+1} = \tilde{\pi}_K^{-1}(\pi_\sigma^{n+1}), s_{L,\sigma}^{n+1} = \tilde{\pi}_L^{-1}(\pi_\sigma^{n+1})$, thus $(P_{K,\sigma}^{n+1}, P_{L,\sigma}^{n+1})$ can be eliminated.

Interface condition system

At the interface Γ we impose

$$Q_{K,\sigma}^{n+1} + Q_{L,\sigma}^{n+1} = 0,$$

$$F_{K,\sigma}^{n+1} + F_{L,\sigma}^{n+1} = 0,$$

and

$$\exists \pi_\sigma^{n+1} \in \tilde{\pi}_K(s_{K,\sigma}^{n+1}) \cap \tilde{\pi}_L(s_{L,\sigma}^{n+1}) \text{ s.t.}$$

$$P_{K,\sigma}^{n+1} - \lambda_{w,1}(\pi_\sigma^{n+1}) = P_{L,\sigma}^{n+1} - \lambda_{w,2}(\pi_\sigma^{n+1}).$$

Proposition

Let $\sigma = K|L \in \Gamma$, and let $\left(s_K^{n+1}, s_L^{n+1}, P_K^{n+1}, P_L^{n+1}\right) \in \mathbb{R}^4$, then there exists $\pi_\sigma^{n+1} \in [\min_i \pi_i(0), \max_i \pi_i(1)]$ satisfying the nonlinear system interface problem.

A priori estimates

A discrete solution satisfies

$L^\infty(\Omega \times (0,1))$ estimate

$$0 \leq s_K \leq 1 \text{ for all } K \in \mathcal{M}$$

and

Energy estimates

$$\sum_{i \in \{1,2\}} |\varphi_i(s_{\mathcal{D}})|^2_{\mathcal{D}} + |P_{\mathcal{D}}|^2_{\mathcal{D}} \leq C$$

Convergence result

Theorem (main result)

Let $(\mathcal{D}_m)_{m\geq 0}$ be a sequence of admissible discretizations of Q such that size$(\mathcal{D}_m)$ tends to 0 and reg$(\mathcal{D}_m)$ remains uniformly bounded as m tends to ∞. We denote by $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m})_m$ a corresponding sequence of discrete solutions

$$s_{\mathcal{D}_m} \to s \text{ strongly in } L^2(Q) \text{ as } m \to \infty,$$

$$P_{\mathcal{D}_m} \to P \text{ weakly in } L^2(Q) \text{ as } m \to \infty.$$

where (s, P) is a weak solution of the continuous problem.

1 Transmission conditions

2 Global pressure formulation

3 Finite volume scheme

4 Numerical results

Numerical results

We consider a model porous medium $\Omega = (0,1)^2$ composed of two layers.

Oil and water mobilities

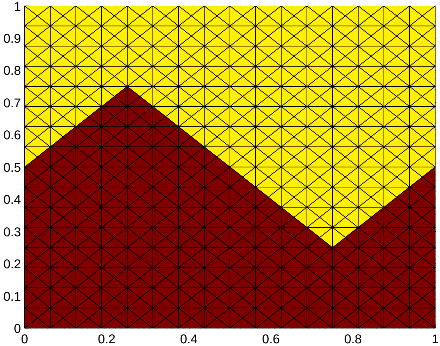
$$\eta_{o,i}(s) = 0.5s^2, \quad \eta_{w,i} = (1-s)^2.$$

Capillary pressure curves

$$\pi_1(s) = s + 0.5, \quad \pi_2(s) = s.$$

Initial saturation is given by

$$s_0(x) = \begin{cases} 0 & \text{if } x \in \Omega_1, \\ 0.3 & \text{otherwise.} \end{cases}$$



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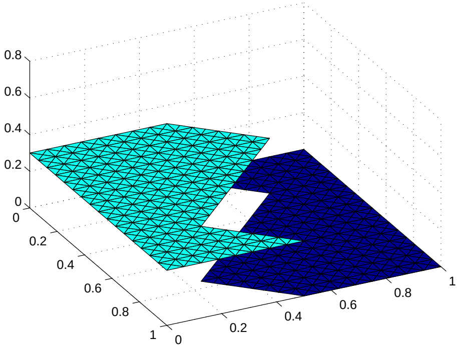
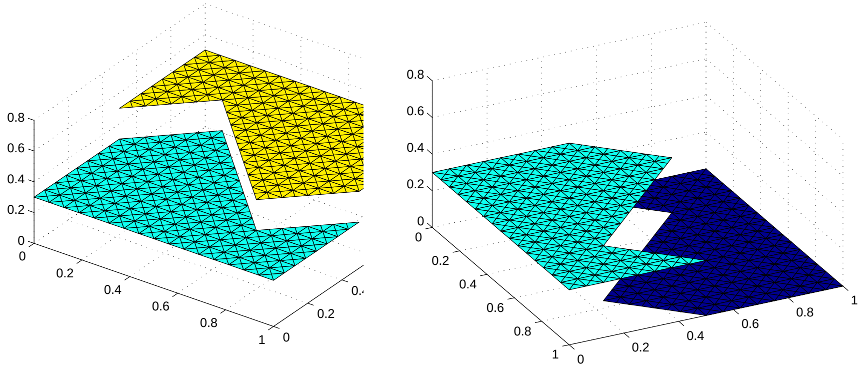
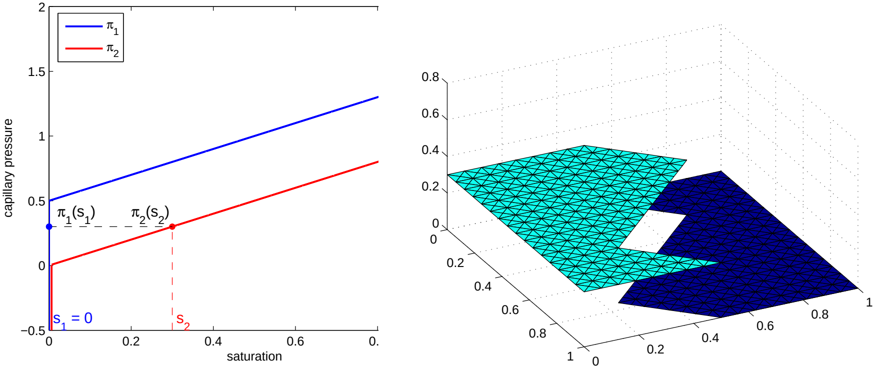


Numerical results



Capillary pressure field (left) and saturation field (right) at time $t = 0$.

Numerical results



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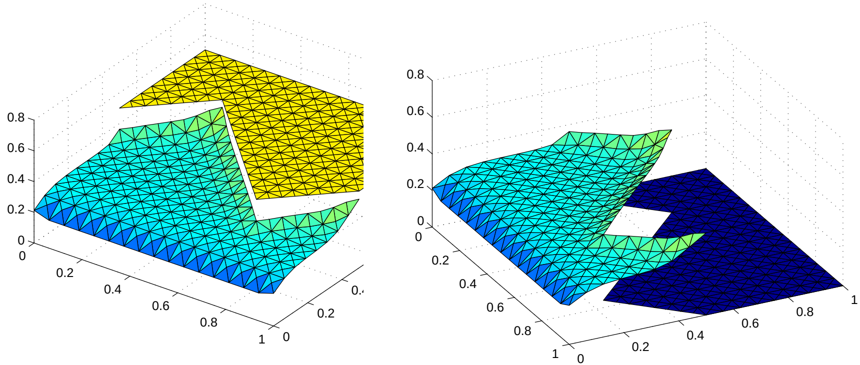
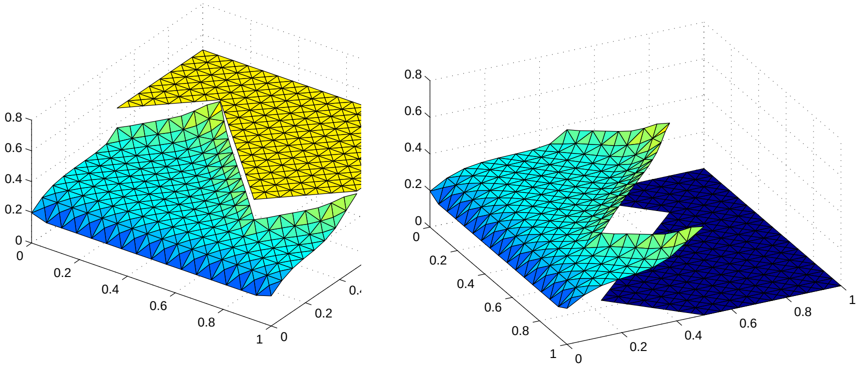


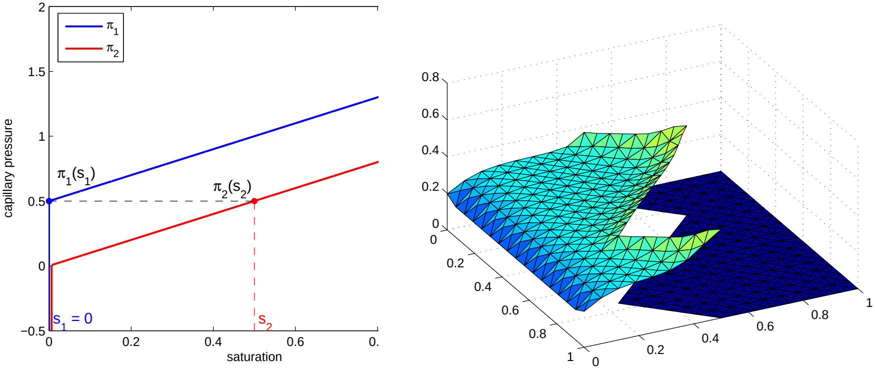
Capillary pressure field (left) and saturation field (right) at time $t = 0.08$.

Numerical results



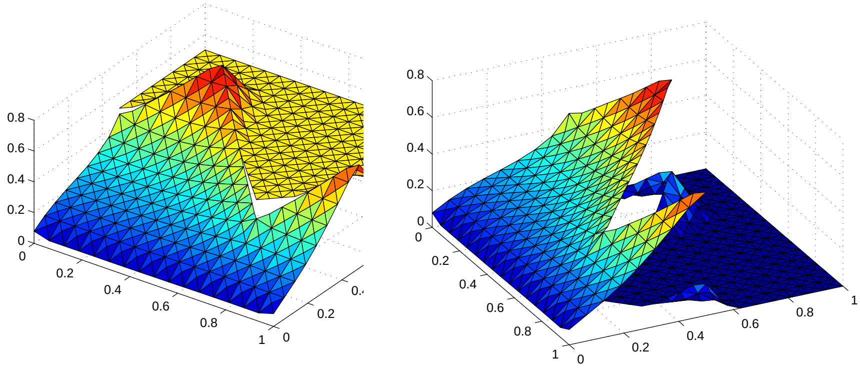
Capillary pressure field (left) and saturation field (right) at time $t = 0.11$.

Numerical results



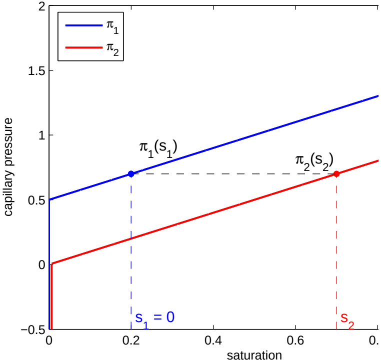
Capillary pressure field (left) and saturation field (right) at time $t = 0.11$.

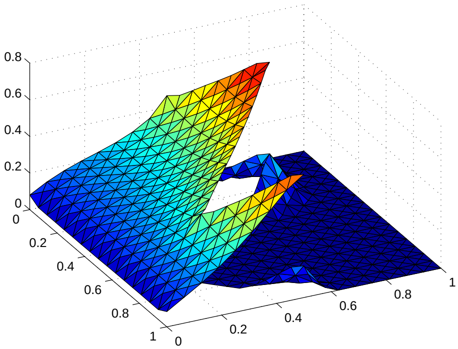
Numerical results



Capillary pressure field (left) and saturation field (right) at time $t = 0.6$.

Numerical results





Capillary pressure field (left) and saturation field (right) at time $t = 0.6$.

Numerical results

We consider a model porous medium $\Omega = (0,1)^2$ composed of two layers.

Oil and water mobilities

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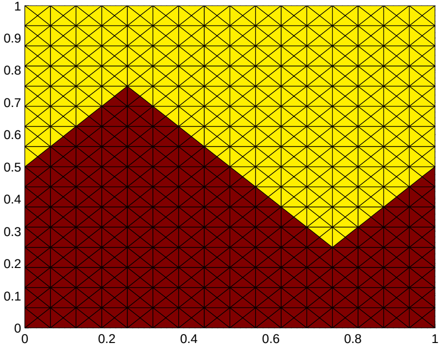

Numerical results

We consider a model porous medium $\Omega = (0,1)^2$ composed of two layers.

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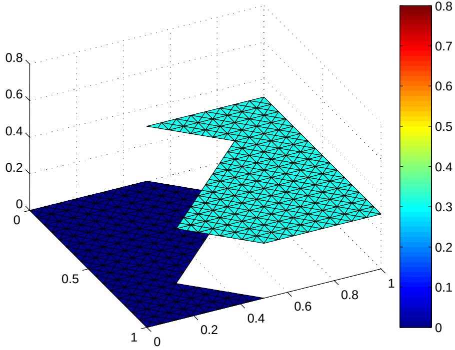
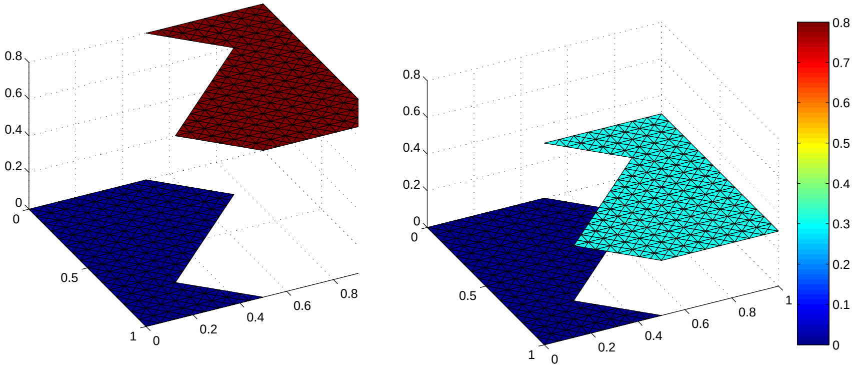
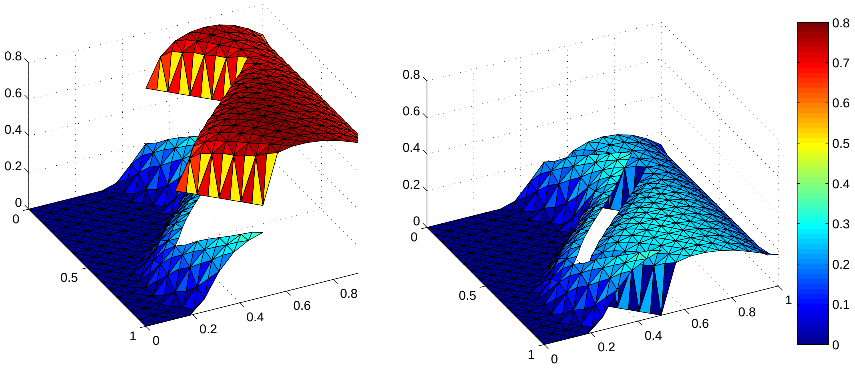


Numerical results



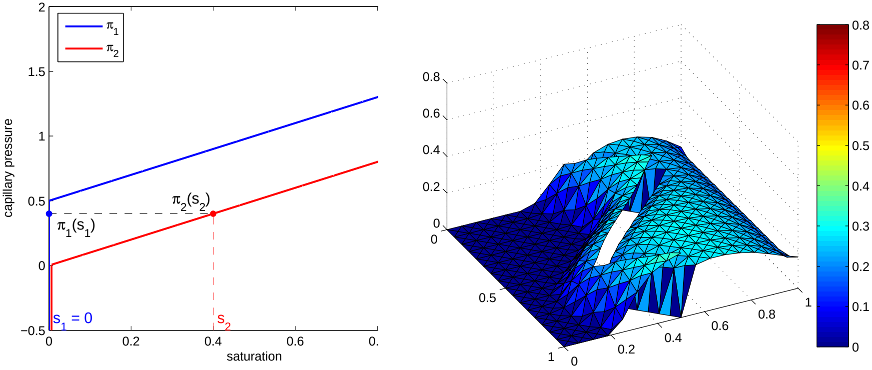
Capillary pressure field (left) and saturation field (right) at time $t = 0$.

Numerical results

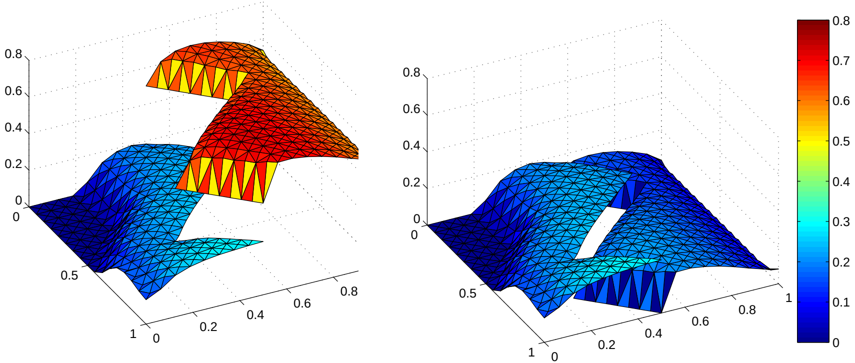


Capillary pressure field (left) and saturation field (right) at time $t = 0.3$.

Numerical results

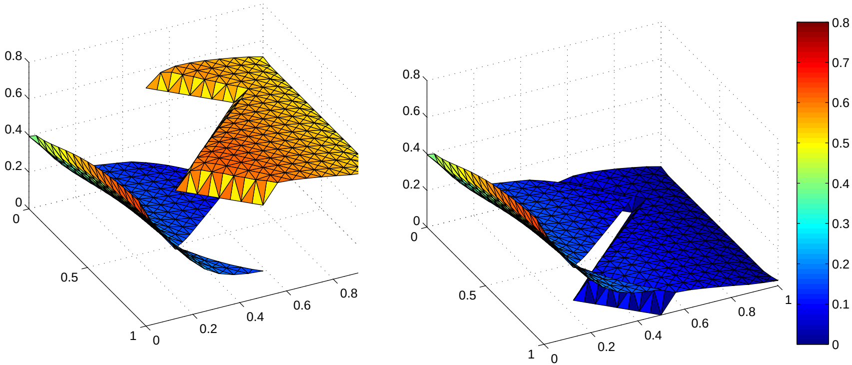


Capillary pressure field (left) and saturation field (right) at time $t = 0.3$.



Capillary pressure field (left) and saturation field (right) at time $t = 1.0$.

Numerical results



Capillary pressure field (left) and saturation field (right) at time $t = 2.9$.

Conclusion and outlooks

Conclusion

- We have a convergent numerical scheme for a very complex problem
- The numerical results shows a 'good' quantitative behavior

Outlooks

- Comparison with other numerical methods
- Extension for the anisotropic problems
- Compressible flow, dissolution

Thank you for you attention!