Absorbing Boundary Conditions for anisotropic elastodynamic media

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Context

Geophysics:

- Hydrocarbons detection: petroleum or natural gas
- Seismic shots: sources and receptors
- Earth medium: seismic waves (body & surface)
- Subsurface case: P-SV waves (compressional/shear)
- Reflection seismology: migration steps

Reverse Time Migration (RTM)

Iterative method based on multiple wave equation resolutions.

To be more realistic:

- (pseudo-)acoustic $\rightarrow$ elastic
- isotropy $\rightarrow$ anisotropy

Introduction

Infinite geophysical domains compared to the wavelengths:

- Reduction of the computational domain to a box (source & receivers)
- Design of efficient boundary conditions (attenuate the reflections)

Two common ways:

Perfectly Matched Layers (PMLs), [Bérenger 1994, 1996]

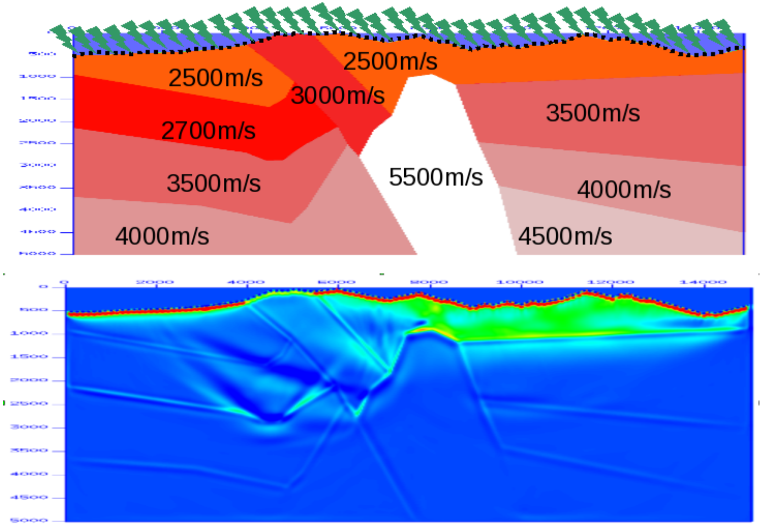
Equation for a layer all around the problem domain
$\rightarrow$ easy to implement, instabilities may appear (anisotropy)

Absorbing Boundary Conditions (ABCs), [Enquist-Majda 1977, 1979]

Equation for the boundary of the problem domain
$\rightarrow$ stable, low-order easy to implement but not very accurate

RTM framework: spurious reflections considered as noise!

Example: RTM in acoustics



Outline

Elastic wave equation

Let us consider:

- Ω, an open bounded domain of $\mathbb{R}^2$
- $\Gamma = \partial\Omega$, the domain boundary; $\mathbf{n}_\Gamma$, the exterior normal
- $\mathbf{x} = (x, z) \in \Omega$ and $t \in [0, T]$, the space and time variables

Velocity-stress formulation [Virieux 1986]

$$\begin{cases} \rho(\mathbf{x})\partial_t v(\mathbf{x}, t) & = \nabla.\underline{\underline{\sigma}}(\mathbf{x}, t) \\ \partial_t \underline{\underline{\sigma}}(\mathbf{x}, t) & = \underline{\underline{C}}(\mathbf{x}) : \underline{\underline{\epsilon}}(v(\mathbf{x}, t)) \end{cases} \quad (1)$$

- $\rho > 0$, the density
- $v \in H^1(\Omega \times [0, T])$, the unknown velocity field
- $\underline{\underline{\sigma}}$ such that $\{\sigma_{ij} \in H^1(\Omega \times [0, T]) \ \forall i, j \in \{x, z\}\}$, the stress tensor
- $\underline{\underline{C}}$, the stiffness tensor (elasticity coefficients)
- $\underline{\underline{\epsilon}}(v) = \frac{1}{2}(\boldsymbol{\nabla} v + (\boldsymbol{\nabla} v)^T)$, the strain tensor (infinitesimal strain theory)

Stiffness tensor

$\underline{\underline{C}}$ properties:

- fourth-rank: C_{ijkl} with $i, j, k, l \in \{x, z\}$
- symmetries: $C_{ijkl} = C_{jikl} = C_{jilk} = C_{klij}$

Matrix view

2D Voigt notation: $ij(\nu)$ with $xx(1),\ zz(2),\ xz(3)$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \tag{2}$$

Geophysics anisotropy

Earth's crust (geological layers of rocks) is assumed to be locally polar anisotropic, also called transversely isotropic (TI).

Isotropic case

Two points of view:

- The two Lamé coefficients $\lambda, \ \mu$

$$C_{11} = \lambda + 2\mu, \ C_{22} = \lambda + 2\mu, \ C_{33} = \mu, \ C_{12} = \lambda$$

- P-waves and S-waves velocities V_p and V_s, and ρ

$$C_{11} = \rho V_p^2, \ C_{22} = \rho V_p^2, \ C_{33} = \rho V_s^2, \ C_{12} = \rho(V_p^2 - 2V_s^2)$$

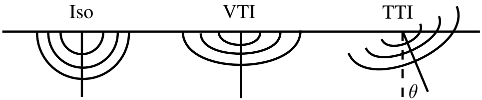


Figure: Wavefronts for isotropy and transverse isotropy (vertical and tilted)

Vertical TI case

Physics parameters:

- P-waves and S-waves velocities V_p and V_s, and ρ
- VTI coefficients $\varepsilon,\ \delta,\ \gamma$ [Thomsen 1986]

$$C_{11} = \rho V_p^2(1+2\varepsilon),\ \ C_{22} = \rho V_p^2,\ \ C_{33} = \rho V_s^2,$$

$$C_{12} = \rho\left(\sqrt{(V_p^2 - V_s^2)^2 + 2\delta V_p^2(V_p^2 - V_s^2)} - V_s^2\right).$$

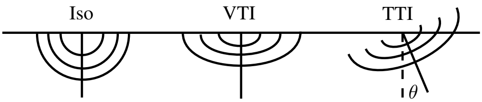


Figure: Wavefronts for isotropy and transverse isotropy (vertical and tilted)

Tilted TI case

Physics parameters:

- P-waves and S-waves velocities V_p and V_s, and ρ
- VTI coefficients ε, δ, γ and TTI angle θ

TTI from VTI

$$\underline{\underline{C}}^{TTI}_{ijkl} = \sum_p \sum_q \sum_r \sum_s R_{pi} R_{qj} R_{rk} R_{sl} \underline{\underline{C}}^{VTI}_{pqrs}, \ \forall \{i,j,k,l\} \quad (3)$$

with the rotation matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

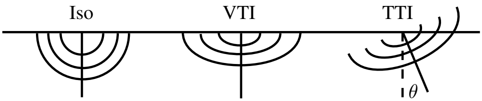


Figure: Wavefronts for isotropy and transverse isotropy (vertical and tilted)

Outline

Space discretization 1/2

Ω_h: polygonal mesh (partition of Ω), composed of triangles K in 2D, with Γ_{in}, the internal boundaries and Γ_{out}, the external boundaries.

Discontinuous Galerkin Method (DGM)

- Approximate v and $\underline{\underline{\sigma}}$ by discontinuous functions such that $\{v, \underline{\underline{\sigma}}_h\} \in L^2(\Omega_h \times [0, T])$ with $\{v|_K, \underline{\underline{\sigma}}|_K\} \in H^1(K \times [0, T])\ \forall K$.
- Integrate against discontinuous test functions w and $\underline{\underline{\xi}}$.

Integration on K

$$\begin{cases} \displaystyle\int_K \rho_K \partial_t v w d\mathbf{x} & = \displaystyle\int_{\partial_K} \underline{\underline{\sigma}} \mathbf{n_K} . w d\mathbf{x} - \int_K \underline{\underline{\sigma}} : \boldsymbol{\nabla} w d\mathbf{x} \\ \displaystyle\int_K \partial_t \underline{\underline{\sigma}} : \underline{\underline{\xi}} d\mathbf{x} & = \displaystyle\int_{\partial_K} (\underline{\underline{C}} : \underline{\underline{\xi}}) \mathbf{n_K} . v d\mathbf{x} - \int_K v . \nabla . (\underline{\underline{C}} : \underline{\underline{\xi}}) d\mathbf{x} \end{cases} \tag{4}$$

Space discretization 2/2

Let us consider $\overline{\Gamma} = \Gamma_{in} \cup \Gamma_{out}$.

$$\begin{cases} \displaystyle\sum_K \int_K \rho_K \partial_t v w d\mathbf{x} = \\ \qquad \displaystyle\sum_{\overline{\Gamma}} \int_{\overline{\Gamma}} \{\!\{\underline{\underline{\sigma}} \mathbf{n_K}\}\!\}.[\![w]\!] d\mathbf{x} - \sum_K \int_K \underline{\underline{\sigma}} : \boldsymbol{\nabla} w d\mathbf{x} \\ \displaystyle\sum_K \int_K \partial_t \underline{\underline{\sigma}} : \underline{\underline{\xi}} d\mathbf{x} = \\ \qquad \displaystyle\sum_{\overline{\Gamma}} \int_{\overline{\Gamma}} [\![(\underline{\underline{C}} : \underline{\underline{\xi}}) \mathbf{n_K}]\!].\{\!\{v\}\!\} d\mathbf{x} - \sum_K \int_K v.\nabla.(\underline{\underline{C}} : \underline{\underline{\xi}}) d\mathbf{x} \end{cases} \tag{5}$$

Semi-discrete form

$$\begin{cases} M_v \partial_t v_h + R_{\underline{\underline{\sigma}}} \underline{\underline{\sigma}}_h & = 0 \\ M_{\underline{\underline{\sigma}}} \partial_t \underline{\underline{\sigma}}_h + R_v v_h & = 0 \end{cases} \tag{6}$$

Time discretization

Time domain $[0, T]$ divided into time steps Δt:

Leap-frog time scheme

Iteration on n:

$$\begin{cases} M_v \dfrac{v_h^{n+1} - v_h^n}{\Delta t} + R_{\underline{\underline{\sigma}}} \underline{\underline{\sigma}}_h^{n+1/2} & = 0 \\ M_{\underline{\underline{\sigma}}} \dfrac{\underline{\underline{\sigma}}_h^{n+3/2} - \underline{\underline{\sigma}}_h^{n+1/2}}{\Delta t} + R_v v_h^{n+1} & = 0 \end{cases} \quad (7)$$

M_v and $M_{\underline{\underline{\sigma}}}$ block-diagonal matrices $\Rightarrow$ quasi-explicit scheme!

Stable under a CFL condition, [Delcourte et al. 2009]

Outline

Domains

Let us consider two cases:

- a rectangle mesh (vertical and horizontal boundaries)
- a rhombus mesh (actually a rectangle mesh rotated by an angle $\pi/6$)

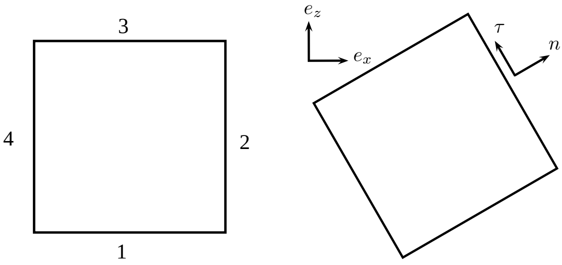


Figure: Rectangle and rhombus meshes

VTI on a rhombus mesh is similar to TTI on a rectangular mesh!

Rectangle VTI ABCs

Basic reasoning:

- To decompose VTI wave equation in P-waves and S-waves only
- To apply Enquist-Majda's methodology [Enquist-Majda 1977, 1979]

Assuming $Cp_{max} = -\sqrt{\rho C_{11}}$, $Cp_{min} = -\sqrt{\rho C_{22}}$ and $Cs = -\sqrt{\rho C_{33}}$:

Boundary	P-waves	S-waves
1	$\sigma_{zz} = -Cp_{min} v_z$	$\sigma_{xz} = -Cs v_x$
2	$\sigma_{xx} = +Cp_{max} v_x$	$\sigma_{xz} = +Cs v_z$
3	$\sigma_{zz} = +Cp_{min} v_z$	$\sigma_{xz} = +Cs v_x$
4	$\sigma_{xx} = -Cp_{max} v_x$	$\sigma_{xz} = -Cs v_z$

Table: VTI ABCs on the rectangle boundaries

Isotropic case → [Tsokga 1999] low-order ABCs (called order 1/2)

Results: isotropic rectangle

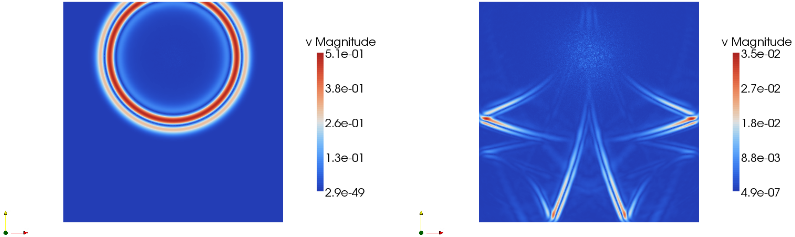


Figure: Isotropic reference case

Results: VTI rectangle

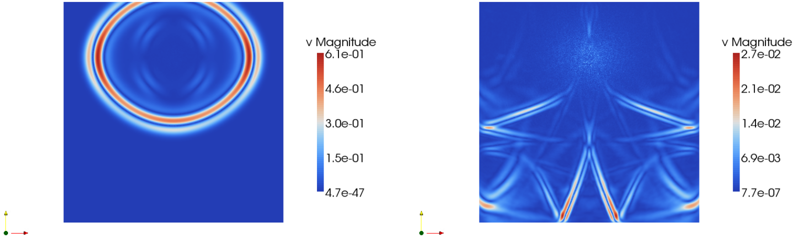


Figure: VTI, rectangle mesh

Arbitrary boundaries

Let us recall:

- $\mathbf{n} = (n_x, n_z)$, normal vector exterior of Γ
- $(\mathbf{n}, \boldsymbol{\tau})$, the associated basis (varying depending on the boundary)

Where does ABC appear in the DGM?

Surface term of the first equation

$$\int_{\Gamma} \underline{\underline{\sigma}} \mathbf{n}_{\Gamma} . w d\mathbf{x} \tag{8}$$

Decomposition of $\underline{\underline{\sigma}}\mathbf{n}$ in the $(\mathbf{n}, \boldsymbol{\tau})$ basis:

$$\underline{\underline{\sigma}}\mathbf{n} = \begin{pmatrix} \sigma_{nn} & \sigma_{n\tau} \\ \sigma_{\tau n} & \sigma_{\tau\tau} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{nn} \\ \sigma_{\tau n} \end{pmatrix}$$

It is sufficient to consider σ_{nn} and $\sigma_{\tau n}$ for ABCs on arbitrary boundaries!

VTI ABC

Idea: develop the boundary integral in the $(\mathbf{n}, \boldsymbol{\tau})$ basis and insert the ABC.

Surface term of the first equation

$$\int_{\Gamma} \underline{\underline{\sigma}} \mathbf{n}_{\Gamma} . w d\mathbf{x} \tag{9}$$

$$\int_{\Gamma} [(Cp_{max} \mathbf{V}.\mathbf{n} n_x - Cs \mathbf{V}.\boldsymbol{\tau} n_z) w_x + (Cp_{min} \mathbf{V}.\mathbf{n} n_z + Cs \mathbf{V}.\boldsymbol{\tau} n_x) w_z] \tag{10}$$

This can be obtained from the ABC VTI formulation:

VTI ABC

$$\begin{cases} \sigma_{nn} &= (Cp_{max} n_x^2 + Cp_{min} n_z^2) \mathbf{V}.\mathbf{n} \\ \sigma_{\tau n} &= -((Cp_{max} - Cp_{min}) n_x n_z) \mathbf{V}.\mathbf{n} + Cs \mathbf{V}.\boldsymbol{\tau} \end{cases} \tag{11}$$

Results: VTI rhombus

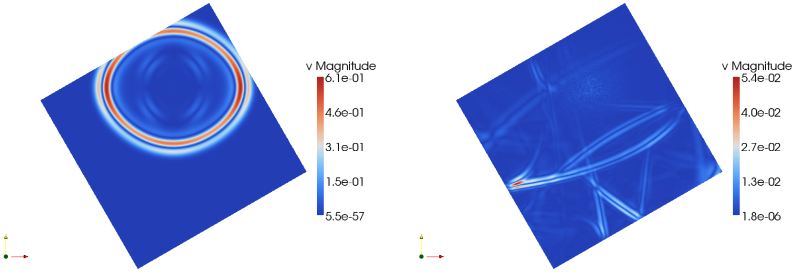


Figure: VTI, rhombus mesh

Outline

α and θ angles

Let us define $\mathbf{n}'$, the VTI direction vector by

$$\begin{pmatrix} n'_x \\ n'_z \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

$\mathbf{n}'$ is independent of the boundary (contrary to $\mathbf{n}$, the face exterior normal).

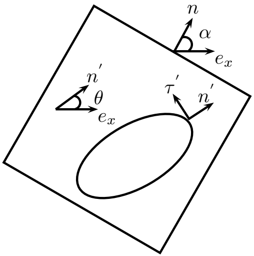


Figure: $(\mathbf{n}', \boldsymbol{\tau}')$ basis, θ and α angles

TTI ABC

Let us introduce (l_x, l_z), the coordinates of $\mathbf{n}$ in the $(\mathbf{n}', \boldsymbol{\tau}')$ basis:

$$\begin{pmatrix} l_x \\ l_z \end{pmatrix} = \begin{pmatrix} \cos(\alpha - \theta) \\ \sin(\alpha - \theta) \end{pmatrix}.$$

VTI ABC

$$\begin{cases} \sigma_{nn} &= (Cp_{max} n_x{}^2 + Cp_{min} n_z{}^2)\mathbf{V}.\mathbf{n} \\ \sigma_{\tau n} &= -((Cp_{max} - Cp_{min}) n_x n_z)\mathbf{V}.\mathbf{n} + Cs\mathbf{V}.\boldsymbol{\tau} \end{cases} \quad (12)$$

TTI ABC

$$\begin{cases} \sigma_{nn} &= (Cp_{max} l_x^2 + Cp_{min} l_z^2)\mathbf{V}.\mathbf{n} \\ \sigma_{\tau n} &= -((Cp_{max} - Cp_{min}) l_x l_z)\mathbf{V}.\mathbf{n} + Cs\mathbf{V}.\boldsymbol{\tau} \end{cases} \quad (13)$$

The coefficients Cp_{max}, Cp_{min}, Cs come from the VTI tensor!

Results: TTI rectangle

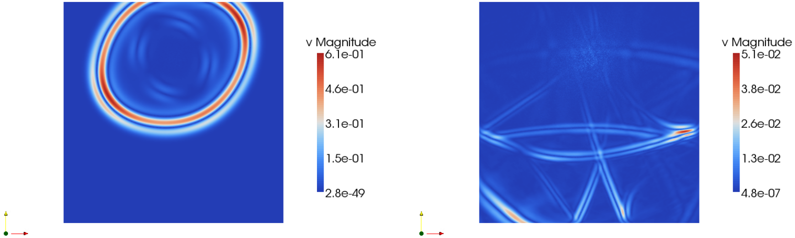


Figure: TTI ($\theta = \frac{\pi}{6}$), rectangle mesh

Conclusion and perspectives

Conclusion:

- first-order elastic wave equation
- new low-order 2D TTI ABC
- stable, cheap and easy integration with the DGM
- spurious reflections (comparable to the isotropic case)
- compatible with the RTM framework

Perspectives:

- high-order 2D ABCs
- 2D RTM test (images)
- 3D TI ABCs & RTM test

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