Algebraic Geometry tools for the study of Kinematics of parallel manipulators

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Abstract

Manipulation of algebraic equations arise frequently in kinematic problems. But in many of these problems it is not necessary to solve the algebraic equations to establish interesting results as sometime only the number of real solutions is important. Fortunately many theorems in algebraic geometry, some of them being not well known, may give some insight on this point. We present some of these theorems and show how they can be applied to demonstrate interesting results in the field of kinematic problems for parallel manipulators.

1 Introduction

Systems of algebraic equations play an important role in kinematics problems as most of these problems can be stated as solving such a system. For some kinematics problems it is not necessary to *solve* the equations but it is more important to determine:

- the maximum number of real roots of the system
- the number of real roots of a polynomial in a given interval

We will present some tools which can be used for these purposes *without* computing the roots of the equations and study their application for some kinematics problems related to parallel manipulators.

2 Order of an algebraic equation

Let F_1 be an algebraic equation in the unknowns x_1, x_2,...x_n

$$F_1(x_1, x_2, ...x_n) = \sum_{i=0}^{i=m} a_i x_1^{u_1^i} x_2^{u_2^i} ... x_n^{u_n^i} = 0 \qquad (1)$$

where u_j^i is the degree of the unknown x_j for the term of coefficient a_i. Let C_i be the sum of the degree of each unknown for the term of F_1 with coefficient a_i

$$C_i = \sum_{j=1}^{j=n} u_i^i \tag{2}$$

The order d_1 of F_1 is defined as the maximum of the C_i's:

$$d_1 = \text{Max}(C_i) \quad i \in \; [0, m] \tag{3}$$

For example the equation

$$F_1(x, y, z) = x^3y^2z + 2\,xy^3z^2 + 3\,x^3y^3z^2 + 2\,x^3y^4z^2 + xy^2z^4 = 0$$

is of order 9 (which come from the term $x^3y^4z^2$)

3 Bezout's theorem

This theorem is one of the most interesting in algebraic geometry. An extensive study of Bezout's theorem can be found in Walker. We give here a simplified version of this theorem:

The intersection of m algebraic equations in n unknowns ($m \geq n$) of degree n_1, n_2,...,n_m is constituted of at most $\prod_{i=1}^{i=m} n_i$ points

In the case of planar algebraic curves a version of Bezout's theorem can be stated as:

two curves of order m, n with no common components have exactly mn intersection points.

4 Circularity

This notion and its application to kinematic problems has been dealt in detail by Hunt.

4.1 Planar case

Bezout's theorem may seem to be rather strange in some case. Let us consider two circle described by algebraic equation of order 2. It is well known that they will have at most two real intersection points....
Let a circle of radius r, with a center at coordinates (a, b) defined by the equation:

$$(x - a)^2 + (y - b)^2 - r^2 = 0$$

By expanding this equation we get:

$$x^2 - 2\,xa + a^2 + y^2 - 2\,yb + b^2 - r^2 = 0$$

The terms of this equation are not homogeneous, i.e. their order with respect to the variables x, y are 2, 1 or 0. Let us rewrite this equation with a new unknown w:

$$(\frac{x}{w} - a)^2 + (\frac{y}{w} - b)^2 - r^2 = 0$$

where w is simply a scaling factor. The previous equation is now homogeneous as it can be written as:

$$(x - aw)^2 + (y - bw)^2 - r^2w^2 = 0$$

or

$$x^2 - 2\,xaw + a^2w^2 + y^2 - 2\,ybw + b^2w^2 - r^2w^2 = 0$$

for which the order of each term with respect to the variables x, y, w is now 2. The system of unknowns x, y, w is called a planar homogeneous system of coordinates.

If $w = 0$ the circle is infinitely enlarged and every point is at infinity. The line $w = 0$ is called the *line at infinity* and this line cross the circle in two points defined by:

$$x^2 + y^2 = 0 \tag{4}$$

i.e. at the points $\mathcal{S}_1$, $\mathcal{S}_2$

$$\mathcal{S}_1 \left\{ \begin{array}{c} w = 0 \\ x = iy \end{array} \right. \qquad \mathcal{S}_2 \left\{ \begin{array}{c} w = 0 \\ x = -iy \end{array} \right.$$

These two imaginary points are called the *the imaginary circular points* and equation (4) defines the *imaginary circle*

As the parameters a, b, r does not appear in the definition of the imaginary circular points they belong to any circle. Therefore they belong too to the intersection of two circles. Accordingly two circles with two real intersection points have also in common the two imaginary circular points i.e. a total of four intersection points, in accordance with Bezout's theorem.

If a planar curve has the points $\mathcal{S}_1$, $\mathcal{S}_2$ as double, triple.. points it will be said that this curve has a *circularity* of 2,3,... Therefore a circle has circularity 1.

4.2 Application of circularity

Let consider the following planar parallel manipulator described in figure 1. The triangular plate BDE is connected to the three fixed points A, C, F by three links

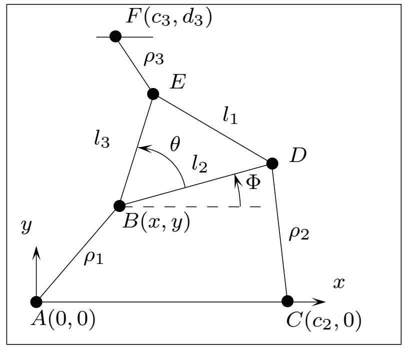

Figure 1: A planar parallel manipulator.

with rotoid joints at each extremity. A linear actuator in each link enables to change the link length and it may be shown that by controlling these lengths the posture of the triangular plate can be adjusted at will. Indeed let us assume that we have fixed the position and orientation of the triangular plate BDE in some reference frame. Therefore the positions of its vertices are also known in this frame. The link lengths corresponding to the given posture is simply the distance between the points AB, CD, EF and we have solved the *inverse kinematic problem* of this manipulator. Formally let us define the reference frame such that: $A : (0, 0)$,$C : (c_2, 0)$,$F : (c_3, d_3)$ and we define the posture of the triangular plate by the coordinates (x, y) in the reference frame of point B and its orientation by the angle Φ between the line BD and the x axis. The link lengths ρ can be computed as:

$$\rho_1^2 = x^2 + y^2 \quad (5)$$

$$\rho_2^2 = (x + l_2 \cos\Phi - c_2)^2 + (y + l_2 \sin\Phi)^2 \quad (6)$$

$$\rho_3^2 = (x + l_3 \cos(\Phi + \theta) - c_3)^2 + (y + l_3 \sin(\Phi + \theta) - d_3)^2 \quad (7)$$

Let $T = \tan(\Phi/2)$. We have:

$$\sin(\Phi) = \frac{2T}{1 + T^2} \qquad \cos(\Phi) = \frac{1 - T^2}{1 + T^2} \quad (8)$$

The previous equations can be written now as:

$$x^2 + y^2 - \rho_1^2 = 0 \quad (9)$$

$$x^2 + x^2T^2 + y^2T^2 + y^2 + a_1x + a_2xT^2 + a_3T^2 + a_4yT^2 + a_5 = 0 \quad (10)$$

$$x^2 + x^2T^2 + y^2T^2 + y^2 + b_1x + b_2xT^2 + b_3T^2 + b_4yT^2 + b_5 = 0 \quad (11)$$

The orders of these equations are 2, 4, 4. Suppose now that the lengths of the links are fixed and that we want to determine the position and orientation of the triangular plate i.e. solve the *direct kinematic problem*. We have therefore to solve the previous system of algebraic equations. Using Bezout's theorem we deduce that this system will have at most 32 (2x4x4) solutions, either real or complex. We will show now that in fact a smaller upper-bound of the number of real solutions can be established. Let us consider another mechanism described in figure 2.

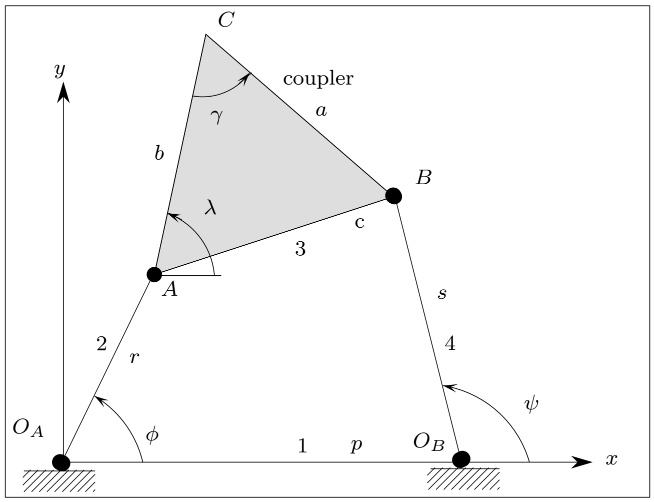

Figure 2: A four-bar mechanism.

This mechanism is called a *four-bar mechanism*. Many authors (see for example Hunt) have shown that point C of this mechanism describes a sixth order curve, a *sextic* with a circularity of 3 (which is the maximum for the circularity of a sextic).

Now let us consider the four-bar mechanism $ABEDC$ in the mechanism of figure 1. E lie on the sextic of the four-bar mechanism but also belongs to the circle centered in F, of radius FE for a valid solution of the direct kinematic problem. E is therefore the intersection point of two algebraic curves of order 2 and 6 and there will be at most 12 intersection points according to Bezout's theorem.

But the intersection will contain the two circular imaginary points $\mathcal{S}_1$, $\mathcal{S}_2$ as triple points. Therefore there will be at most 6 *real* intersection points and therefore this is upper bound of the number of posture of the direct kinematic problem.

This has been confirmed by Gosselin which has shown that the system of equations (5, 6, 7) can be reduced to a 6th order polynomial in one variable. Indeed let substract equation (5) from equations (6), (7). We get a linear system of two equations in the two unknowns x, y which can be solved, the result being substituted in equation 5. The only unknown in this equation is now Φ. Using the susbstitution described by equation (8) the remaining equation becomes a 6th order polynomial in T:

$$a_6T^6 + a_5T^5 + a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0 = 0 \quad (12)$$

4.3 Spatial case

Let us consider now the intersection of two spheres i.e. two surfaces of degree 2 which, according to Bezout's theorem, must intersect along a curve of order 4.

But it is known that the intersection of two spheres is a circle of degree 2. We must therefore find a conic at infinity which explains the missing degree. We rewrite the equation of the sphere in homogeneous coordinates:

$$(x - aw)^2 + (y - bw)^2 + (z - cw)^2 - r^2w^2 = 0$$

The plane $w = 0$ is called the plane at infinity and the intersection of the sphere and this plane is found as:

$$x^2 + y^2 + z^2 = 0 \quad (13)$$

As none of the parameters a, b, r appear in this equation this curve of order 2 belong to all the spheres and therefore to the intersection of any two spheres. Equation 13 defines an imaginary cone whose intersection with the plane at infinity is the *imaginary spherical circle* which belong to all the spheres.

The imaginary cone intersects the plane $z = 0$ along two imaginary lines defined by $x = \pm iy$ and therefore the circular imaginary points belong to the cone. As a consequence there cannot be more than 2 real intersection points between a sphere and a circle.

The circularity of a surface is then defined as the order of multiplicity with which it contains the imaginary spherical circle. A sphere has therefore a circularity 1. For example it may be shown that a general torus (fourth-order surface) has a circularity 2 (maximal circularity).

4.4 Application for a kinematic problem

Let us consider the spatial mechanism described in figure 3. A triangular plate $B_1B_2B_3$ is connected to three fixed points A_1, A_2, A_3 by three links which have a rotoid joint at point A and a ball-and-socket joint at point B. We assume that the lengths of links A_1B_1, A_2B_2, A_3B_3 are fixed and we want to determine the possible locations of the triangular plate $B_1B_2B_3$ i.e. solve the direct kinematic problem for this mechanism.

We will consider the spatial mechanism obtained when we dissociate one of the B_i. We get the mechanism described in figure 4 which is known under the name $RSSR$.

We use now one of Cayley's theorem (see Hunt):

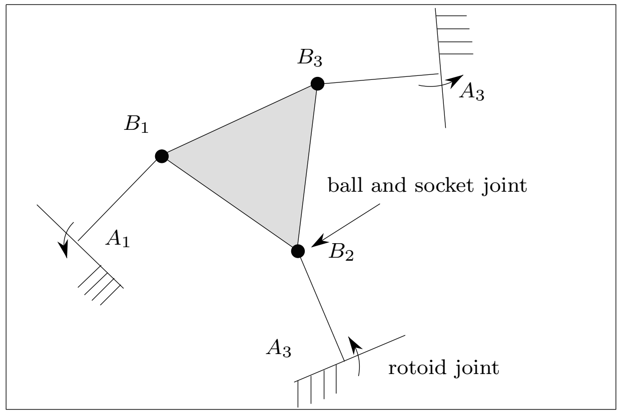

Figure 3: A general spatial mechanism.

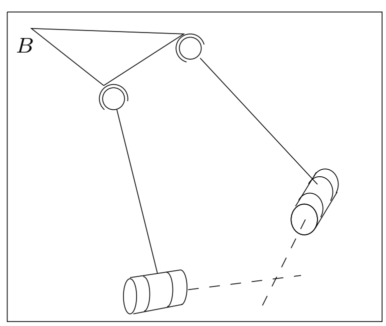

Figure 4: The $RSSR$ mechanism

A line with two points C, D lying on two algebraic curves of degree n_c, n_d and circularity p_c, p_d describes a ruled surface of degree $2n_c(n_d - p_d) + 2n_d(p_c - n_c)$

For the $RSSR$ we have two points lying on two circles i.e. $n_c = n_d = 2, p_c = p_d = 1$. The order of the surface is therefore 8 and B lie on a surface of order 16 (as point B can freely rotate around the line joining the centers of the ball-and-socket joints). It may be shown that the circularity of this surface is 8 (see Merlet). For a valid posture of the mechanism described in figure 3 point B_1 belongs to such a surface but also to the circle centered in A_1 whose radius is the link length . According to Bezout's theorem there is 32 intersection points (either complex or real) between the surface and the circle and according to the circularity of the surface and the circle 16 points among these 32 intersection point are the circular imaginary points. Therefore there is at most 16 real intersection points and consequently the direct kinematic problem as at most 16 solutions.

Now let us consider a parallel manipulator (figure 5). In this manipulator the 6

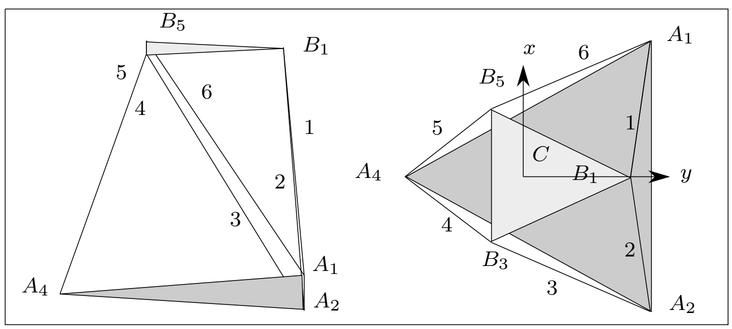

Figure 5: A parallel manipulator.

legs have their extremities connected to the plates by ball-and-socket joints. Their lengths can be modified in order to control the position and orientation of the upper plate.

Suppose that the legs have a known fixed length. The direct kinematics problem consist in determining the maximum number of postures of the upper plate for these leg lengths. Let us consider two legs with a common point B on the upper plate. As the leg lengths are fixed B will lie on a circle whose center and radius can be computed from the leg lengths and the position of the joint centers on the fixed plate. We can therefore substitute the two legs by a virtual leg whose only possible motion is a rotation around a fixed axis.

This can be done for any pair of leg sharing a common point on the upper plate and therefore the parallel manipulator upper plate will have the same possible postures as the mechanism described in figure 3: its maximum number of postures will be 16.

A more tedious way to demonstrate this result is to combine the algebraic equations describing the inverse kinematic problem to get a polynomial in one variable which order shall be 16 or less. A sixteen order polynomial has been first found by Charentus and Renaud and later by many authors, for example Dedieu (which give additional result about the convexity of the solution), Griffis, Innocenti. Using this result an example of manipulator with 16 possible postures for the end-effector has been presented by Merlet and Dedieu. In the former reference it has been shown that this result can be extended to many others manipulators as soon as they have a triangular end-effector.

5 Number of real roots of a polynomial in a given interval

Systems of algebraic equations arising in some kinematic problem can be reduced to the analysis of a polynomial in only one variable which shall furthermore lie in a given interval. Therefore it is of interest to consider a polynomial in one variable and to determine the number of its real roots in a given interval.

5.1 Sturm's method

An excellent and practical introduction to this method can be found in the book of Mineur.

Principle

Let a polynomial of degree n in x:

$$f_0(x) = \sum_{i=0}^{i=n} a_n x^n = 0$$

We consider the first derivative of this polynomial with respect to x:

$$f_1(x) = f_0'(x)$$

We denote by $\text{Rem}(f_{i-1}(x), f_i(x))$ the remainder of the Euclidian division of $f_{i-1}(x)$ by $f_i(x)$

We build a sequence of function by:

$$f_{i+1} = -\text{Rem}(f_{i-1}(x), f_i(x)) \quad i \in [1, n-1]$$

The last function of this sequence does not depend upon x. Let $[x_1, x_2]$ be the interval in which we are looking for the real roots of $f_0(x) = 0$.

Sturm's theorem

The number of real roots of the equation $f_0(x) = 0$ in the interval $[x_1, x_2]$ is obtained as the number of sign changes in the sequence $f_i(x_1), f_{i+1}(x_1), i \in [0, n-1]$ minus the number of sign changes in the sequence $f_i(x_2), f_{i+1}(x_2), i \in [0, n-1]$.

5.2 Application example

We consider a particular case of the planar parallel mechanism described in a previous section (figure 6). The equations giving the links lengths for a given posture

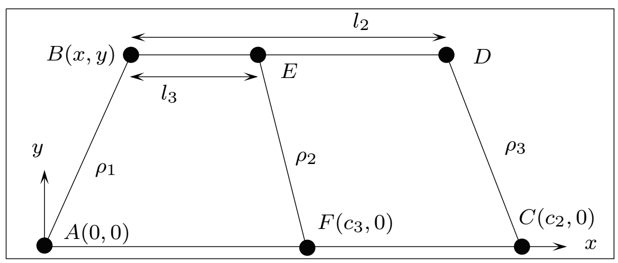

Figure 6: A special case of planar parallel manipulator.

of the end-effector are:

$$\begin{aligned}
\rho_1^2 &= x^2 + y^2 \\
\rho_2^2 &= (x + l_2 \cos\Phi - c_2)^2 + (y + l_2 \sin\Phi)^2 \\
\rho_3^2 &= (x + l_3 \cos\Phi - c_3)^2 + (y + l_3 \sin\Phi)^2
\end{aligned}$$

By manipulating these equations in a similar manner as in section 4.2 they can be reduced to a polynomial in one variable:

$$f_0(T) = a_3 T^3 + a_2 T^2 + a_1 T + a_0 = 0 \tag{14}$$

with $T = \cos\Phi$. Therefore they can be up to 3 real roots to this polynomial and as each root define two values for Φ we may think that an upper bound of the number of postures for the direct kinematic problem is 6. We will show now that in fact there will be at most 4 solutions to this problem.

To solve the direct kinematic problem we have to find the real roots of the polynomial but we are looking only for the roots in the interval [-1,1]. It may be shown that $f_0(1), f_0(-1)$ are always strictly positive (their symbolic values are squares)

If we build the Sturm's sequence we get four functions f_0, f_1, f_2, f_3 where f_3 is a constant. We are looking for sequences such that the number of real roots in the interval [-1,1] is maximal. This number will be maximum in four cases:

	f_0	f_1	f_2	f_3	number of sign changes
$x=-1$	+	-	+	+	2
$x=1$	+	+	+	+	0

	f_0	f_1	f_2	f_3	number of sign changes
$x=-1$	+	+	-	+	2
$x=1$	+	+	+	+	0

	f_0	f_1	f_2	f_3	number of sign changes
$x=-1$	+	-	+	-	3
$x=1$	+	+	+	-	1

	f_0	f_1	f_2	f_3	number of sign changes
$x=-1$	+	-	+	-	3
$x=1$	+	-	-	-	1

In all cases the number of roots will be at most 2 and therefore the direct kinematic problem will have up to 4 solution.

6 Huat's Theorem

Let a polynomial equation of degree n in x:

$$f_0(x) = \sum_{i=0}^{i=n} a_n x^n = 0$$

with real coefficients.

Theorem

If the roots of $f_0(x)$ are all real the square of all the non extremal coefficients is necessarily greater than the product of its neighbour coefficients

$$a_k^2 > a_{k-1}a_{k+1} \quad \forall k \in [1, n-1]$$

In fact Huat's theorem is a result of Newton inequalities (see Hardy), which state that if $f_0(x)$ has only real roots then:

$$k(n-k)a_k^2 \geq (k+1)(n-k+1)a_{k-1}a_{k+1} \ \ \forall \ k \in [1, n-1]$$

6.1 Application in kinematics

Let us consider the planar parallel manipulator described in figure 1. We have seen in a previous section that for a fixed set of link lengths there can be a maximum of 6 different postures for the triangular mobile plate. We are considering now a robot with a given geometry and are looking for a set of link lengths such that the direct kinematic problem has effectively 6 solutions i.e. 6 postures of the end-effector can be found.

To solve this problem we may choose randomly three link lengths, compute the coefficients of the 6th order polynomial (12) and then solve the polynomial until we find a set of link lengths such that all the 6 roots of the polynomial are real. Although this method has worked in practice (an example of solution is given in Merlet) the computation time may be great.

A faster way is to choose randomly only two of the three link lengths, say ρ_1, ρ_2 and then compute the 7 coefficients a_i of the forward kinematics polynomial (12) as functions of the unknown link length ρ_3.

Then we compute the square of all the non extremal coefficients minus the product of their neighbour coefficients i.e. $a_1^2 - a_0a_2, a_2^2 - a_1a_3, a_3^2 - a_2a_4, a_4^2 - a_5a_3, a_5^2 - a_4a_6$ which happen to be all fourth order polynomials in ρ_3.

The roots of these 5 polynomials P_j are computed and are used to determine for each polynomial the intervals of ρ_3 such that the polynomial is positive.

If the intersection $I_\cap$ of these intervals is empty then there is no value of ρ_3 such that the direct kinematic problem will have 6 solutions for the link lengths ρ_1, ρ_2.

At the opposite if the intersection is not empty the possible solutions for ρ_3 will lie in the interval $I_\cap$. Therefore random values for ρ_3 can be tested but only in $I_\cap$.

Such an algorithm has been implemented using the symbolic computation program MAPLE.

7 Conclusion

Dealing with algebraic equations is the "essence" of kinematic problems but many of these problems can be solved in an elegant way without determining the roots of these equations. By using basic theorems of algebraic geometry we have shown that many powerful results can be established in the field of parallel manipulators kinematic problems. These results have been established in most cases by dealing either with the geometrical aspect of the problem or with manipulation on the symbolic value of the coefficient of the algebraic equations which arise during the resolution of the problem. Therefore we have avoided to use numerical procedures in which numerical errors may introduce spurious results. Unfortunately many of these algebraic geometry theorems are not well known and are missing in many textbooks.

References