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# On the infinitesimal motion of a parallel manipulator in singular configurations

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**Abstract**

*As serial robots parallel manipulator may be in a singular configuration. In these configurations the inverse jacobian matrix is singular and the end-effector may move although the articular velocities are equal to zero. The determination of the loci of these singular configurations is an important problem because in such configuration the articular forces may go to infinity and yield important mechanical damages. In a preceding paper we have proposed a geometrical approach for finding the singular configurations loci. We consider here a specific parallel manipulator and find what are the features of the infinitesimal motions associated to each of the singular configurations.*

## 1 Introduction

We consider in this paper the 6 d.o.f. parallel manipulator described in Figure 1 and called the MSSM (it is in fact the well known Bricard's octahedron [2]). Basically it consists in two plates connected by 6 articulated links. In the following sections the smaller plate will be called the *mobile* and the larger (which is in general fixed) will be called the *base*. A ball-and-socket joint connect the links to the mobile and an universal joint connect the links to the base plate. In each link there is one linear actuator and by changing the lengths of the links we are able to control the position and orientation of the gripper. We introduce an absolute frame R with origin $O$ and a relative frame $R_b$ fixed to the mobile with origin $C$ (Figure 1). The rotation matrix relating a vector in $R_b$ to the same vector in R will be denoted by $M$. Each link will be numbered from 1 to 6.

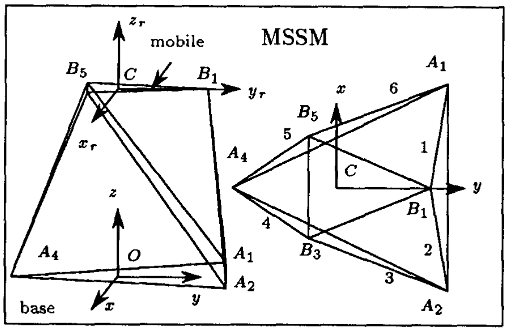


Figure 1: A parallel manipulator : the MSSM (perspective and top view).

The center of the articulation on the base for link $i$ will be denoted $A_i$ and that on the mobile $B_i$. The length of link $i$ will be noted $\rho_i$, and the unit vector of this link $\mathbf{n_i}$. The coordinates of $C$ in the reference frame are $\mathbf{X_C} = (x_c, y_c, z_c)$. We use the Euler's angles $\mathbf{\Omega_C} = (\psi, \theta, \phi)$ to represent the orientation of the mobile. A vector which components are expressed in the relative frame will be denoted by the subscript $_r$.

We will consider the case where each set of articulation points of both the base and the mobile lie in a plane.

## 2 Singularities

### 2.1 Inverse and Direct kinematics

Let us calculate the fundamental relations between the links lengths and the position of the mobile. For a given link we have :

$$\mathbf{AB} = \rho \mathbf{n} \tag{1}$$

$$\mathbf{AB} = \mathbf{AO} + \mathbf{OC} + \mathbf{CB} \quad \mathbf{CB} = M\mathbf{CB_r} \tag{2}$$

where $\mathbf{CB_r}$ means the coordinates of the articulation points with respect to frame $R_b$. $\mathbf{n}$ being a unit vector we have :

$$\rho = ||\mathbf{AO} + \mathbf{OC} + M\mathbf{CB_r}|| \tag{3}$$

If the position of the mobile is given we are able to calculate the components of the right side vector of the above equation and thus the length of the segment. Therefore the inverse kinematics is straightforward (this is in fact a general feature of parallel manipulators) and is defined by the above 6 equations which constitute a system of non-linear equations denoted by $\mathcal{S}$.

At the opposite the direct kinematics is much more complicated. Indeed to find the position of the mobile for a given set of links lengths we have to solve the system $\mathcal{S}$. It has been shown in [8] that in general the solution is not unique (there can be up to 16 solutions).

## 2.2 An analytical approach of singular configurations

Let us assume that for a given set of links lengths $\rho$ we know a solution $\mathbf{X_0}$ of the system $\mathcal{S}$. From the rank theorem we know that in a neighborhood of $\mathbf{X_0}$ the solution of $\mathcal{S}$ is unique if the rank of the jacobian matrix $J$ of this system is equal to 6 with:

$$J = ((\frac{\partial \rho}{\partial \mathbf{X}})) \tag{4}$$

where $\mathbf{X}$ is the position parameters vector. Note that this matrix is in fact the inverse jacobian (in a robotics sense) of the manipulator. It relates the linear and angular velocities of the mobile to the actuators velocities.

Now let us assume that $J$ is singular: this means that the mobile plate may have an *infinitesimal motion* around $\mathbf{X_0}$ without any change in the links lengths. In that case we will say that $\mathbf{X_0}$ is a *singular configuration* of the manipulator. In other words the velocity of the mobile plate may be different from zero although the actuator's velocities are all equal to zero. This means also that in these configurations the manipulator *gains* some degrees of freedom (at the opposite of the singular configurations of serial manipulator where it loses degrees of freedom).

An important point is that in the case of the MSSM (but also for very different mechanical architectures of parallel manipulators) the $i^{\text{th}}$ row $J_i$ of the jacobian $J$ can be written as :

$$J_i = ((\mathbf{n_i} \qquad \mathbf{CB_i} \wedge \mathbf{n_i})) \tag{5}$$

Therefore a row is the Plücker vector of the line associated to link $i$.

## 2.3 A mechanical approach of singular configurations

The previous approach indicates that in a singular configuration the manipulator is no more controllable. But a mechanical approach will give another insight into these configurations. Let $\tau$ denotes the articular forces vector (i.e. the traction-compression stresses in the links) and $\mathbf{F}$ an external wrench applied on the mobile plate. It is well known that we have:

$$\mathbf{F} = J^T \tau \tag{6}$$

If $J$ is singular then no $\tau$ can be found to equilibrate a set of wrenches. Furthermore in the vicinity of a singular configuration the articular forces may go to infinity. Practically this imply that if the mobile plate is "close" from a singular configuration the robot will suffer mechanical damages and this explain why the determination of the loci of these configurations is an important problem.

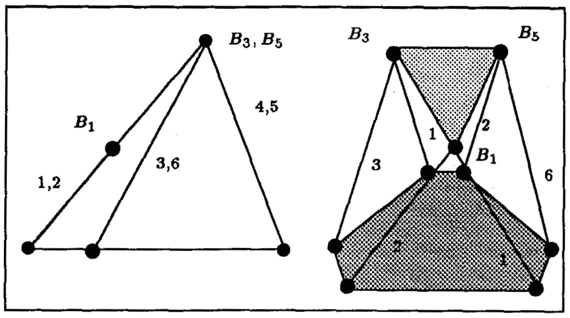


Figure 2: Hunt's singular configuration.

## 2.4 Determination of the loci of the singular configurations

### 2.4.1 The determinant of $J$

From this point finding the loci of the singular configurations as a function of $\mathbf{X_C}, \mathbf{\Omega_C}$ seems obvious : as the matrix $J$ is fully known we may calculate its determinant and then find its roots in $\mathbf{X_C}, \mathbf{\Omega_C}$. In fact even the first step of this method, the symbolic computation of this determinant, is rather tedious. To get an idea of its complexity the computation of the determinant of a MSSM involves 16 powers, 485 multiplications, 28 additions and 40 subtractions.....

### 2.4.2 Previous works

Few researchers have addressed the problem of determining the loci of the singular configurations. Using a mechanical analysis Hunt [4] has determined a singular configuration for a parallel manipulator similar to the MSSM but with a triangular mobile plate and an hexagonal base plate (Figure 2). In this configuration all the segments intersect one line (line $B_3B_5$) and an external torque around this line cannot be equilibrated by the actuator forces. Fichter [3] describes another singular configuration for a MSSM in which the mobile and base plate are equilateral triangle. It is obtained when the mobile plate is rotated around its $z$ axis with an angle of $\pm\frac{\pi}{2}$. This result was obtained by noticing that in this case two lines of the determinant were constant. But outside these two particular configurations no systematic method was proposed to find *all* the singular configurations of a parallel manipulator.

### 2.4.3 A geometrical approach

We have proposed in [7] a geometrical approach to this problem. If we suppose that the jacobian matrix $J$ is singular this means that there is a linear dependency between a subset of the Plücker vectors which constitute this matrix. But Grassmann has shown that such a dependency involves some kind of geometrical relationship between the lines associated to the Plücker vectors. For example if we consider three Plücker vectors and assume that they are linearly dependent then the associated lines will belong to a flat pencil of lines (all the lines will lie in a plane and meet the same point). Thus a singularity of $J$ can be expressed by some geometrical constraints on some sets of lines associated to the links. A previous analysis [9] of the MSSM has shown that among the geometrical properties described by Grassmann only three of the them can be fulfilled by the lines associated to the links of this manipulator, namely:

- 3d case : 4 lines are coplanar
- 5a case : all the 6 lines belong to a general complex
- 5b case : all the 6 lines intersect a line of space (special complex)

This result was obtained under the assumption that a link cannot lie in the base plane and we keep this assumption in this paper. We will now investigate the infinitesimal motion associated to these singular configurations. As stated by the reviewers the results we present in this paper are specific to the MSSM but the principle of the analysis can be applied for many others parallel manipulators. We focus on the MSSM only because it is the most simple manipulator.

## 3 Infinitesimal motion associated to the singular configuration of a MSSM

### 3.1 Case 3d

We consider a set of 4 lines $\mathcal{S}$ and investigate for which configurations of the mobile plate they will belong to some plane $\mathcal{P}$. First we notice that the set of lines must be such that the number of their distinct articulation points $A_i$ is less than three. Indeed in the opposite case the plane $\mathcal{P}$ will be defined by the points $(A_1, A_2, A_3)$ i.e. the plane $\mathcal{P}$ will be the base plane and therefore the links will lie on the base plane. Thus the only possible sets for $\mathcal{S}$ are (1,2,3,6), (2,3,4,5), (1,4,5,6). Let us notice that in each case the points $B_1, B_3, B_5$ belong to $\mathcal{P}$ i.e. this plane is the mobile plane and we get the singular configuration described by Hunt. By changing the base and relative frame we can consider that all these sets can be reduced to the set (1,2,3,6). The coordinates of the articulation points on the base in the reference frame are :

$$A_6 = A_1 = [\ xa_0 \quad ya_0 \quad 0\ ]^T$$
$$A_3 = A_2 = [\ xa_1 \quad ya_0 \quad 0\ ]^T$$
$$A_5 = A_4 = [\ 0 \quad ya_3 \quad 0\ ]^T$$

The coordinates of the articulation points on the mobile in the relative frame are :

$$B_{2_r} = B_{1_r} = [\ 0 \quad y_0 \quad 0\ ]^T$$
$$B_{3_r} = B_{4_r} = [\ x_2 \quad y_2 \quad 0\ ]^T$$
$$B_{5_r} = B_{6_r} = [\ x_3 \quad y_2 \quad 0\ ]^T$$

The coplanarity of lines (1,2,3,6) can be expressed by two equations:

$$(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_3B_3} = 0$$
$$(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_6B_6} = 0 \qquad (7)$$

which constitute a system of two linear equations in $y_c, z_c$ in which $x_c$ does not appear. The determinant $\Delta$ of this system is:

$$\Delta = -\sin(\theta)\sin(\psi)(xa_0 - xa_1)^2(-x_3 + x_2)(-y_2 + y_0) \qquad (8)$$

If $\Delta$ is not equal to zero we get $y_c, z_c$ but they are such that:

$$\mathbf{A_1B_1} \wedge \mathbf{A_2B_2} = \mathbf{0} \qquad (9)$$

which means that lines (1,2) are colinear and lie in the base plane. Therefore we have to consider the case where the determinant $\Delta$ is equal to zero. This happen for $\sin\theta = 0$ i.e. for $\theta = 0$ or $\theta = \pi$. But in both cases the only possible solution for the equations (7) is $z_c = 0$; therefore the links lie in the base plane. The determinant $\Delta$ may also vanish for $\sin\psi = 0$ i.e. for $\psi = 0$ or $\psi = \pi$. In this case we get:

$$\psi = 0 \text{ and } z_c = -\frac{(ya_0 - y_c)\sin(\theta)}{\cos(\theta)} \qquad (10)$$

$$\psi = \pi \text{ and } z_c = \frac{(ya_0 - y_c)\sin(\theta)}{\cos(\theta)} \qquad (11)$$

For $\psi = 0$ a basis [1] for the null space of the linear transformation defined by the matrix $J$ is given by a vector $\mathbf{A}$. If $\phi \neq 0$ we have:

if $\phi \neq 0$

$$\mathbf{A} = [0, \frac{y_2\sin\theta}{\sin\phi\cos\theta}, -\frac{y_2}{\sin\phi}, \frac{\cos\phi}{\sin\phi\cos\theta}, 1, \frac{\sin\theta}{\cos\theta}]^T$$

if $\phi = 0$

$$\mathbf{A} = [0, \sin(\theta)y_2, -\cos(\theta)y_2, 1, 0, 0]^T \qquad (12)$$

For $\psi = \pi$ we get :

if $\phi \neq 0$

$$\mathbf{A} = [0, \frac{y_2\sin\theta}{\sin\phi\cos\theta}, \frac{y_2}{\sin\phi}, \frac{\cos\phi}{\sin\phi\cos\theta}, 1, -\frac{\sin\theta}{\cos\theta}]^T$$

if $\phi = 0$

$$\mathbf{A} = [0, \sin(\theta)y_2, \cos(\theta)y_2, 1, 0, 0]^T \qquad (13)$$

The velocity of a point $M$ can be calculated by :

$$\mathbf{V_M} = \mathbf{V_C} + \mathbf{MC} \wedge \mathbf{\Omega} \qquad (14)$$

Using this formula and equation (12) or equation (13) it is easy to show that:

$$\mathbf{V_{B_4}} = \mathbf{V_{B_5}} = 0 \qquad (15)$$

Therefore the corresponding motion is a rotation around the line going through $B_4, B_5$.

[1] This basis is obtained from $J$ by using the **kernel** function of the algebraic computation system Maple.

### 3.2 Case 5a

A set of 5 skew lines in space defines a linear complex. Any line whose Plücker vector is a linear combination of the Plücker vectors of the 5 lines belongs to the complex. An interesting property of the linear complex of lines is that all the lines of a complex which are coplanar belong to a flat pencil of lines i.e. they intersect all the same point [6], [5]. Another interesting property of a complex is that the corresponding infinitesimal motion is a screw motion around the complex axis [1], [4], [5].

Let us investigate now in which configuration of the mobile plate the 6 lines constitute a complex. Let us consider the lines of the flat pencils spanned by (1,6), (2,3),(4,5). These lines belong to the complex. Among these lines consider the three lines $D_1, D_2, D_3$ which lie on the base plane. If the 6 lines belong to a linear general complex the lines $D_1, D_2, D_3$ belong to a flat pencil of lines of center $M$, whose coordinates in the reference frame are $(x, y, 0)$. If $\mathbf{v_{ij}}$ denotes the normal vector to the pencil of lines spanned by line $i, j$ we must have:

$$\begin{aligned} \mathbf{A_1M.v_{16}} &= 0 &(16)\\ \mathbf{A_3M.v_{23}} &= 0 &(17)\\ \mathbf{A_5M.v_{45}} &= 0 &(18)\end{aligned}$$

These three equations are linear in term of $x, y$. We use the first two to calculate these unknowns and put their values in the last equation which yield to a constraint equation. This equation is of order 3 in $z_c$, 2 in $x_c, y_c$. An interesting point is that for $\theta = 0$ or $\theta = \pi$ this equation can be reduced to:

$$\begin{aligned}\tan(\psi) &= (y_0xa_0 - y_0xa_1 - x_2ya_0 + ya_3x_2 - ya_3x_3\\ &\quad -xa_0y_2 + xa_1y_2 + ya_0x_3)/\\ &\quad (x_2xa_1 - xa_0x_3) \qquad (19)\end{aligned}$$

And if the mobile plate is symmetric ($x_3 = -x_2, xa_1 = -xa_0$) equation(19) is reduced to :

$$\cos(\psi) = 0 \qquad (20)$$

Thus in this case we get a singular configuration for $\psi = \pm\frac{\pi}{2}$ whatever are $x_c, y_c, z_c$ : we find the singular configuration described by Fichter. Let us determine the instantaneous screw axis (ISA) of the motion. If point $P(x, y, z)$ belongs to the ISA then :

$$\mathbf{V_P} = \alpha\mathbf{\Omega} \qquad (21)$$

where $\mathbf{V_P}$ is the velocity of point $M$ and $\alpha$ is a constant. We have:

$$\mathbf{V_P} = \mathbf{V_C} + \mathbf{PC} \wedge \mathbf{\Omega} \qquad (22)$$

which yield to:

$$\mathbf{V_P} \wedge \mathbf{\Omega} = \mathbf{V_C} \wedge \mathbf{\Omega} + (\mathbf{PC} \wedge \mathbf{\Omega}) \wedge \mathbf{\Omega} = 0 \qquad (23)$$

or:

$$\mathbf{V_C} \wedge \mathbf{\Omega} - (\mathbf{\Omega.\Omega})\mathbf{PC} + (\mathbf{\Omega.PC})\mathbf{\Omega} = 0 \qquad (24)$$

Equation (24) defines two planes and the ISA is the intersection between these planes. The pitch $h$ of the screw motion is then obtained by:

$$\mathbf{V_P} = h\frac{\mathbf{\Omega}}{||\mathbf{\Omega}||} \qquad (25)$$

For $\psi = \frac{\pi}{2}$ a basis for the null space of the linear transformation defined by the matrix $J$ is given by a vector $\mathbf{A}$:

$$\begin{aligned} A[1] &= 0\\ A[2] &= -\frac{x_2xa_0 - y_2ya_0 + y_2ya_3}{ya_0 - ya_3}\\ A[3] &= ((ya_0 - y_c)(ya0 - ya_3)(y_2y_0 + y_2^2) + y_cx_2xa_0\\ &\quad (y_0 - y_2) + ya_0x_2xa_0y_2 - y_0ya_3x_2xa_0)/\\ &\quad ((ya_0 - ya_3)(y_0 - y_2)z_c)\\ A[4] &= \frac{x_2xa_0 + y_2ya_3 + ya_0x_c - y_2ya_0 - ya_3x_c}{z_c(ya_0 - ya_3)}\\ A[5] &= -\frac{y_2y_c - y_2ya_0 + x_2xa_0 + y_0ya_0 - y_0y_c}{(y_0 - y_2)z_c}\\ A[6] &= 1 \qquad (26)\end{aligned}$$

Let $l_m = y_0 - y_2$, $l_b = ya_0 - ya_3$. Although the ISA equation is too long to be given here the pitch is:

$$\begin{aligned} h &= (x_2xa_0 + l_ml_b)x_2xa_0/(x_2^2xa_0^2(l_b^2 + l_m^2)\\ &\quad +2x_2xa_0l_ml_b(l_b(ya_0 - y_c) + l_m(x_c - y_2)) +\\ &\quad 7l_b^2l_m^2((x_c - y_2)^2 + (l_b - y_c)^2\\ &\quad +ya_3(ya_3 - 2y_c + 2l_b) + z_c^2)^{\frac{1}{2}} \qquad (27)\end{aligned}$$

If we suppose that the mobile plate and the base plate are equilateral triangles and that the radii of the circles circumscribed about these triangles are $R_m, R_b$ then a basis of the null space is:

$$\begin{aligned} &\text{if } x_c \neq 0\\ &\mathbf{A} = [0, 0, -\frac{R_mR_b}{2x_c}, \frac{y_c}{x_c}, \frac{z_c}{x_c}, 1,]^T\\ &\text{if } x_c = 0 \qquad (28)\\ &\mathbf{A} = [0, 0, -\frac{R_mR_b}{2z_c}, 0, \frac{y_c}{z_c}, 1,]^T \qquad (29)\end{aligned}$$

The pitch of the corresponding screw motion is:

$$\begin{aligned} x_c &\neq 0 \quad \Rightarrow h = -\frac{R_mR_bz_c}{2x_c\sqrt{x_c^2 + y_c^2 + z_c^2}} \qquad (30)\\ x_c &= 0 \quad \Rightarrow h = -\frac{R_mR_b}{2\sqrt{y_c^2 + z_c^2}} \qquad (31)\end{aligned}$$

If $x_c \neq 0$, $y_c \neq 0$ the ISA is defined by:

$$x = -\frac{x_c^2R_mR_b - 2x_cz_c^2y - 2x_c^3y + y_c^2(R_mR_b - 2x_cy)}{2y_c(x_c^2 + y_c^2 + z_c^2)}$$

$$z = -\frac{z_c(R_m R_b x_c - 2y_c^2 y - 2z_c^2 y - 2x_c^2 y)}{2y_c(x_c^2 + y_c^2 + z_c^2)}$$

If $x_c = 0$ the ISA is defined by:

$$x = -\frac{R_m R_b y_c}{2(y_c^2 + z_c^2)} \quad y = \frac{y_c z}{z_c}$$

If $y_c = 0$ the screw motion axis is the platform z-axis as shown by Fichter.

### 3.3 Case 5b

In this case the six lines associated to the link intersect all the same line $(D)$ in space. An analysis has shown that this may happen in two cases:

- $(D)$ is an edge of the mobile plate and four lines are coplanar. This is Hunt's singular configuration already studied in a previous section.
- 3 lines are coplanar and $(D)$ is defined the two articulations points which are not common to two of the coplanar lines. For example if lines (2,3,4) are coplanar the line $(D)$ defined by $A_4, B_1$ intersects lines (1, 2, 3, 4, 5). Then by rotating the mobile plate around $B_1, B_4$ we may find a configuration in which line 6 intersect $(D)$ (Figure 3).

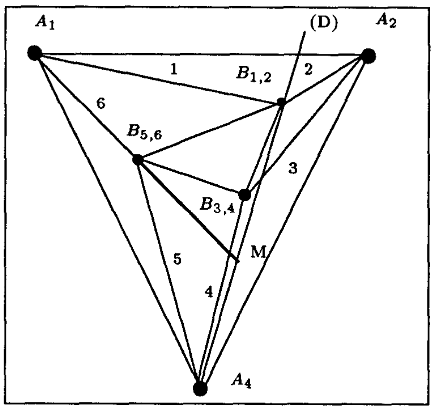


Figure 3: In this example the links (2,3,4) are coplanar and the line $(D)$ going through $A_4, B_1$ intersects lines (1,2,3,4,5). By rotating the mobile around its edge $B_1, B_3$ line 6 intersects (D) at point $M$. Therefore the 6 lines intersect $(D)$ and the manipulator is in a singular configuration.

It may be shown that by changing the frames the different cases are equivalent to the case where the lines (1,2,3) are coplanar. An example of such singular configuration is given in Figure 4. If lines (1,2,3) are coplanar then we have:

$$(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}).\mathbf{A_3B_3} = 0 \tag{32}$$

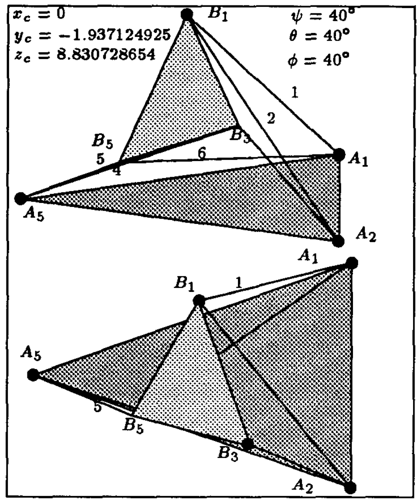


Figure 4: An example of singular configuration of type 5b for the MSSM: lines 1,2,3 are coplanar and line $A_5B_5$ intersects line $A_1B_3$.

Then line $(D)$ is defined by $A_1, B_3$ and intersect the lines (1,2,3,4,6). We may write that this line intersects line 5 by:

$$\mathbf{A_1B_3}.(\mathbf{OA_5} \wedge \mathbf{OB_5}) + \mathbf{A_5B_5}.(\mathbf{OA_1} \wedge \mathbf{OB_3}) = 0 \tag{33}$$

Equations (32)(33) constitute a linear system in $y_c, z_c$ which is solved. A basis of the null space is:

$$\begin{aligned}
A[1] &= \frac{x_2 y_0 \sin\psi}{y_0 \cos\phi - x_2 \sin\phi} \\
A[2] &= \frac{y_0 x a_0 x_2 \sin\psi}{y a_3 (y_0 \cos\phi - x_2 \sin\phi)} \\
A[3] &= -(x_2 y_0 \sin\psi(x a_0 \cos\phi \sin\psi + x a_0 \cos\psi \\
&\quad \cos\theta \sin\phi - y a_3 \sin\psi \cos\theta \sin\phi + \\
&\quad y a_3 \cos\psi \cos\phi))/(\sin\phi \sin\theta \\
&\quad (y_0 \cos\phi - x_2 \sin\phi) y a_3) \\
A[4] &= (((\sin\psi \cos\theta \sin\phi - \cos\psi \cos\phi) \\
&\quad (x_2 y a_3 \sin\phi - y_0 y a_3 \cos\phi) + x a_0 y a_0 \sin\psi))/ \\
&\quad (\sin\phi \sin\theta (y_0 \cos\phi - x_2 \sin\phi) y a_3) \\
A[5] &= -(x_2 \cos\psi \cos\theta \sin\phi + x_2 \sin\psi \cos\phi - \\
&\quad y_0 \cos\theta \cos\phi \cos\psi + y_0 \sin\psi \sin\phi)/ \\
&\quad ((y_0 \cos\phi - x_2 \sin\phi) \sin\theta) \\
A[6] &= 1
\end{aligned} \tag{34}$$

It can then be shown that :

$$\mathbf{V_{B_3}} = \mathbf{V_{A_1}} = 0 \tag{35}$$

Therefore the motion associated to this singular configuration is a rotation around the line going through these points.

## 4 Conclusion

Every singular configuration for a MSSM is in fact obtained when the 6 lines associated to the links of the manipulator belong to a linear complex of lines. Consequently we know that the corresponding infinitesimal motion is a screw motion. If the complex is singular (as in Hunt's configuration) the pitch of this motion is equal to zero and therefore the resulting motion is a pure rotation around the ISA (which is the line intersecting all the 6 lines of the manipulator). If the complex is general we have determined the ISA and the pitch of the screw motion when the base and mobile plates are parallel.

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