# A generic numerical continuation scheme for solving the direct kinematics of cable-driven parallel robot with deformable cables

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***Abstract*— Solving the direct kinematics (DK) of parallel robots, i.e. finding all the possible poses of the platform for given input values, is most of the time a difficult problem. This is evidently true for cable-driven parallel robots (CDPR) that are more complex than classical parallel robot because of the unilateral nature of the cable actions but also if cable deformations are taken into account. Furthermore there are many different deformable cable models and developing a DK algorithm for each of them will be a tremendous task. Consequently using a numerical continuation scheme that starts from the DK solutions for non-deformable cables and then moves toward the solution for deformable cables appears to be an interesting approach. To use this scheme we have to assume that the cable model relies on physical parameters that have limit values for which the deformable cable acts like a non-deformable one. Under that assumption we are then able to derive a possible maximum of solutions for the DK, whatever is the cable model. We then apply this approach for a specific complex cable model, the catenary case, to show that this approach is computer efficient but requires to address difficult theoretical issues in order to obtain a solving algorithm that is guaranteed to determine all DK solutions.**

## I. INTRODUCTION

Cable-driven parallel robot (CDPR) have the mechanical structure of the Gough platform with rigid legs except that the legs are cables whose length may be controlled. Numerous prototypes of CDPRs have been developed e.g. large scale maintenance studied in the European project Cablebot [1], large telescope system [2], control of aerial robot [3], rescue robot [4], [5] and transfer robot for elderly people [6] to name a few. We will assume that the output of the coiling system for cable $i$ is a single point $A_i$, while the cable is connected at point $B_i$ on the platform. A cable may be assumed to be mass less and *non-deformable* i.e. the cable shape is the linear segment going from $A$ to $B$ and its length does not change whatever is the tension in the cable or may be *deformable* i.e. the previous assumptions on the cable shape and/or its lengths do not hold. For example figure 1 presents a robot with sagging cables. In this paper we will assume that the platform is *suspended* i.e. there is no cable that is pulling the platform downward except possibly for the action of the cable own mass. The kinematics analysis of parallel robot with rigid legs leads usually to simple inverse kinematics and complex direct kinematics that involves solving a square system of equations that is derived from the geometry of the robot.

Let us now consider now CDPR and assume first that the cable are not deformable and have no mass. As far as DK is concerned it can be shown that for suspended CDPRs there will be at most 6 cables under tension simultaneously whatever is the total number of cables and furthermore a DK solution may have any number between 1 and 6 of cable(s) under tension, the other cables being slack. Hence a full DK analysis implies to solve the DK for all possible combinations of cables whose number lies between 1 and 6. Unfortunately for combination with less than 6 cables static equilibrium has to be taken into account into the DK solving as there is more unknowns (the pose parameters) than geometrical equations. This addition of equations/unknowns that are not present in the rigid legs case leads to a larger number of equations than in the rigid case [7], [8]. and consequently to solving problems [9], [10], [11], [12], [13], [14]. Still solving (i.e. finding all solutions) is possible and upper bounds for the total number of solutions $N$ has been determined according to the number of cables under tension, see table I [15]. Note that for the cases of 3, 4 and 5 cables under tension no example with $N$ solutions has never be found up to now and consequently the given $N$ may be overestimated.

If we assume now that the cables may be elastic and/or deformable, then the DK becomes even more complex. Taking this deformation into account in the kinematics implies to have a *cable model* that describes the cable characteristics (shape and tension) according to the tension to which it

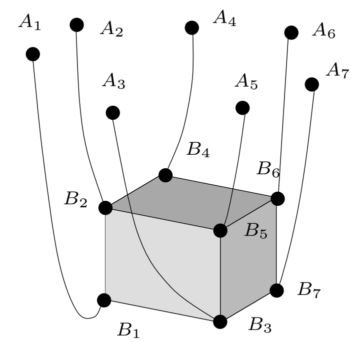


Fig. 1. Cable driven parallel robots with sagging cables

| number of cables | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| solutions | 24 | 156 | 216 | 140 | 40 |

TABLE I

MAXIMAL NUMBER OF SOLUTIONS OF THE DK OF CDPR ACCORDING TO THE NUMBER OF CABLES UNDER TENSION

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is submitted and to the location of the anchor points of the cable.The shape of the cable at the anchor point $B$ is important as it defines the line of action of the cable on the platform and a mechanical equilibrium of the platform shall result from the cable actions.

Some numerical kinematics algorithm have been proposed for simple cable model [16], [17], [18], [19] or for a catenary cable model [20]. However these analysis are rather computer intensive and to the best of the author knowledge no bound on the maximum number of solution of the DK has been established up to now. The purpose of this paper is to propose another solving approach that may be sued with any cable model under very minimal assumptions and may allow to determine a generic upper bound on the maximum number of solutions.

## II. Cable model

We denote by $\rho$ the length of the cable after deformation, $l_0$ is length at rest, $\mathbf{n}$ its tangent vector at $B$ and by $\tau$ the cable tension measured at point $B$. A physical cable model relies on a set $\mathcal{P}$ of parameters that allows one to describe the physical properties of the cable material with respect to deformation under tension. A cable model is a set of relations

$$\mathbf{T}(A, B, \rho, l_0, \mathbf{n}, \tau, \mathcal{P}) = 0 \tag{1}$$

that allows one to determine the values of $\mathbf{n}, \tau$ for given values of $A, B, l_0, \mathcal{P}$. Note that a valid cable model must provide a unique solution for $\mathbf{n}, \tau$ in that case.

As example of cable model we may mention the Irvine sagging cable model that is valid for elastic cable with mass [21]. Comparison of the model and a real CDPR have shown a very good agreement [22]. In this model we consider the vertical plane that includes the cable and assume that the cable is attached at point $A$ with coordinates (0,0) while the other extremity is attached at point $B$ with coordinates $(x_b \geq 0, z_b)$. The vertical and horizontal forces $F_z, F_x$ are exerted on the cable at point $B$ and the cable length at rest is $L_0$. With this notation the coordinates of $B$ are related to

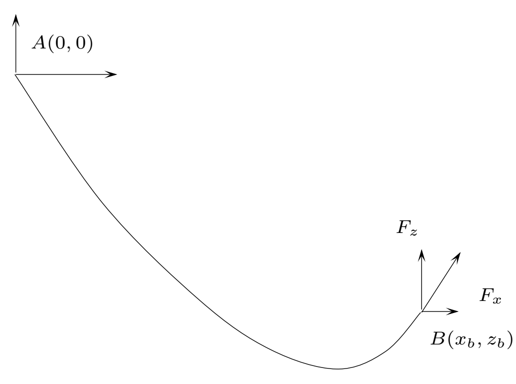


Fig. 2. A deformed cable

the forces $F_x, F_z$ [21] by the $C^n$ functions:

$$x_b = F_x\left(\frac{L_0}{EA_0} + \frac{sinh^{-1}(F_z) - sinh^{-1}((F_z - \frac{\mu g L_0}{F_x})}{\mu g}\right) \tag{2}$$

$$z_b = \frac{\sqrt{F_x^2 + F_z^2} - \sqrt{F_x^2 + (F_z - \mu g L_0)^2}}{\mu g} + \frac{F_z L_0}{EA_0} - \frac{\mu g L_0^2}{2EA_0} \tag{3}$$

where $E$ is the Young modulus of the cable material, $\mu$ its linear density, $A_0$ the surface of the cable cross-section and $F_x > 0$. Note that within the DK context the coordinates of the $B$ will be unknown so that this model involves the two variables $F_x, F_z$ but also the unknown rotation angle $\alpha$ around the $z$ axis that allow to write the above equations in the vertical cable plane. If $x_b^r, y_b^r, z_b^r = z_b$ are the coordinates of $B$ in the reference frame, then we must have:

$$y_b = \sin\alpha x_b^r + \cos\alpha y_b^r = 0 \tag{4}$$

while $x_b$ will be obtained as $x_b = \cos\alpha x_b^r - \sin\alpha y_b^r$ There are 2 solutions to equation (4) but we will retain the one leading to a positive $x_b$. Hence the cable model is constituted of the 3 equations (2,3,4) that involve $F_x, F_z, \alpha$. Note that for known positions for the $A, B$ and length at rest $L_0$ the system of equations (2,3) admits a single solution although it is constituted of non linear and non algebraic equations.

## III. DK kinematico-static equations

We consider a CDPR with $m$ deformable cables, numbered from 1 to $m$, whose lengths are supposed to be known. To parametrize the CDPR pose we use as unknowns the 3 coordinates of four $B_i$ points (that may be assumed to be points $B_1, B_2, B_3, B_4$). Such choice allows for simpler Jacobian and Hessian matrices (and therefore faster to evaluate) and larger uniqueness region for the Kantorovitch theorem. A consequence of such a modeling is that the coordinates of the remaining $B$ points may be written as

$$\mathbf{OB_j} = \sum_{k=1}^{k=4} \beta_k \mathbf{OB_k} \quad \forall j \in [5, m]$$

where the $\beta_k$ are constants that can be determined beforehand. A pose is therefore parametrized by 12 unknowns but we have 6 additional constraints:

$$||\mathbf{B_i B_j}||^2 = d_{ij}^2 \qquad \forall i, j \in [1, 4], j > i \tag{5}$$

where $d_{ij}$ is the known distance between $B_i, B_j$. For solving the DK we consider the constraints due to the cable model (1) that introduces $n$ additional unknowns while providing $n$ constraints. Additionally we have to consider the mechanical equilibrium of the CDPR. If $\mathbf{F}$ is the external force applied on the platform and $\tau$ the vector of the amplitudes of the tensions applied by the cables on the platform at $B$ we have

$$\mathbf{F} = \mathbf{J^{-T}} \tau \tag{6}$$

where $\mathbf{J^{-T}}$ is the transpose of the inverse kinematic jacobian matrix, that is pose dependent. The $j$-th row $J_j$ of $\mathbf{J^{-T}}$ is given by

$$J_j = ((\mathbf{n_j} \quad \mathbf{CB_j} \times \mathbf{n_j}))$$

where $\mathbf{n_j}$ is the unit vector of the line of action of the cable $j$ at point $B_j$. Thus the total number of unknowns is 12 (the pose parameters) $+m \times n$ while we have as constraints the 6 equations (6)+ the 6 constraints equations (5) + $m \times n$

(equation 1). Hence we always end up with a square system of equations.

## IV. Numerical continuation

A numerical continuation method [23], [24], [25] is an approach to solve $n$ non linear equations $\mathbf{F}(\mathbf{X}) = 0$, where $\mathbf{X}$ is a $n$ dimensional vector (of reals in our case), by using a higher-dimensional embedding and solution tracing. For that purpose a continuous differential mapping $\mathbf{H}$ is used, that depends on a set of parameters $\lambda$ so that

- $\mathbf{H}(\mathbf{X}, \lambda_0) = 0$ has known solutions in $\mathbf{X}$
- $\mathbf{H}(\mathbf{X}, \lambda_f) = \mathcal{F}(\mathbf{X})$

The solutions of $\mathbf{F}$ are obtained by starting from the solutions of $\mathbf{H}(\mathbf{X}, \lambda_0)$ and tracing these solutions by modifying $\lambda$ by small increments until $\lambda$ reaches $\lambda_f$. For example a simple continuation mapping is given by

$$\mathbf{H}(\mathbf{X}, \lambda) = \lambda\mathcal{F}(\mathbf{X}) + (1-\lambda)(\mathbf{X} - \mathbf{U}) \quad (7)$$

where $\lambda_0 = 0$ and $\lambda_f = 1$ and $\mathbf{U}$ is a known vector. If we assume that $\mathcal{F}(\mathbf{X}, \mathcal{P})$ is dependent of a set of parameters $\mathcal{P}$ and that solutions of $\mathcal{F} = 0$ can be found for $\mathcal{P} = \mathcal{P}_0$ while solution have to be found for $\mathcal{P} = \mathcal{P}_f$ we may use as mapping:

$$\mathbf{H}(\mathbf{X}, \lambda) = \lambda\mathcal{F}(\mathbf{X}, \mathcal{P}_f) + (1-\lambda)\mathcal{F}(\mathbf{X}, \mathcal{P}_0) \quad (8)$$

For solving the DK with numerical continuation the following assumptions will be required:

1) the set $\mathbf{T}$ is constituted of continuous and differentiable functions,
2) the number of relations in $\mathbf{T}$ should be such that the DK equations including the cable model must be a square system
3) for each parameter in $\mathcal{P}$ there is a limit value such that the cable model will be asymptotically identical to the non deformable cable model

For example the catenary cable model (2,3) and all kinetostatic equations satisfy the first assumption. Regarding the second assumption the limit of $x_b$ when $\mu \to 0, E \to \infty$ is obtained from equation (2) as $x_b = L_0 F_x/\sqrt{(F_x^2 + F_z^2}$. In the same way the limit of $z_b$ in (3) when $\mu \to 0, E \to \infty$ is $z_b = F_z L_0/\sqrt{(F_x^2 + F_z^2}$. These limit values are coherent with the non deformable cable model and therefore the DK equations with the catenary as cable model satisfy both above assumptions.

## V. Applying numerical continuation

According to the implicit function theorem being given a solution $\mathbf{X_0}$ for the DK equations there will be an unique solution in the vicinity of $\mathbf{X_0}$ for the DK equations obtained for a "small" change in the equations, provided that they are not singular. The amplitude of the perturbation that will lead to a unique solution of the DK that is close to $\mathbf{X_0}$ may be obtained through the Kantorovitch theorem [26]. Provided that the jacobian of the new system has an inverse at $\mathbf{X_0}$ and that some conditions are satisfied for the norm of the equations at $\mathbf{X_0}$, for the norm of the inverse jacobian and for the norm of the Hessian matrix of the system for any $\mathbf{X}$ in a ball centered at $\mathbf{X_0}$, then the theorem ensures that there is a single solution of the new system in this ball and guarantees that the Newton-Raphson scheme will converge toward this solution. Using this result we may apply successive small incremental changes in the equations that will "push" them toward the final set of DK equations and therefore will provide us a solution for the DK.

### A. The inverse model

In this section we will assume that the cable model is the catenary one and that the homotopy mapping is based on changes in the parameters $E, \mu$ with

$$E = E_0 - \lambda_1(E_s - E_0) \quad \mu = \mu_0 - \lambda_1(\mu_s - \mu_0) \quad (9)$$

applied on the DK equations with the catenary model. Here $E_s, \mu_s$ have respectively a very large and a very small value, $E_0, \mu_0$ are the objective values while $\lambda_{1,2}$ are in the range [0,1]. Hence for $\lambda_1 = \lambda_2 = 0$ the DK equations are the one with the real cables while for $\lambda_1 = \lambda_2 = 1$ we will get equations that should have a solution close to the non-deformable cables case. Physically at each step of the continuation scheme this mapping is equivalent to fix the platform in its current pose and to substitute the existing cables by cables that have a larger $E$ and a smaller $\mu$ and then let the platform go to a new pose. There will always be such a pose but according to the changes in $E, \mu$ the change in pose may exceed the conditions of the Kantorovitch theorem or the CDPR may cross a singularity.

We will also assume that a solving algorithm for the DK with the real cables has allowed us to determine the DK solution(s) for given cable lengths. Starting the scheme from one of these solutions and provided that it does not encounter a singularity it should converge toward a DK solution that is very close to a solution obtained for the DK for non-deformable cables. Note that for a CDPR with $m$ cables this solution may have from 1 to 6 cables under tension, the other one being slack, whatever is $m \geq 6$.

Regarding singularity we will assume the following conjecture:

**Conjecture** *singularity occurs only at isolated poses or for bifurcation points so that the crossing of two kinematic branches will lead to no more than two new kinematic branches*.

Under that conjecture the continuation scheme that starts from the $r$ DK solutions of the CDPR with deformable cables will lead to at more $r$ DK solutions of the CDPR with non-deformable cables. Note that our continuation scheme involves non-algebraic equations and shall be a multi-parameter homotopy as we may have to manage as many $E, \mu$ variables as cables.

### B. The direct model and the maximum number of DK solutions

We now revert the process explained in the inverse model for obtaining the DK solutions for the CDPR with deformable cables. The starting poses of the continuation scheme will be the DK solutions for the non-deformable cables and we will use a homotopy mapping that is similar to (9) except that we revert the role of $E_0, E_s$ and $\mu_0, \mu_s$

$$E = E_s - \lambda_1(E_0 - E_s) \quad \mu = \mu_s - \lambda_1(\mu_0 - \mu_s) \quad (10)$$

so that for $\lambda_{1,2} = 0$ we have a cable model that is very close to the non-deformable case while for $\lambda_{1,2} = 1$ we have the current CDPR. We may assume that $E_s$ is large enough and $\mu_s$ is small enough so that the Newton scheme with as initial guess a non-deformable DK solution will converge toward a solution of the DK equations with the catenary cable model. The physical meaning of this homotopy mapping is the same than for the inverse model and it may be thought that just changing the cables at each step of the continuation will always lead to a CDPR pose that is close to the previous one.

Using the same reasoning than for the inverse model and under the same conjecture, each solution of the DK with non-deformable cables among the $p$ possible one will lead to at most one solution of the DK with deformable cables. Consequently the number of DK solutions with the deformable cables cannot exceed $p$.

Note that this scheme starts with a non-deformable cables state with possibly some slack cables. However during the iterations it may perfectly happen that an initially slack cable, which therefore does not support the platform, becomes supportive and vice-versa.

For the CDPR with non-deformable cables, being given a distribution of slack and under tension cables there is always a finite number of solutions to the DK problem whose upper bound is provided in table I. Hence the total number of solutions of the DK with deformable cables cannot exceed the total number of slack/under tension combinations multiplied by their maximal number of DK solutions. Consequently the upper bound of the number of DK solutions for a CDPR with $m$ cables may be established as 40 $C_6^m$ (6 cables under tension) +140 $C_5^m$ (5 cables under tension) + 216 $C_4^m$ (4 cables under tension) + 156 $C_3^m$ (3 cables under tension) + 24 $C_2^m$ (2 cables under tension). Note that we do not consider the case of only one cable under tension as this is a very special case. For a CDPR with 8 cables the above formula leads to a maximum of 33488 solutions. Note however that for the time being we have only an example of 8-cables CDPR with 19 DK solutions [27].

## VI. Detailed study of an example

We consider the large scale robot developed by LIRMM and Tecnalia as part of the ANR project Cogiro [16]. (figure 3). This robot is a suspended CDPR (i.e. there is no cable pulling the platform downward) with 8 cables, whose $A_i$ coordinates are given in table II. Note however that the same method may be used for fully constrained CDPR. The cables are assumed to follow Irvine cable model (2,3) with the following characteristics: $E = 100^9 N/m^2$ (Young modulus), $\mu$ = 0.346 kg/m (linear density) and 10 mm diameter. The mass $m$ of the platform is 10 kg. We will assume that the non-deformed cables lengths are the one

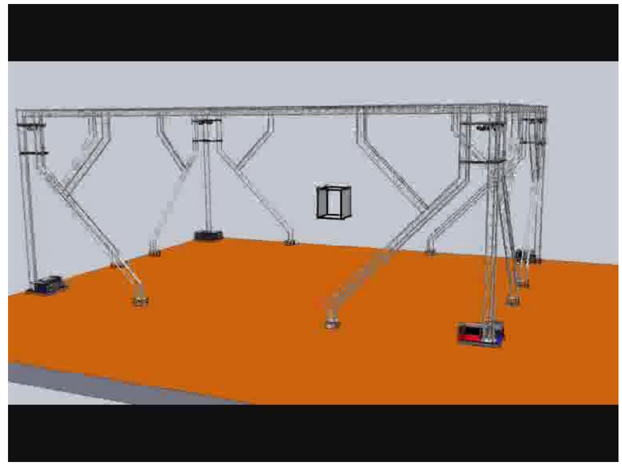

Fig. 3. The robot developed for the the ANR Cogiro project. Although the robot is real we present a CAD drawing that allows one to better figure out the CDPR.

| x | y | z | x | y | z |
|---|---|---|---|---|---|
| -7.175 | -5.244 | 5.462 | -7.316 | -5.1 | 5.47 |
| -7.3 | 5.2 | 5.476 | -7.161 | 5.3 | 5.485 |
| 7.182 | 5.3 | 5.488 | 7.323 | 5.2 | 5.499 |
| 7.3 | -5.1 | 5.489 | 7.161 | -5.27 | 5.497 |

TABLE II

Coordinates of the attachment points on the base (in meters)

given in table III, that correspond to the pose (1,0,2) in meters and for an orientation of the platform such that the reference frame and the mobile frame are aligned. The DK solutions for the sagging cable haven been determined in [27] using a specific algorithm based on interval analysis which has led to 19 solutions for the DK. For using this algorithm we have to assume that we are looking for solutions within a bounded domain for the unknowns. Being given the coordinates of the $A$ points and the cable lengths we may safely provide a bounds for the 12 coordinates of the $B_1, B_2, B_3, B_4$ points. As for the 8 angles $\alpha_i$ that appear in equation (4) we may use as bounds the interval $[0, 2\pi]$. However regarding the unknowns $F_x, F_z$ we have as only known constraint $F_x > 0$. In [27] we have assumed an upper bound of $10mg$ for $F_x$ while $F_z$ was assumed to lie in the range $[-10mg, 10mg]$. Furthermore to speed up the solving algorithm we have assumed that there was no other DK solution(s) within an hyper-cube centered at a DK solution with edge of length 0.2.

For determining all the DK solutions for the non-

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| 10.48215 | 9.838952 | 10.16035 | 8.96827 |
| 5 | 6 | 7 | 8 |
| 10.310003 | 8.421629 | 8.663245 | 8.655556 |

TABLE III

Non-deformed cable lengths for the DK

deformable cables model and for all possible combinations of cable under tension we have also used an interval analysis-based solving algorithm. Note that we retain only the solutions such that for all the cables that are not under tension the distance between the $A$ and $B$ point is lower than the nominal cable lengths (this additional constraint is included in the solver which allow to speed up the determination of the DK solutions). We get DK solutions for the following combinations (the number of solution(s) is between parenthesis): [1,3,4,7] (1), [1,3,4,8] (1), [1,4,5,7] (2), [1,4,5,8] (1), [1,4,7,8] (1), [2,3,6,7] (1), [2,4,5,7] (1), [2,5,6,8] (1), [3,4,5,8] (1), [3,4,7,8] (1), [3,6,7,8] (1), [4,5,7,8] (1), [5,6,7,8] (1), [1,2,3,4,7] (2), [1,2,3,6,7] (2), [1,2,3,6,8] (3), [1,2,5,6,7] (1), [1,3,4,5,7] (1), [1,3,4,5,8] (3), [1,3,4,6,7] (1), [1,3,4,7,8] (1), [1,3,6,7,8] (1), [1,4,5,7,8] (1), [1,5,6,7,8] (1), [2,3,4,5,8] (1), [2,3,4,6,8] (1), [2,4,5,6,7] (1), [2,4,5,6,8] (1), [2,4,5,7,8] (1), [2,5,6,7,8] (1), [3,4,5,6,8] (1), [3,4,6,7,8] (1), [4,5,6,7,8] (1),[1,2,3,4,7,8] (1),[1,2,3,5,6,8] (1), [1,2,3,6,7,8] (3),[1,2,4,6,7,8] (3), [1,2,5,6,7,8] (2), [1,3,4,5,6,7] (1), [1,3,4,5,7,8] (1), [1,4,5,6,7,8] (2), [2,3,4,5,6,7] (1), [2,3,4,5,6,8] (1), [2,3,4,6,7,8] (1), [2,4,5,6,7,8] (1), for a total of 58 solutions.

The continuation mapping (10) has been implemented in `Maple` and all calculations are performed with a 300 digits accuracy. We have been able to retrieve 17 out of the 19 DK solutions for the CDPR with deformable cables while finding 10 new solutions. But among the 58 branches the continuation scheme stops in a singularity in 28 cases.

We have tested several other continuation mapping. One of them was to change first the $\mu$ variables, one at a time. For determining which $\mu^i$ (i.e. the value of $\mu$ for cable $i$) should be modified by $\Delta\mu$ we compute the absolute value $d$ of the determinant of the jacobian of the DK equations for the current value of $\mu^i$ and then calculate first the DK solution for the system where only one of the $\mu$ is modified, $\mu^k$. For each change we calculate the absolute value $d_m^k$ of the determinant of the jacobian of the DK equations. The cable which will be submitted to a change in its $\mu$ is determined according to the following rules:

- if there is at least one $k$ such that $d_m^k > d$, then we select the $k$ that maximizes $d_m^k$
- otherwise choose the $k$ that minimizes $d - d_m^k$

As soon as one of the $\mu^k$ reaches its objective value then it is no more modified. When all the $\mu^k$ have reached their objective we then modify the $E$ until it too reaches its objective. This approach has allowed us to find 4 additional solutions and to retrieve one of the missing solution. Among the 28 paths that have led to a singularity using the (10) mapping 11 have been successfully solved with the new mapping. The 16 paths that still led to a singularity are obtained for the combinations: [1,3,4,8], [1,4,5,7], [3,6,7,8], [4,5,7,8], [5,6,7,8], [1,2,3,4,7], [1,2,5,6,7], [1,3,4,5,7], [1,3,4,5,8], [1,3,6,7,8], [1,4,5,7,8], [1,5,6,7,8], [2,3,4,6,8], [2,4,5,7,8], [1,2,3,6,7,8].

We have therefore found a total of 14 new solutions that are presented in table IV that leads to a total of 36 solutions. An examination of these new solutions has shown that either at least one of the $F_x, F_z$ was outside the range used by our initial solving scheme or that the solution is very close to a previously found one. However one of the solution is still missing (table V) for which cables 3, 6, 7 are supportive while the other cables lie partly below their $B$ points on the platform. Using the inverse model and a very specific mapping we have been able to identify the non-deformable case [3,6,7,8] as solution. A test case for further analysis of DK algorithm will be to determine if we can recover this missing solution.

It must be noted that the continuation scheme is significantly faster than the interval analysis based solving algorithm even if the computation time of the non-deformable cables combinations is included. The computation is however still relatively intensive with a total computation time of about 6 minutes to get all solutions. Note however that real-time DK solving for which we are looking for a solution close to a pose that has been determined a few milliseconds before may be safely performed using the approach proposed in [28].

| solution | $xb_1$ | $yb_1$ | $zb_1$ | $xb_2$ | $yb_2$ | $zb_2$ |
|---|---|---|---|---|---|---|
| 1 | 0.69 | 0.75 | 2.66 | 1.04 | -0.71 | 3.3513 |
| 2 | 0.71 | 0.42 | 4.01 | 1.48 | -1.04 | 3.9514 |
| 3 | 0.91 | 0.89 | 4.14 | 1.36 | -0.70 | 4.2539 |
| 4 | 1.01 | 1.27 | 4.84 | 1.27 | -0.36 | 4.9155 |
| 5 | 1.17 | 1.07 | 4.97 | 1.07 | -0.27 | 3.9947 |
| 6 | 1.24 | 0.80 | 4.01 | 0.32 | 0.08 | 5.1909 |
| 7 | 1.24 | 0.64 | 3.84 | 0.44 | 0.34 | 5.2519 |
| 8 | 1.27 | 0.65 | 4.41 | -0.04 | -0.31 | 4.1096 |
| 9 | 1.40 | 0.51 | 3.70 | 0.02 | -0.37 | 3.9712 |
| 10 | 1.47 | -0.31 | 2.51 | 0.53 | 0.42 | 3.6533 |
| 11 | 1.58 | -0.15 | 3.48 | 0.34 | 0.07 | 2.4052 |
| 12 | 1.66 | 0.38 | 5.14 | 0.46 | 0.72 | 4.0576 |
| 13 | 1.79 | -0.03 | 4.40 | 0.33 | -0.72 | 4.0919 |
| 14 | 1.94 | -0.09 | 4.95 | 0.42 | -0.26 | 4.3155 |
| solution | $xb_3$ | $yb_3$ | $zb_3$ | $xb_4$ | $yb_4$ | $zb_4$ |
| 1 | 0.56 | -0.23 | 2.39 | 1.27 | -0.51 | 2.3910 |
| 2 | 1.24 | -0.16 | 4.68 | 0.84 | -0.81 | 4.6906 |
| 3 | 0.58 | 0.00 | 3.74 | 0.40 | -0.41 | 4.3558 |
| 4 | 1.72 | 0.66 | 5.27 | 2.00 | 0.28 | 4.6719 |
| 5 | 1.68 | 0.73 | 4.14 | 1.95 | 0.05 | 4.3744 |
| 6 | 1.28 | -0.09 | 4.54 | 1.32 | 0.14 | 5.2599 |
| 7 | 1.31 | -0.07 | 4.58 | 1.45 | 0.35 | 5.1984 |
| 8 | 1.10 | -0.37 | 4.39 | 0.85 | -0.48 | 3.6761 |
| 9 | 0.42 | 0.74 | 3.95 | 0.03 | 0.44 | 3.3711 |
| 10 | 1.03 | 0.62 | 2.61 | 0.29 | 0.44 | 2.6742 |
| 11 | 0.60 | -0.44 | 3.44 | 0.23 | 0.23 | 3.3930 |
| 12 | 1.62 | 0.83 | 4.22 | 1.14 | 1.41 | 4.3372 |
| 13 | 1.15 | -0.04 | 3.60 | 1.10 | -0.79 | 3.4418 |
| 14 | 1.01 | -0.34 | 5.33 | 0.83 | -0.95 | 4.9229 |

TABLE IV

THE 14 NEW SOLUTIONS FOUND WITH THE NUMERICAL CONTINUATION

| $xb_1$ | $yb_1$ | $zb_1$ | $xb_2$ | $yb_2$ | $zb_2$ |
|---|---|---|---|---|---|
| 1.04 | 0.90 | 3.63 | 0.54 | -0.12 | 4.8247 |
| $xb_3$ | $yb_3$ | $zb_3$ | $xb_4$ | $yb_4$ | $zb_4$ |
| 1.52 | 0.33 | 4.34 | 1.27 | -0.37 | 4.1751 |

TABLE V

THE MISSING SOLUTION

Still for several kinematic branches the continuation scheme ends up in a singular state although we have tried numerous continuation mapping, including several one working in the complex field.

## VII. Discussion

The previous example shows that numerical continuation may provide DK solutions for any cable model but that it raises multiple issues. Clearly we have to figure out what happen at the singular points of the DK equations. A first singularity may occur if the wrench system that is applied on the platform is linearly dependent. A numerical instability may also occur if a $F_x$ comes close to 0. For the purpose we have investigated the case where the initial set of non-deformable cables under tension is [4,5,7,8] while the remaining cables are slack. The DK equations are denoted $\mathcal{F}(\mathbf{X}, E, \mu)$ with $\mathcal{F}_0(\mathbf{X}, E, \mu)$ the system obtained for a large value of $E$ and a small value of $\mu$ for which we have been able to determine a solution using as initial guess the solution of the non-deformable cables DK. The system $\mathcal{F}_1(\mathbf{X}, E_g, \mu_g)$ is the one obtained with the objective value for $E, \mu$. The continuation mapping that was used is

$$\mathcal{H}(\mathbf{X}, \lambda) = (1-\lambda)\mathcal{F}_0 + \lambda\mathcal{F}_1 \tag{11}$$

A singularity occurs close to $\lambda = 0.01351913$ where the determinant value of the jacobian of $\mathcal{H}$ is $-1.254445e-10$ while for $\lambda = 0$ this determinant is of order $1e27$. Close to the singularity the supportive cables are [1,4,5,7,8] and there are two cables that have a large value for $F_x$.

We have first investigated the singular values of the mechanical equilibrium equations and have found that all of them were larger than 0.12. We have also considered the pairs of equations (2,3) for each cable and have found that the determinant of the jacobian of the pair have a very low value for the cables having a high $F_x$.

We have then used the continuation mapping proposed in ( 10). A singularity occurs for $\mu = 5.660946e-03$, a value that is not significantly changed by modifying $E$. At this step the supportive cables are [4,5,7,8] and the $F_x$ of cables 5 and 8 are large while the determinant of the jacobian of the DK equations is lower than $1e-11$ while initially its value was of order $1e27$. As for the previous continuous mapping the determinant of the jacobian of the pairs of equations (2,3) is very low for cables 5 and 8. Hence although different continuation mapping are used we still end up in a singular configuration.

We have then tried the mapping ( 10) for one of the solution of the [1,2,5,6,7,8] combination where $xb_1 = 0.8258$, $yb_1 = 1.242$, $zb_1 = 3.51553$. The continuation scheme stopped at $\mu = 2.672010e-04$ with as supportive cables [1,2,5,6,7,8]. In that case the $F_x$ have all a reasonable value. On the other hand one of the singular values of the equilibrium equations have a low value that may suggest that we are close to a parallel singularity. However the value of this singular value is not sufficient to explain the extremely low value of the determinant of the jacobian of the DK equations. Note that a faster decrease in $E$ toward its goal leads to a singularity at $\mu = 2.672113e-04$.

We have then tried the mapping (11) that stops for $\lambda = 1.059387e-03$. The supportive cables are the same than for the previous mapping and the $F_x$ have also reasonable values. The analysis of the singular values of the jacobian of the mechanical equilibrium equations shows that one of the singular value is low.

This experimental study raises several issues:

- it appears that a singularity may happen because of the bad conditioning of the pair of equations (2,3) when $F_x$ is large and $\mu$ is small. This may motivate the use of a continuation mapping that monitor the value of $F_x$ and choose a path that minimize their maximum value.
- another approach for avoiding the bad conditioning will be to use a simplified cable model (such as the one proposed in [18]) while the $E$ is high enough and the $\mu$ quite low and to switch to the full catenary model as soon as the $\mu$ reaches some threshold
- the continuation parameters may be non only $E, \mu$ but also the cable length at rest. This suggest to test a continuation mapping that plays on $E, \mu, L_0$ but our initial test have shown that modifying slightly the $L_0$ after each successful iteration of the mapping still leads to a singularity.
- the second example shows that the poor conditioning of equations (2,3) is not sufficient to explain the singularity in the continuation scheme. It is however unclear if the closeness to a parallel singularity is sufficient to explain the non convergence of the continuation scheme
- we are dealing with a multi-parameter continuation problem with a high number of parameters (for a CDPR with 8 cables we may consider as parameters the $E, \mu, L_0$ for a total of 24 parameters). Therefore there is a very large variety of continuation mappings and finding the one that avoid singularity is a very complex problem

Note that continuation may also be applied to the inverse kinematics problem (IK). We have already proposed a specific algorithm for solving the IK [20] and our first trial for using a continuation scheme, that will be presented in another paper, have shown that although we have been able to recover the IK solutions it is still necessary to used different continuation mapping to avoid singularity.

As mentioned by the reviewers the above example assumes a low platform mass so that the weights of the cables have an as strong influence as the the platform. However other examples have been investigated with a much higher platform mass (up to 100 kg) and still multiple solutions have been found.

## VIII. Conclusion

We have investigated the use of a numerical continuation scheme to determine the DK solutions of a CDPR with deformable cables by starting from it DK solutions for non-deformable cables, whatever is the cable model, provided that this model satisfies minimal assumptions. We also exhibit a bound for the maximal number of DK solutions under the assumption that all DK solutions for deformable cables may be found by following the kinematic branches issued from the DK solutions for non deformable cables. As expected for multi-parameters continuation scheme we have to manage the singularity problem: several kinematic branches has led us to a singularity, When comparing the solution found with a simple continuation scheme and solutions $\mathcal{S}$ that have been found by a dedicated solving algorithm all the solutions of $\mathcal{S}$ have been found except for one solution, while new solutions have been discovered, thereby leading to a CDPR that 36 solution for its DK. Recovering all known solutions has required to use different continuation mapping but in some cases all the tested mappings has not allowed us to avoid to encounter a singularity. It appears that in some cases the singularity is inherent to the poor conditioning of the cable model but in other cases the singularity cannot be attributed to the cable model. In the same way it may happen that the continuation scheme stops in the pose that is very close to a parallel singularity but is is unclear if this proximity is sufficient to explain the non convergence of the continuation. Clearly the singularity of the DK system have to be investigated from a theoretical view point, an issue that has to be investigated anyway for a safe control of the CDPR. Finally note that the proposed approach is also relevant for solving the inverse kinematic problem, which will be the topic of another paper.

## References


[1] A. Pott *et al.*, "IPAnema: a family of cable-driven parallel robots for industrial applications," in *1st Int. Conf. on cable-driven parallel robots (CableCon)*, Stuttgart, September, 3-4, 2012, pp. 119–134.

[2] L. Hui, "A giant sagging-cable-driven parallel robot of FAST telescope:its tension-feasible workspace of orientation and orientation planning," in *14th IFToMM World Congress on the Theory of Machines and Mechanisms*, Taipei, October, 27-30, 2015.

[3] Y. Takeda, K. Maeda, and S. Tadokoro, "Attitude stability of a cable driven balloon robot," in *IEEE Int. Conf. on Intelligent Robots and Systems (IROS)*, Beijing, October, 9-15, 2006, pp. 3504–3509.

[4] S. Tadokoro *et al.*, "A portable parallel manipulator for search and rescue at large-scale urban earthquakes and an identification algorithm for the installation in unstructured environments," in *IEEE Int. Conf. on Intelligent Robots and Systems (IROS)*, Kyongju, October, 17-21, 1999, pp. 1222–1227.

[5] J.-P. Merlet and D. Daney, "A portable, modular parallel wire crane for rescue operations," in *IEEE Int. Conf. on Robotics and Automation*, Anchorage, May, 3-8, 2010, pp. 2834–2839. [Online]. Available:

[6] J.-P. Merlet, "MARIONET, a family of modular wire-driven parallel robots," in *ARK*, Piran, June 28- July 1, 2010, pp. 53–62. [Online]. Available:

[7] T. Bruckman *et al.*, *Parallel manipulators, New Developments*. ITECH, April 2008, ch. Wire robot part I, kinematics, analysis and design, pp. 109–132.

[8] Q. Jiang and V. Kumar, "The inverse kinematics of 3-d towing," in *ARK*, Piran, June 28- July 1, 2010, pp. 321–328.

[9] G. Abbasnejad and M. Carricato, "Real solutions of the direct geometrico-static problem of underconstrained cable-driven parallel robot with 3 cables: a numerical investigation," *Meccanica*, vol. 473, no. 7, pp. 1761–1773, 2012.

[10] A. Berti, J.-P. Merlet, and M. Carricato, "Solving the direct geometrico-static problem of underconstrained cable-driven parallel robots by interval analysis," *Int. J. of Robotics Research*, vol. 35, no. 6, pp. 723–739, 2016.

[11] M. Carricato and J.-P. Merlet, "Direct geometrico-static problem of under-constrained cable-driven parallel robots with three cables," in *IEEE Int. Conf. on Robotics and Automation*, Shangai, May, 9-13, 2011, pp. 3011–3017. [Online]. Available:

[12] M. Carricato and G. Abbasnejad, "Direct geometrico-static analysis of under-constrained cable-driven parallel robots with 4 cables," in *1st Int. Conf. on cable-driven parallel robots (CableCon)*, Stuttgart, September, 3-4, 2012, pp. 269–286.

[13] M. Carricato and J.-P. Merlet, "Stability analysis of underconstrained cable-driven parallel robots," *IEEE Trans. on Robotics*, vol. 29, no. 1, pp. 288–296, 2013. [Online]. Available:

[14] A. Pott, "An algorithm for real-time forward kinematics of cable-driven parallel robots," in *ARK*, Piran, June 28- July 1, 2010, pp. 529–538.

[15] A. Berti, "Kinematics and statics of cable-driven parallel robots by interval-analysis-based methods," Ph.D. dissertation, University of Bologna, Bologna, April, 21, 2015.

[16] M. Gouttefarde *et al.*, "Simplified static analysis of large-dimension parallel cable-driven robots," in *IEEE Int. Conf. on Robotics and Automation*, Saint Paul, May, 14-18, 2012, pp. 2299–2305.

[17] K. Kozak *et al.*, "Static analysis of cable-driven manipulators with non-negligible cable mass," *IEEE Trans. on Robotics*, vol. 22, no. 3, pp. 425–433, June 2006.

[18] D. Nguyen *et al.*, "On the simplification of cable model in static analysis of large dimension cable-driven parallel robots," in *IEEE Int. Conf. on Intelligent Robots and Systems (IROS)*, Tokyo, November, 3-7, 2013, pp. 928–934.

[19] N. Riehl *et al.*, "Effects of non-negligible cable mass on the static behavior of large workspace cable-driven parallel mechanisms," in *IEEE Int. Conf. on Robotics and Automation*, Kobe, May, 14-16, 2009, pp. 2193–2198.

[20] J.-P. Merlet, "The kinematics of cable-driven parallel robots with sagging cables: preliminary results," in *IEEE Int. Conf. on Robotics and Automation*, Seattle, May, 26-30, 2015, pp. 1593–1598.

[21] H. M. Irvine, *Cable Structures*. MIT Press, 1981.

[22] N. Riehl *et al.*, "On the determination of cable characteristics for large dimension cable-driven parallel mechanisms," in *IEEE Int. Conf. on Robotics and Automation*, Anchorage, May, 3-8, 2010, pp. 4709–4714.

[23] E. Allgower, *Numerical Continuation Methods: an introduction*. Springer Series in Computational Mathematics, 1990.

[24] B. N. Lundberg and A. B. Poore, "Numerical continuation and singularity detection methods for parametric nonlinear programming," *SIAM Journal on Optimization*, vol. 3, no. 1, pp. 134–154, 1993.

[25] D. Wolf and S. Sanders, "Multi-parameter homotopy methods for finding dc operating points of nonlinear circuits," in *Circuits and Systems, 1993., ISCAS '93, 1993 IEEE International Symposium on*, May 1993, pp. 2478–2481 vol.4.

[26] R. Tapia, "The Kantorovitch theorem for Newton's method," *American Mathematic Monthly*, vol. 78, no. 1.ea, pp. 389–392, 1971.

[27] J.-P. Merlet, "The forward kinematics of cable-driven parallel robots with sagging cables," in *2nd Int. Conf. on cable-driven parallel robots (CableCon)*, Duisburg, August, 24-27, 2014, pp. 3–16. [Online]. Available:

[28] ——, "On the real-time calculation of the forward kinematics of suspended cable-driven parallel robots," in *14th IFToMM World Congress on the Theory of Machines and Mechanisms*, Taipei, October, 27-30, 2015.