Geometry and Kinematic Singularities of closed-loop manipulators

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Abstract

The determination of the singularities of closed-loop manipulators is in general a difficult problem. We consider in this paper specific classes of closed-loop manipulators and we show how we can define a singular configuration. The behaviour of the robot in the vicinity of these configurations is examined. We describe then a geometrical approach which enables to find the loci of the singular configurations together with a geometrical description of the robot in these configurations. This approach is illustrated by the complete analysis of a given manipulator.

1 Parallel manipulator

1.1 Introduction

Parallel manipulators are closed-loop mechanism in which all the links are connected both at the base and at the gripper of the robot. Manipulators of this type have been designed or studied for a long time. The first one, to the author's knowledge, was designed for testing tyres (see Mc Gough in Stewart paper [20]). But this mechanical architecture is mainly used for the flight simulator (see for example Stewart [20], Baret [2]). The first design as a manipulator system has been done by Mac Callion in 1979 for an assembly workstation [13]. Some other researchers have also addressed this problem: Arai [1], Fichter [6], Gosselin [7], Hervé [8],Inoue [12], Reboulet [19], Yang [23], Zamanov [24].

To illustrate our approach we will consider a specific mechanical architecture called a SSM described in Figure 1. Basically it consists in two plates connected by 6 articulated links. In the following sections the smaller plate will be called the *mobile* and the larger (which is in general fixed) will be called the *base*. In each articulated link there is one linear actuator and by changing the lengths of the links we are able to control the position [1] of the gripper. The SSM can be simplified if the hexagonal mobile plate is changed to a triangular plate (we will call this kind of manipulator a TSSM). A further simplification is obtained when both plates are triangles and the resulting manipulator is called a MSSM (Figure 8).

[1] In this paper position means position and orientation

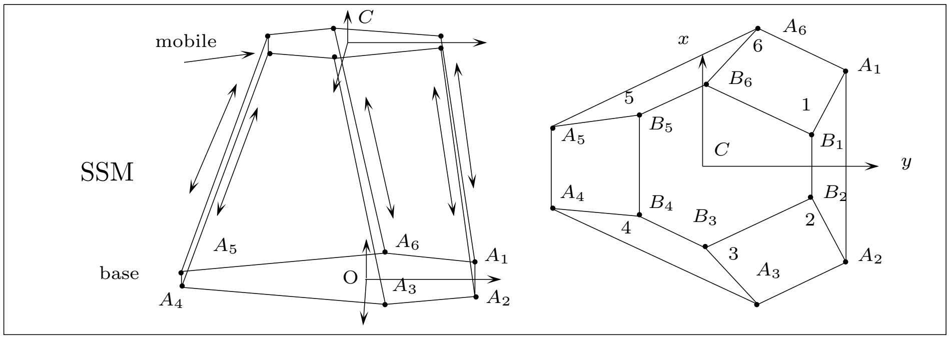

Figure 1: A parallel manipulator: the SSM. Links i is articulated at point A_i, B_i. (perspective and top view)

1.2 Notation

We introduce an absolute frame R with origin O and a relative frame R_b fixed to the mobile with origin C (see Figure 1). The rotation matrix relating a vector in R_b to the same vector in R will be denoted by M.

The center of the articulations on the base for link i will be denoted A_i and that on the mobile B_i. The length of link i will be noted ρ_i, and the unit vector of this link $\mathbf{n_i}$. The coordinates of A_i in frame R are (xa_i, ya_i, za_i), the coordinates of B_i in frame R_b are (x_i, y_i, z_i) and the coordinates of C, the origin of the relative frame, $\mathbf{X_c} = (x_c, y_c, z_c)$. We use the Euler's angles $\mathbf{\Omega_c} = (\psi, \theta, \phi)$ to represent the orientation of the mobile.

For the sake of simplicity the subscript i is omitted whenever it is possible and vectors will be noted in **bold** character. A vector with coordinates expressed in the relative frame will be denoted by the subscript $_r$.

We will consider the case where each set of articulation points of both the base and the mobile lie in a plane. In this case, without loss of generality, we will define R such that $za_i = 0$ and R_b such that $z_i = 0$.

1.3 Inverse and Direct kinematics

Let us calculate the fundamental relations between the links lengths and the position of the mobile. For a given link we have :

$$\mathbf{AB} = \rho\mathbf{n} \quad \mathbf{AB} = \mathbf{AO} + \mathbf{OC} + \mathbf{CB} \qquad \mathbf{CB} = M\mathbf{CB_r} \tag{1}$$

where $\mathbf{CB_r}$ means the coordinates of the articulation points with respect to the frame R_b. $\mathbf{n}$ being a unit vector we have :

$$\rho = ||\mathbf{AO} + \mathbf{OC} + M\mathbf{CB_r}|| = ||\mathbf{U}|| \tag{2}$$

If the position of the mobile is given we are able to calculate the components of $\mathbf{U}$ and thus the length of the segment. Therefore the inverse kinematics is straightforward (this is in fact a general feature of parallel manipulators) and is defined by the above 6 equations which constitute a system of non-linear equations denoted by $\mathcal{S}$.

At the opposite the direct kinematics is much more complicated. Indeed to find the position of the mobile for a given set of links lengths we have to solve the system $\mathcal{S}$. It has been shown in [17] that in general the solution is not unique : if the mobile plate is a triangle up to 16 solutions can exist and in the case of the SSM it has been shown that an upperbound of the number of solution is 352 although a numerical study has yield to at most 12 solutions.

2 Singularities

2.1 An analytical approach

Let us assume that for a given set of links lengths ρ we know a solution $\mathbf{X_0}$ of the system $\mathcal{S}$. From the rank theorem we know that in a neighborhood of $\mathbf{X_0}$ the solution of $\mathcal{S}$ is unique if the rank of the jacobian matrix J of this system is equal to 6 with:

$$J = ((\frac{\partial \rho}{\partial \mathbf{X}})) \tag{3}$$

where $\mathbf{X}$ is the position parameters vector. Note that this matrix is in fact the inverse jacobian (in a robotics sense) of the manipulator. Now let us assume that J is singular : this means that the mobile plate may have an *infinitesimal motion* around $\mathbf{X_0}$ without any change in the links lengths. In that case we will say that $\mathbf{X_0}$ is a *singular configuration* of the manipulator. In other words the velocity of the mobile plate may be different from zero although the actuator's velocities are all equal to zero. This means also that in these configurations the manipulator *gains* some degrees of freedom (at the opposite of the singular configurations of serial manipulator where it loses degrees of freedom).

2.2 A mechanical approach

The previous approach indicates that in a singular configuration the manipulator is no more controllable. But a mechanical approach will give another insight of these configurations. Let τ denotes the articular forces vector (i.e. the traction-compression stress in the links) and $\mathbf{F}$ an external wrench applied on the mobile plate. It is well known that we have:

$$\mathbf{F} = J^T \tau \tag{4}$$

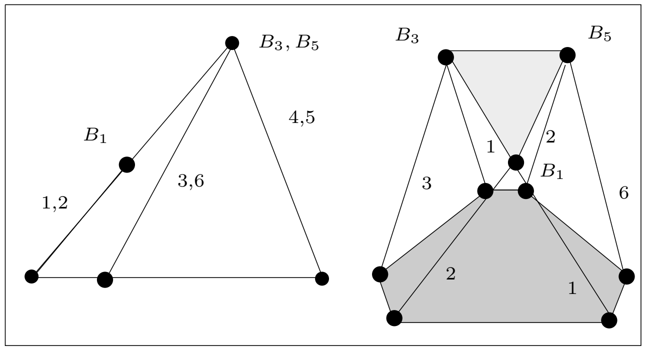

Figure 2: Hunt's singular configuration for the TSSM.

If J is singular then no τ can be found to equilibrate a set of wrench. Furthermore in the vicinity of a singular configuration the articular forces will tend to infinity. Practically this imply that if the mobile plate is "close" from a singular configuration the robot will suffer mechanical damages and this explain why the determination of the loci of these configurations is an important problem.

2.3 The determinant of J

From this point the solution to this problem seems obvious : as the matrix J is completely determined we calculate its determinant and find its roots in $\mathbf{X}$. In fact the symbolic computation of this determinant is rather tedious (see [14] for the formulation of this determinant). To get an idea of its complexity the computation of the determinant of a SSM involves 29 powers , 21530 multiplications, 915 additions and 907 subtractions.....

2.4 Previous works

Few researchers have addressed the problem of determining the loci of the singular configurations.

Using a mechanical analysis Hunt [10] has determined a singular configuration for a TSSM (Figure 2). In this configuration all the segments intersect one line (line B_3B_5) and an external torque around this line cannot be equilibrated by the actuator forces.

Fichter [6] describes another singular configuration which is obtained when the mobile plate is rotated around the z axis with an angle of $\pm\frac{\pi}{2}$. This configuration was obtained by noticing that in this case two lines of the determinant

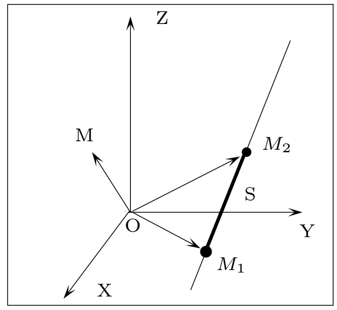

Figure 3: Plücker coordinates.

were constant. But outside these two particular configurations no systematic method was proposed to find *all* the singular configurations of a parallel manipulator.

3 A geometrical approach for finding the singularities

3.1 Plücker coordinates of lines

It is well known that a line can be described by its Plücker coordinates. Let us introduce briefly these coordinates. We consider two points on a line, say M_1 and M_2, and a reference frame R_0 whose origin is O (see Figure 3).

Let us consider now the two three dimensional vectors **S** and **M** defined by:

$$\mathbf{S} = \mathbf{M_1M_2} \qquad \mathbf{M} = \mathbf{OM_1} \wedge \mathbf{OM_2} = \mathbf{OM_2} \wedge \mathbf{S} = \mathbf{OM_1} \wedge \mathbf{S}$$

If we assemble these vectors to form a six-dimensional vector we get the Plücker vector $\mathbf{L_p}$ of this line.

$$\mathbf{L_p} = [S_x, S_y, S_z, M_x, M_y, M_z]$$

Let us assume now that the Plücker vectors belong to a vector space V_6 and we consider the one-dimensional subspaces of V_6 as points of a projective P_5. Then every line g in P_3 corresponds to exactly one point **G** in P_5.

It is well known that point **G** belongs to a quadric Q_p (see [4], [22], [3]). Indeed we have for every line of P_3 :

$$S_xM_x + S_yM_y + S_zM_z = 0$$

This equation defines the quadric Q_p which is called the *Grassmannian* or the *Plücker quadric*. At this point we have defined a one-to-one relation between the set of lines in the real P_3 and the quadric Q_p in P_5. The rank of this mapping is 6 (there is at most 6 independent Plücker vectors).

Let us consider now the various sub-spaces of P_5 (or more precisely their intersection with Q_p). We get various varieties whose rank ranges from 0 to 6. As a matter of example a point in P_5 (rank=1) corresponds to a line in P_3. As for Q_p (which represents the set of line of P_3) it is defined through 6 linearly independent Plücker vectors and is therefore of rank 6.

3.2 Plücker vectors of the links and matrix J

Consider the equilibrium of the mobile plate under the effect of an external wrench $\mathbf{T} = (\mathbf{F}, \mathbf{M})$ and the actuators force vector τ. The equilibrium conditions are:

$$\mathbf{F} = \sum_{i=1}^{i=6} \tau_i \mathbf{n_i} \qquad \mathbf{M} = \sum_{i=1}^{i=6} \mathbf{CB_i} \wedge \tau_i \mathbf{n_i} \tag{5}$$

which can be written as :

$$\mathbf{T} = ((\mathbf{n_i} \quad \mathbf{CB_i} \wedge \mathbf{n_i}))^T \tau \tag{6}$$

Using equation (4) we get :

$$J = ((\mathbf{n_i} \quad \mathbf{CB_i} \wedge \mathbf{n_i})) \tag{7}$$

and therefore row i of matrix J is equal to the Plücker vector of the line associated to link i. Although we have obtained this result for a SSM it can be extended to various kind of parallel manipulator [18] (even for manipulator with less than 6 degrees of freedom).

From this results we deduce that a degeneracy of matrix J imply a linear dependence between the six 6-dimensional Plücker vectors of the line associated to the links or in other words that the variety spanned by these lines has a rank less than 6.

4 Grassmann Geometry

The varieties spanned by a set of lines has been studied by H. Grassmann (1809-1877). The purpose of his study was to find geometric characterizations of each varieties i.e. find all the geometric conditions on a set of m lines such that these lines spanned a variety of rank n with $n < m \leq 6$. We will introduce now the various results which can be found in [5] or, with more mathematical justifications, in [22].

Let us begin with the linear varieties of rank 0 through 3 (Figure 4). We have first the empty set of rank 0. Then the *point* (rank=1), which is a line

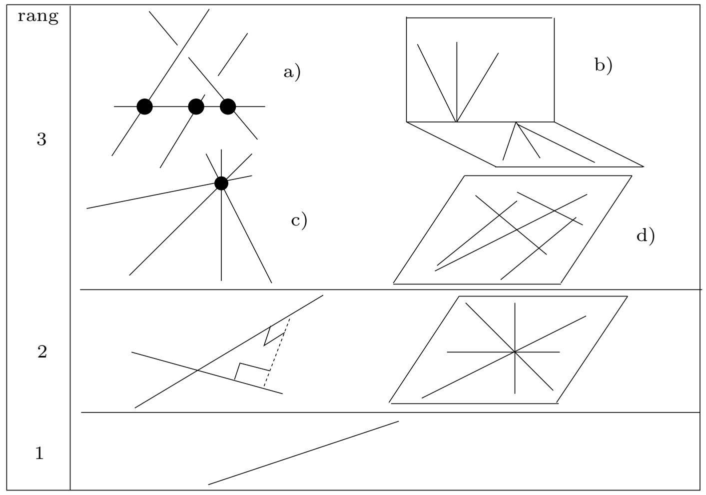

Figure 4: Grassmann varieties of rank 1,2,3.

in the 3D space. The *lines* (rank=2) are either a pair of skew lines in R^3 or a flat pencil of lines: those lying in a plane and passing through some point on that plane.

The *planes* (rank=3) are of four types:

- all lines in a plane (3d)
- all lines through a point (3c)
- the union of two flat pencils having a line in common but lying in distincts planes and with distinct centers (3b)
- a regulus (3a)

Let us define the regulus. Take three skew lines in space and consider the set of lines which intersect these three lines : this set of lines build a surface which is an hyperboloid of one sheet (a quadric surface) and is called a **regulus**. Each line belonging to the regulus is called a *generator* of the regulus. It is shown in [9],[22] that this surface is *doubly ruled*. This means that there exist two reguli (a regulus and its "complementary" regulus) which generate the same surface or that each point on the surface is on more than one line.

Therefore there are two families of straight lines on the hyperboloid and each family covers the surface completely. A line on this surface is dependent on the lines of either the regulus or the complementary regulus. An interesting

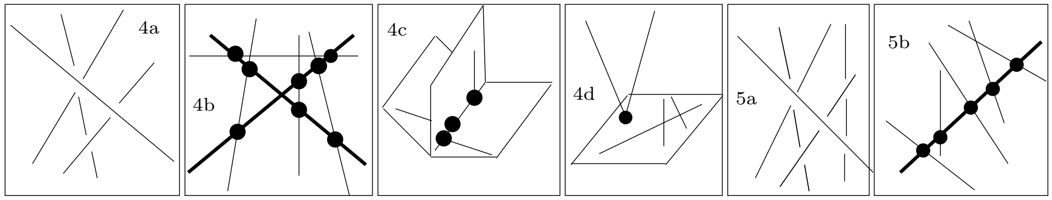

Figure 5: Grassmann varieties of rank 4,5.

property is that a line of one family intersects all the lines of the other family and that any two lines of the same family are mutually skew (see [21] for the hairy details).

Let us describe now the linear varieties of higher rank of the Grassmann geometry (Figure 5). Linear varieties of dimension 4 are called linear *congruences* and are of four types:

- a linear spread generated by four skew lines i.e. no line meet the regulus generated by the three others lines in a proper point (*elliptic congruence*, 4a)
- all the lines concurrent with two skew lines (*hyperbolic congruence*, 4b)
- a one-parameter family of flat pencil, having one line in common and forming a variety (*parabolic congruence*, 4c)
- all the lines in a plane or passing through one point in that plane (*degenerate congruence*, 4d)

Linear varieties of dimension 5 are called linear *complexes* and are of two types:

- *non singular* (or *general*): generated by five independent skew lines (5a)
- *singular* (or *special*): all the lines meeting one given line (5b)

The geometric characterization of a general linear complex is that through any point of the space there is one and only one flat pencil of line such that all the lines which belong to the pencil belong also to the complex. In other words all the lines of a linear complex which are coplanar intersect one point.

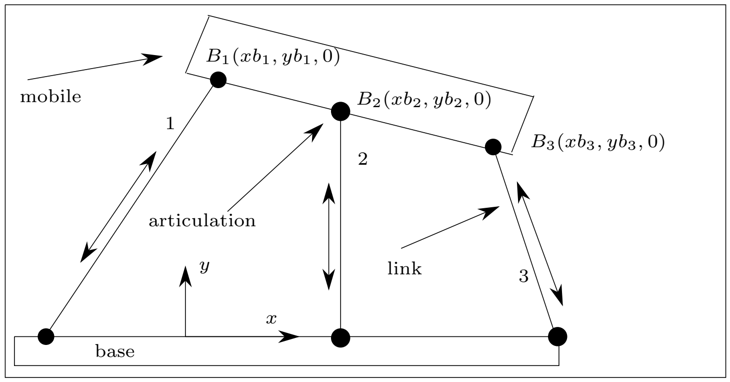

Figure 6: A 2D parallel manipulator. By changing the lengths of links (1,2,3) the position and orientation of the mobile can be controlled.

5 Application of Grassmann geometry to the determination of the singularities

If we consider the example of a SSM in a singular configuration the rank of the variety spanned by the lines associated to the links will be less than 6. This imply that it exists at least a set of $m(m \leq 6)$ lines which spanned a variety of rank $m-1$. Therefore to find the position of the mobile plate such that the SSM is in a singular configuration we will examine each set of m lines ($m \in [3,6]$) and determine the position of the mobile plate such that the geometric condition given by Grassmann geometry for a variety of rank $m-1$ is fulfilled. For example we will consider all the set of 4 lines and determine the position of the mobile plate for which these 4 lines have a common point: in this case the rank of the variety spanned by the four lines is 3 and the matrix J is singular.

5.1 A basic example: the 2D parallel manipulator

Let us consider a basic example: a 2D parallel manipulator (Figure 6). The equilibrium condition can be written in matrix form as:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} n_{1_x} & n_{2_x} & n_{3_x} \\ n_{1_y} & n_{2_y} & n_{3_y} \\ n_{1_y}xb_1 - yb_1 n_{1_x} & n_{2_y}xb_2 - yb_2 n_{2_x} & n_{3_y}xb_3 - yb_3 n_{3_x} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad (8)$$

Let us consider the three column vectors $\mathcal{T}_i$ of the above 3x3 matrix. If one of them is linearly dependent from the two others then the manipulator is in a singular configuration. For each of these vectors we may build an augmented 6 dimensional vector $\mathcal{S}_i$ by adding three 0 :

$$\mathcal{S}_i = [n_{i_x}, n_{i_y}, 0, 0, 0, n_{i_y} x b_i - y b_i n_{i_x}]$$

Clearly if one the $\mathcal{T}_i$ is linearly dependent from the two others then the corresponding $\mathcal{S}_i$ will also be linearly dependent from the two others and the opposite is also true. It must then be noticed that the $\mathcal{S}_i$ is the Plücker vector of the line associated to link i. Therefore in this case we have three Plücker vectors and we are looking for a configuration of the manipulator such that the rank of the variety spanned by these 3 lines is 2.

By reference to Figure 4 we can see that the only possibility for a system of three coplanar bars to be a 2-rank Grassmann variety is obtained when the three lines cross the same point (Figure 7). Therefore the loci of the singular

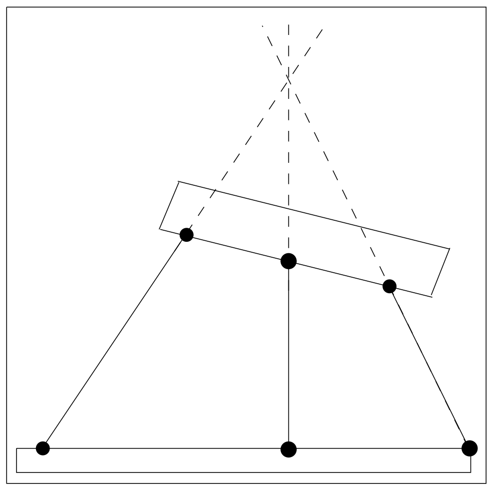

Figure 7: Singular configuration for the 2D parallel manipulator : the three line associated to the links have a common intersection point.

configurations expressed as position of the mobile plate can be easily described from a geometrical view point.

6 Study of the MSSM

We will deal now with a more complete example of a 6 d.o.f. manipulator called the MSSM (Figure 8). A previous analysis [15] has shown that among the Grassmann conditions only three can be satisfied for some configurations of the MSSM namely :

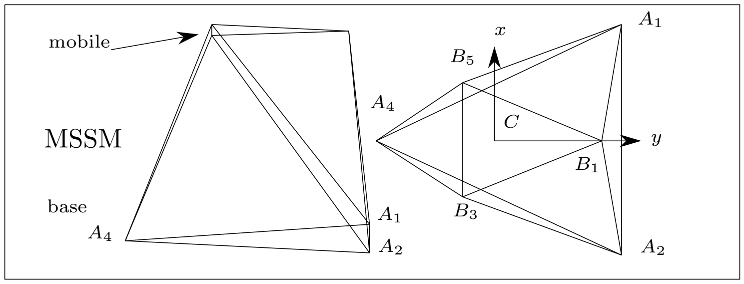

Figure 8: The MSSM parallel manipulator (perspective and top view).

- 3d case : 4 lines are coplanar
- 5a case : all the 6 lines belong to a general complex
- 5b case : all the 6 lines intersect a line of space (special complex)

In this analysis we assume that it is not possible that a link lie in the base plane.

6.1 Case 3d

We consider a set of 4 lines $\mathcal{S}$ and investigate for which configurations of the mobile plate they can belong to a plane $\mathcal{P}$. First we notice that the set of lines must be such that the number of their distinct articulation points A_i is less than three. Indeed in the opposite case the plane $\mathcal{P}$ will be defined by the points (A_1, A_2, A_3) i.e. the plane $\mathcal{P}$ will be the base plane and therefore the links will lie on the base plane. Thus the only possible sets for $\mathcal{S}$ are (1,2,3,6), (2,3,4,5), (1,4,5,6). Let us notice that in each case the points B_1, B_3, B_5 belong to $\mathcal{P}$ i.e. this plane is the mobile plane. By changing the base and relative frame we can consider that all these sets can be reduced to the set (1,2,3,6) (Figure 9). The coordinates of the articulation points on the base in the reference frame are :

$$A_6 = A_1 = \begin{pmatrix} xa_0 \\ ya_0 \\ 0 \end{pmatrix} \qquad A_3 = A_2 = \begin{pmatrix} xa_1 \\ ya_0 \\ 0 \end{pmatrix} \qquad A_5 = A_4 = \begin{pmatrix} 0 \\ ya_3 \\ 0 \end{pmatrix}$$

The coordinates of the articulation points on the mobile in the relative frame are :

$$B_2 = B_1 = \begin{pmatrix} 0 \\ y_0 \\ 0 \end{pmatrix} \qquad B_3 = B_4 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \qquad B_5 = B_6 = \begin{pmatrix} x_3 \\ y_2 \\ 0 \end{pmatrix}$$

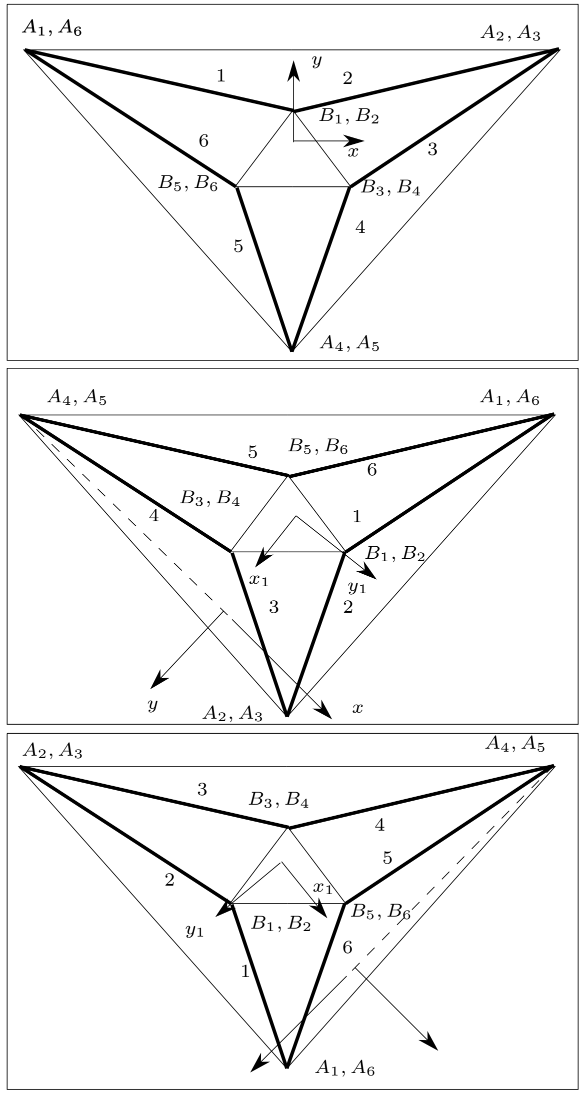

Figure 9: Various frames which can be used to study the case 3d.

The coplanarity of lines (1,2,3,6) can be expressed by two equations:

$$(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_3B_3} = 0 \quad (\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_6B_6} = 0 \tag{9}$$

which constitute a system of two linear equations in y_c, z_c in which x_c does not appear. The determinant Δ of this system is:

$$\Delta = -\sin(\theta)\sin(\psi)(xa_0 - xa_1)^2(-x_3 + x_2)(-y_2 + y_0) \tag{10}$$

Therefore in the case where Δ is not equal to zero we solve this linear system and we get a singularity condition described by :

$$y_c = H_{3d_1}(\mathbf{\Omega_c}) \qquad z_c = H_{3d_2}(\mathbf{\Omega_c}) \qquad \forall x_c \tag{11}$$

But in that case it is possible to show that lines (1,2) are colinear and lie in the base plane (in that case we have $\mathbf{A_1B_1} \wedge \mathbf{A_2B_2} = \mathbf{0}$).

The determinant Δ may vanish for $\sin\theta = 0$ i.e. for $\theta = 0$ or $\theta = \pi$ but in both cases the only possible solution for the equations (9) is $z_c = 0$: therefore the links lie in the base plane. The determinant Δ may also vanish for $\sin\psi = 0$ i.e. for $\psi = 0$ or $\psi = \pi$. If $\psi = 0$ both equations yield to:

$$z_c = -\frac{(ya_0 - y_c)\sin(\theta)}{\cos(\theta)} \tag{12}$$

If $\psi = \pi$ both equations yield to:

$$z_c = \frac{(ya_0 - y_c)\sin(\theta)}{\cos(\theta)} \tag{13}$$

Therefore we get two others singularity conditions:

$$\psi = 0 \qquad z_c = H_{3d_3}(y_c, \theta) \qquad \forall x_c \tag{14}$$

$$\psi = \pi \qquad z_c = -H_{3d_4}(y_c, \theta) \qquad \forall x_c \tag{15}$$

An example of this kind of singular configurations is presented in Figure 10. We may notice that we get the singular configuration described by Hunt.

6.2 Case 5a

In this case all the 6 lines belong to a general complex. This means also that every lines of the complex which are coplanar must intersect the same point. Let us consider the lines of the pencils spanned by (1,6), (2,3),(4,5). These lines belong to the complex. Among these lines consider the three lines D_1, D_2, D_3 which lie on the base plane. If the 6 lines belong to a linear general complex the lines D_1, D_2, D_3 must intersect the same point M, whose coordinates are

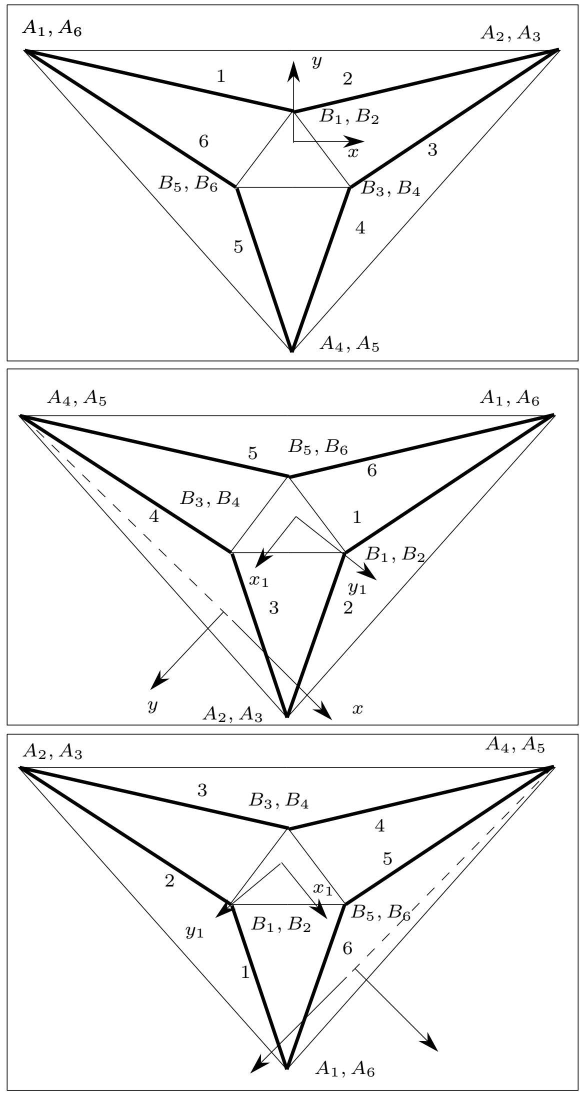

Figure 10: An example of singular configuration of type 3d for the MSSM: 4 links are coplanar.

$(x, y, 0)$. If $\mathbf{v_{ij}}$ denotes the normal vector to the pencil of lines spanned by line i, j we must have:

$$\mathbf{A_1 M}.\mathbf{v_{16}} = 0 \quad (16)$$

$$\mathbf{A_3 M}.\mathbf{v_{23}} = 0 \quad (17)$$

$$\mathbf{A_5 M}.\mathbf{v_{45}} = 0 \quad (18)$$

These three equations are linear in term of x, y. We use the first two to calculate these unknowns and put their values in the last equation which yield to a constraint equation. This equation is of order 3 in z_c, 2 in x_c, y_c. Therefore we get three possible singularity conditions :

$$a_3(x_c, y_c, \mathbf{\Omega_c})z_c^3 + a_2(x_c, y_c, \mathbf{\Omega_c})z_c^2 + a_1(x_c, y_c, \mathbf{\Omega_c})z_c + a_0(x_c, y_c, \mathbf{\Omega_c}) = 0 \quad (19)$$

$$b_2(x_c, z_c, \mathbf{\Omega_c})y_c^2 + b_1(x_c, z_c, \mathbf{\Omega_c})y_c + b_0(x_c, z_c, \mathbf{\Omega_c}) = 0 \quad (20)$$

$$c_2(y_c, z_c, \mathbf{\Omega_c})x_c^2 + c_1(y_c, z_c, \mathbf{\Omega_c})x_c + c_0(y_c, z_c, \mathbf{\Omega_c}) = 0 \quad (21)$$

Figure 11 shows three examples of singular configurations of type 5a obtained for fixed $x_c, y_c, \mathbf{\Omega_c}$.

An interesting point about equation (19) is that for $\theta = 0$ or $\theta = \pi$ the coefficients a_i are :

$$a_0 = a_1 = a_2 = 0$$

$$a_3 = 2x_2 y_0 \cos(\psi - \phi)(x a_0 y_0 + x_2 y a_3) \quad (22)$$

Thus in this case we get a singular configuration for $\psi = \pm\frac{\pi}{2}$ whatever are x_c, y_c, z_c : we find the singular configuration described by Fichter.

A particular case has to be considered : let us assume that in the set of equations (16,17,18) there are only two independent equations which can be used to determine x, y. As a consequence the last equation will not yield to a singularity condition. But if we consider the two dependent equations we will get such a condition by writing that their determinant is equal to zero if the two equations are coherent. The determinants of equations (16,17), (16,18), (18,17). are first order polynomials in x_c. Thus we are able to get x_c and put its value back in the equations. It can then be shown that the two equations are not coherent.

7 Case 5b

In that case the 6 lines intersect one line of space. A previous analysis has shown that five lines may intersect one line of space in two cases :

- the intersection line is an edge of the mobile and four lines of the manipulator are coplanar : this is Hunt's singular configuration

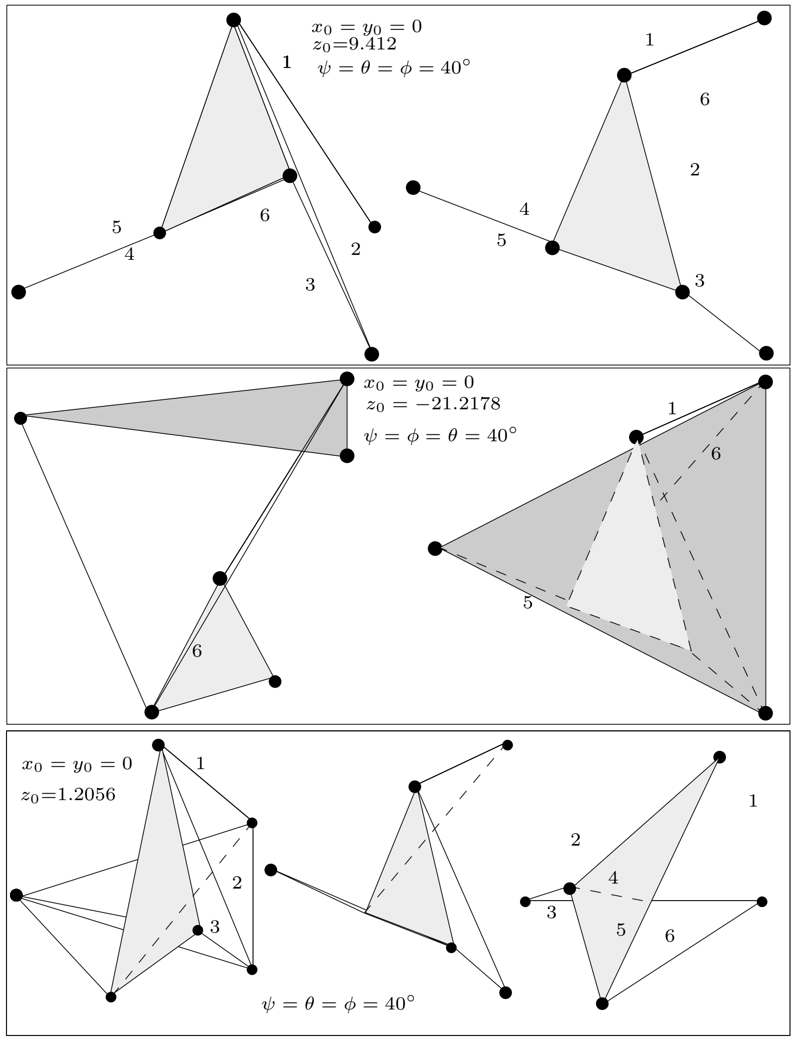

Figure 11: Three examples of singular configuration of type 5a for the MSSM.

- three lines are coplanar and the intersection line (D) is defined by the two articulation points which are not common to two of the coplanar lines. For example in Figure 12 lines (2,3,4) are coplanar and the line (D) defined by A_4, B_1 intersects (1,2,3,4,5).

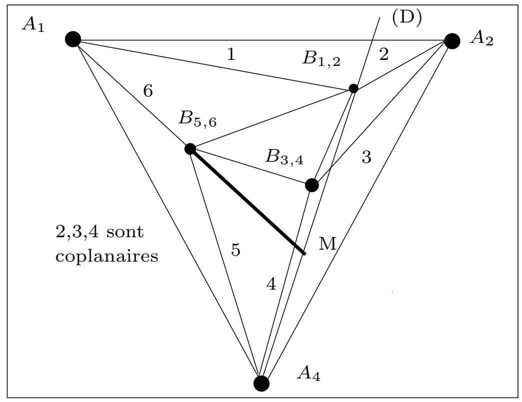

Figure 12: In this example the links (2,3,4) are coplanar and the line (D) going through A_4, B_1 intersects the lines (1,2,3,4,5).

Let us consider this last case. By rotating the mobile plate around the edge of the mobile defined by the articulation points of the three coplanar lines (in the example edge B_1, B_3) we may find a configuration where the last line (6 in the example) will intersect (D) (Figure 13).

Such a configuration is fully determined by its set of coplanar lines. These 3 coplanar lines must share only two articulation points on the base (in the opposite case the plane on which they lie will be the base plane). In the same manner they must also share only two articulation points on the mobile; indeed if they share three points the plane will be the mobile plane and any line which is not in this set and which has a common articulation point with one line of the set will therefore be in the plane; thus we get 4 coplanar lines and this case has been considered

Therefore the only possible set of three coplanar lines is (1,2,3), (1,5,6), (2,3,4), (3,4,5), (4,5,6). As in the previous section we can choose the reference and relative frame so that we have to consider only the set (1,2,3).

These frames are defined in Figure 14. The coordinates in the reference frame of the articulation points on the base are :

$$A_6 = A_1 = \begin{pmatrix} xa_0 \\ 0 \\ 0 \end{pmatrix} \qquad A_3 = A_2 = \begin{pmatrix} xa_1 \\ 0 \\ 0 \end{pmatrix} \qquad A_5 = A_4 = \begin{pmatrix} 0 \\ ya_3 \\ 0 \end{pmatrix}$$

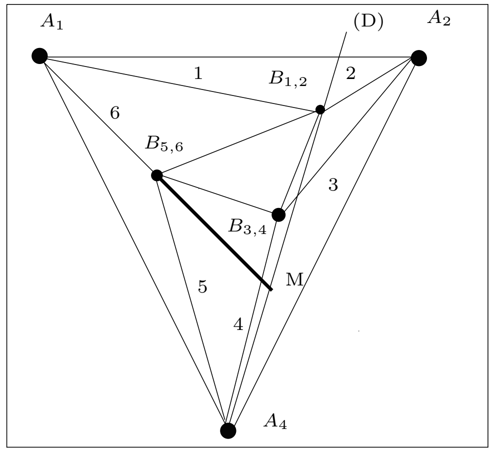

Figure 13: In this example the links (2,3,4) are coplanar and the line (D) going through A_4, B_1 intersects the lines (1,2,3,4,5). By rotating the mobile around its edge B_1, B_3 line 6 intersects (D) at point M.

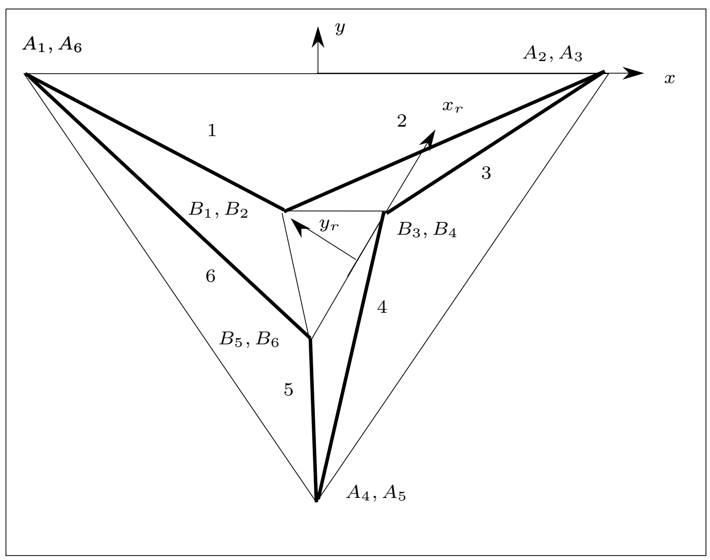

Figure 14: Notation and frame for singularity 5b.

The coordinates in the relative frame of the articulation points on the mobile are :

$$B_2 = B_1 = \begin{pmatrix} 0 \\ y_0 \\ 0 \end{pmatrix} \qquad B_3 = B_4 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} \qquad B_5 = B_6 = \begin{pmatrix} x_3 \\ 0 \\ 0 \end{pmatrix}$$

The coplanarity of lines 1, 2, 3 is defined by the equation:

$$(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_3B_3} = 0 \tag{23}$$

Then we can express that the line going through A_5, B_5 intersects the line going through A_1, B_3 by the equation:

$$\mathbf{A_1B_3}.(\mathbf{OA_5} \wedge \mathbf{OB_5}) + \mathbf{A_5B_5}.(\mathbf{OA_1} \wedge \mathbf{OB_3}) = 0 \tag{24}$$

Equations (23)(24) constitute a linear system in y_c, z_c. The determinant Δ of this system is:

$$\begin{aligned}\Delta \;=\; & \sin(\theta)(xa_0 - xa_1)(x_2 - x_3)(x_2 \sin(\phi)^2 ya_3 \sin(\psi)\cos(\theta) - \\ & x_2 \sin(\phi) ya_3 \cos(\psi)\cos(\phi) - \cos(\phi) y_0 ya_3 \sin(\psi)\cos(\theta)\sin(\phi) + \\ & xa_0 y_0 \sin(\psi) + \cos(\phi)^2 y_0 ya_3 \cos(\psi))\end{aligned}$$

If the determinant is not equal to zero we get then two singularity conditions:

$$y_c = H_{5b_1}(x_c, \mathbf{\Omega_c}) \qquad z_c = H_{5b_2}(x_c, \mathbf{\Omega_c}) \tag{25}$$

An example of such singular configuration is given in Figure 15.

The determinant may be equal to zero if $\sin\theta = 0$. For $\theta = 0$ equations (23)(24) are reduced to:

$$-z_c(-xa_1 + xa_0)(-sin(\psi)x_2 + cos(\psi)y_0) = 0 \tag{26}$$

$$z_c(ya_3 cos(\psi) + xa_0 sin(\psi))(-x_3 + x_2) = 0 \tag{27}$$

The case where $z_c = 0$ means that all the lines lie on the base plate. The other case is obtained for :

$$\tan\psi = \frac{y_0}{x_2} = -\frac{ya_3}{xa_0}$$

This case can hold only for a specific geometry of the robot.

For $\theta = \pi$ equations (23)(24) are reduced to:

$$z_c(-xa_1 + xa_0)(-sin(\psi - \phi)x_2 + y_0 cos(\psi - \phi)) = 0 \tag{28}$$

$$-z_c(sin(\psi - \phi)xa_0 - cos(\psi - \phi)ya_3)(-x_3 + x_2) = 0 \tag{29}$$

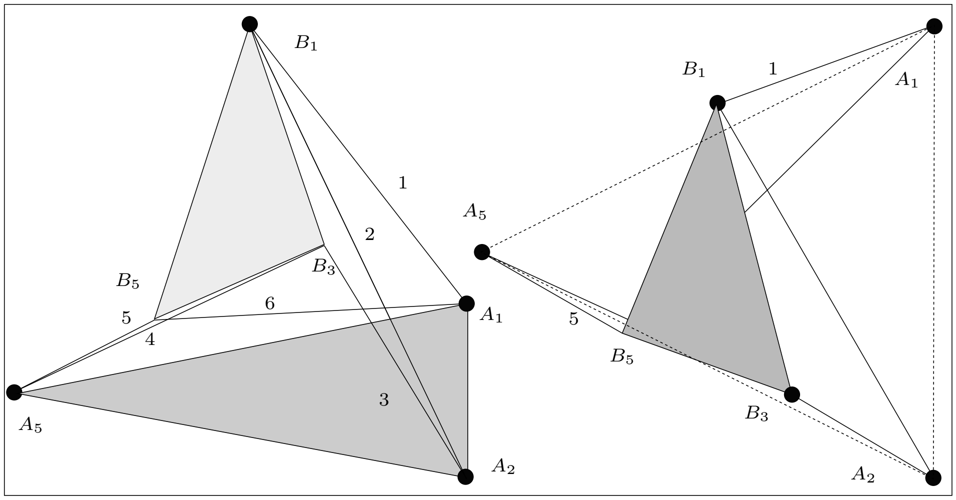

Figure 15: An example of singular configuration of type 5b for the MSSM: lines 1,2,3 are coplanar and line A_5B_5 intersects line A_1B_3.

the case where $z_c = 0$ means that all the lines lie on the base plate. The other case is obtained for :

$$\tan(\psi - \phi) = \frac{y_0}{x_2} = \frac{ya_3}{xa_0}$$

As before this case can hold only for a specific geometry of the robot.

The last case where the determinant is equal to zero is obtained for :

$$(x_2 \sin(\phi)^2 ya_3 \sin(\psi)\cos(\theta) - x_2 \sin(\phi) ya_3 \cos(\psi)\cos(\phi) + xa_0 y_0 \sin(\psi) + \cos(\phi)^2 y_0 ya_3 \cos(\psi) - \cos(\phi) y_0 ya_3 \sin(\psi)\cos(\theta)\sin(\phi)) = 0 \quad (30)$$

which can be solved in ψ :

$$\psi = -\arctan\left(\frac{ya_3 \cos(\phi)(\cos(\phi)y_0 - x_2 \sin(\phi))}{x_2 ya_3 \cos(\theta)\sin(\phi)^2 + xa_0 y_0 - \cos(\phi) y_0 ya_3 \cos(\theta)\sin(\phi)}\right) \quad (31)$$

Then we can get z_c from equation (23) and x_c from equation (24). Therefore we get three singularity conditions :

$$\psi = H_{5b_3}(\theta, \phi) \qquad z_c = H_{5b_4}(y_c, \theta, \phi) \qquad x_c = H_{5b_5}(y_c, \theta, \phi) \quad (32)$$

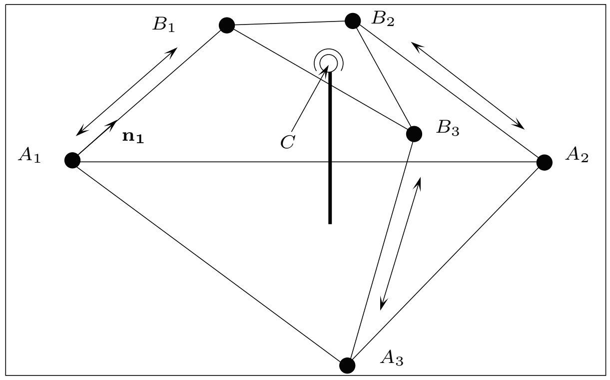

Figure 16: A 3 d.o.f parallel wrist. The mobile plate rotates around a fixed ball and socket joint R whose center is C.

8 Summary of the singularity conditions for a MSSM

case	singularity conditions		
3d	$\psi = 0$	$z_c = H_{3d_3}(y_c, \theta)$	$\forall x_c$
	$\psi = \pi$	$z_c = -H_{3d_4}(y_c, \theta)$	$\forall x_c$
5a	$\sum_{i=0}^{i=3} a_i(x_c, y_c, \mathbf{\Omega_c}) z_c^i = 0$		
	$\sum_{i=0}^{i=2} b_i(x_c, z_c, \mathbf{\Omega_c}) y_c^i = 0$		
	$\sum_{i=0}^{i=2} c_i(y_c, z_c, \mathbf{\Omega_c}) x_c^i = 0$		
	$\theta = \phi = 0$	$\psi = \pm\frac{\pi}{2}$	$\forall(\mathbf{X_c})$
5b	$y_c = H_{5b_1}(x_c, \mathbf{\Omega_c})$	$z_c = H_{5b_2}(x_c, \mathbf{\Omega_c})$	
	$\psi = H_{5b_3}(\theta, \phi)$	$z_c = H_{5b_4}(y_c, \theta, \phi)$	$x_c = H_{5b_5}(y_c, \theta, \phi)$

9 Analysis of a 3 d.o.f. parallel wrist

The purpose of this section is to show that our geometric approach can be applied to manipulator with less than 6 d.o.f. We consider the 3 d.o.f. parallel wrist presented in Figure 16. The mobile plate is articulated on a ball and socket joint R which is fixed with respect to the base. Three variable length links articulated at point A_i, B_i enable to control the orientation of the mobile plate. If C is the center of the ball and socket joint and $\mathbf{n_i}$ the unit vector of link i the articular velocity $\dot{\rho}$ is related to the angular velocity of the mobile plate $\dot{\omega}$ by :

$$\dot{\rho} = J^{-1}\dot{\omega} \tag{33}$$

where row i of matrix J^{-1} is defined by:

$$J_i^{-1} = ((\mathbf{CB_i} \wedge \mathbf{n_i})) \tag{34}$$

The singular configurations of this wrist are obtained when the matrix J^{-1} is singular. Let us denote by $\mathbf{F_c}$ the force vector applied on the ball and socket joint, τ the articular force vector and $\mathbf{T} = (\mathbf{F}, \mathbf{M})$ an external wrench applied on the mobile plate. We have :

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{M} \end{pmatrix} = J_f \begin{pmatrix} \mathbf{F_c} \\ \tau \end{pmatrix} \tag{35}$$

where J_f is a 6x6 matrix defined by:

$$\begin{pmatrix} & \vdots & \mathbf{n_1} & \mathbf{n_2} & \mathbf{n_3} \\ \mathbf{I_3} & \vdots & & & \\ & & & & \\ \mathbf{0} & \vdots & \mathbf{CB_1} \wedge \mathbf{n_1} & \mathbf{CB_2} \wedge \mathbf{n_2} & \mathbf{CB_3} \wedge \mathbf{n_3} \end{pmatrix} \tag{36}$$

where $\mathbf{I_3}$ is the 3x3 identity matrix. It is easy to see that J^{-1} and J_f have the same determinant. Then we notice that the three first columns of J_f are the Plücker vectors of the lines crossing C and parallel to the axis of the reference frame. The last three column are simply the Plücker vectors of the link. We may thus apply our geometrical approach to this 6 lines to find the singular configurations of the wrist.

10 Conclusion

We have described a geometrical approach to determine the singular configurations of closed-loop manipulator. This approach is in general much more simpler than the classical approach which use the determinant of a jacobian matrix. Another advantage is that we get also a geometrical description of the singular configurations.

A complete analysis of a parallel manipulator has been presented. The geometrical approach enables to determine all the relations between the position and orientation parameters of the mobile plate which define the singular configurations.

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