Geometrical determination of the workspace of a constrained parallel manipulator

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Abstract: Our purpose is to determine through a geometrical method the workspace of a parallel manipulator taking into account the limitations on the articular coordinates, the mechanical limits of the joints and the possible interference between the links. We show that we can determine planar cross sections of this workspace when the end-effector moves in a plane and its orientation is constant.

1 Introduction

Let us consider the parallel manipulator described in figure 1. Two plates are connected through 6 articulated

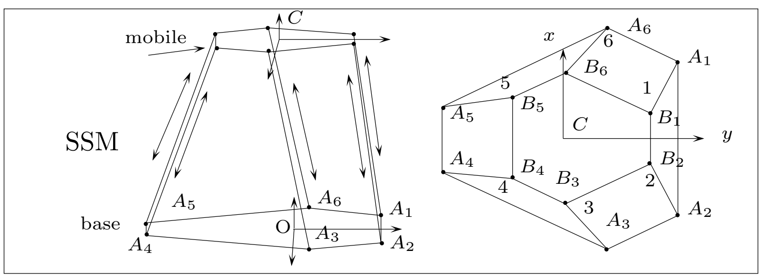

Figure 1: A parallel manipulator with 6 D.O.F.

links in which a linear actuator enables to change the links lengths. By controlling these lengths we are able to control the position and orientation of the upper-plate with respect to the base plate. Usually the joints linking the links to the plates are universal joint for the base plate and ball-and-socket joint for the mobile plate. One of the main criteria for designing a parallel manipulator is clearly its workspace. This workspace is limited by the range of the linear actuators, the mechanical limits on the joints and the possible interference between the links. Unfortunately the cartesian coordinates are coupled for the workspace. For example the translatory limits are dependent from the orientation of the mobile plate. To overcome this difficulty it is usual to represent this workspace for a constant orientation and the resulting 3D workspace is determined by calculating slices of the workspace obtained by assuming that the center C, which coordinates in some reference frame are (x_c, y_c, z_c), of the mobile plate moves in a given plane. The reference frame is chosen such that the position of C in the plane is defined by x_c, y_c. For example the slices are calculated when the end-effector moves in a horizontal plane and a discretisation method enable to display the non-empty slices for various altitudes of the end-effector.

2 Previous works

Many researcher have addressed the problem of determinating the border of the workspace slices. We have described a method for this problem [5] which uses a discretisation procedure in the plane of the slice. A central point is chosen and polar coordinates (r, θ) are used. For a given θ the position of the border is determinated by increasing the value of r, calculating the links lengths (it is easy to calculate the lengths as soon as the cartesian coordinates are known), until the constraints on the links lengths are not satisfied. Then the value of θ is increased. Similar methods have been developed [2], some of them taking into account the mechanical limits on the joints [4], or every constraints [1].

Unfortunately this kind of method is time-consuming due to the large computation involved. Furthermore we will see that the slice may have "holes" and therefore one has to deal very carefully with the discretisation on the polar coordinates. Another drawback is that the result of the method is a large list of points and is therefore not really easy to store and manipulate.

A completely different approach has been proposed by C. Gosselin [3] for determining the border of the workspace due to the limited range of the linear actuators. If we consider one mobile-plate joint center B_i the links lengths constraints imply that B_i lie between two spheres S_{i_i}, S_{e_i} whose center is the base-plate joint center A_i and whose radii are the minimum and maximum links lengths (figure 2). As the end-effector center C moves in a plane and

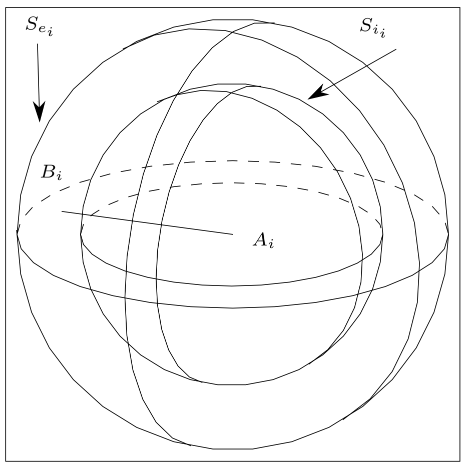

Figure 2: The links lengths constraints imply point B_i lie between two spheres.

the orientation is constant, B_i moves also in a plane and must therefore lie inside the intersection of this plane with the spheres i.e. a region C_{c_i} which is either a circle or the region between two circles. As the orientation is constant by translating C_{c_i} by the constant vector $\mathbf{B_iC}$ we get a region such that if C lie in this region the links length of link

i will be between its minimum and maximum value. By doing that for the 6 links we get 6 regions and the workspace is clearly the intersection of these regions. Therefore the workspace border is constituted by a list of arc of circle. Figure 3 shows an example (for clarity only three regions have been used). This method is very fast, exact and en-

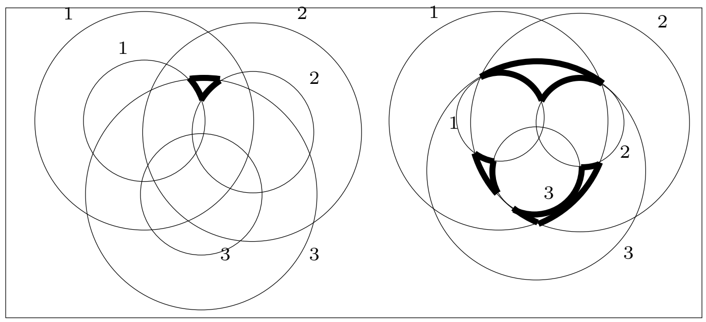

Figure 3: In thick line the border of the workspace obtained by intersecting three annular regions.

ables to calculate easily the area of the workspace. The storage of the representation of the border is efficient as few data are needed.

Unfortunately this method does not take into account the mechanical limits on the joints. We propose now an extension which will enable to consider these constraints. For this purpose we suppose that the mechanical limits of a joint can be described through the definition of a pyramid which apex is the joint center and which faces are such that if the joint constraints are satisfied the link will be in the interior of the pyramid as shown in figure 4 a). This pyramid may be defined by the normal to its faces.

Let us suppose we have defined such pyramids for the base-plate joints. If the joint limits for link i are satisfied then point B_i will lie inside the intersection of the pyramid with the plane in which lie B_i, i.e. in a polygon P_{b_i}. We may also define a pyramid $\mathcal{P}$ for the mobile-plate joint which center is B_i and such that if the limits on the joints are satisfied then point A_i will lie inside the pyramid. For symmetry reason we may calculate from $\mathcal{P}$ an equivalent pyramid $\mathcal{P}'$ which apex is A_i such that if A_i is inside $\mathcal{P}$ then B_i is inside $\mathcal{P}'$ (figure 4 b)). If the mobile-plate joint limits for link i are satisfied then point B_i will lie inside the intersection of the equivalent pyramid with the plane in which lie B_i, i.e. in a polygon P_{m_i}. Therefore if for link i the length and joints limits are satisfied point B_i will lie inside the intersection of $C_{c_i}, P_{b_i}, P_{m_i}$, a region which border is a generalized polygon i.e. a polygon whose edges may be arc of circle and which may have forbidden regions whose borders are also generalized polygons. We have designed an algorithm to calculate the intersection of such geometrical object [6] and the resulting region defines an allowable zone for B_i. As previously we calculate the allowable zones for the 6 B_i, translate them by the constant vector $\mathbf{B_iC}$ and obtain 6 allowable zones for C. The workspace is then the intersection of these 6 regions and our algorithm can be applied to find its border.

3 Dealing with links interference

Using the above method we have calculated the border of the workspace $\mathcal{W}$. Suppose now we want also to check the link interference. For that purpose we introduce a safety distance l and we try to find the zone in the workspace such that the distance between two links is l. First we write that the distance between the lines associated to the links is equal to l which yield to a constraint equation on

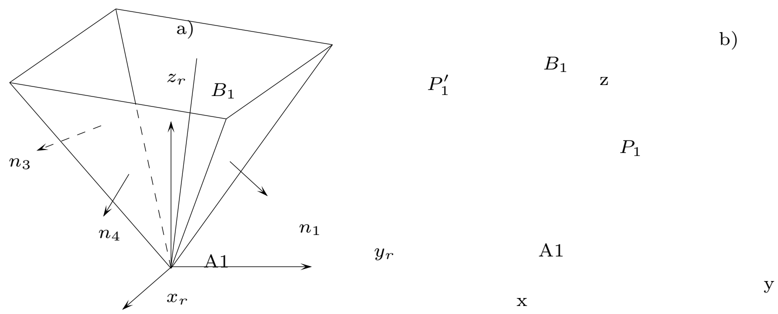

Figure 4: In a) the joint limits are defined through a pyramid which apex is the center of the joint and which faces are such that if the joints satisfy the limits the link will be in the interior of the pyramid. In b) A pyramid for the mobile-plate joint and its equivalent pyramid.

the variable x_c, y_c which is:

$$a_1 x_c^2 + a_2 y_c^2 + a_3 x_c y_c + a_4 x_c + a_5 y_c + a_6 = 0 \qquad (1)$$

where the a_i are constant. Therefore if the distance between the lines is l then C moves along a conic $\mathcal{C}$, called the *safety conic* in its plane. If $\mathcal{C}$ does not intersect the workspace $\mathcal{W}$ then the distance between the links is either always smaller than l or always greater than l and any point in the workspace can be used to verify in which case we are. If the distance is greater then for any point in $\mathcal{W}$ the distance between the links is always greater than l and therefore there is no links interference. Then we have two consider two cases:

- $\mathcal{C}$ does not intersect the workspace but the distance between the lines is lower than l
- $\mathcal{C}$ intersects the workspace

Indeed the minimal distance between the link, which are only a part of the line is always greater or equal to the distance between the lines. It can be shown that the minimal distance between the links is either:

- the distance between the lines if the points of the common perpendicular to the lines lie on the link
- the minimal distance between the set of points defined by an extremity of a link and its projection point on the other link if the projection point lie on the link

If we write that the distance between an extremal point of a link to its projection on the opposite line is l we get also a constraint equation on x_c, y_c which states that C is on a conic, the *point conic*. Therefore we have four point conics (one for each of the A, B) and one safety conic. On the safety conic and on on the point conics we have *critical points*:

- on the safety conic the critical points are the position of C such that at least one of the point of the common perpendicular lying on the line is an extremal point of the link.
- on the point conics the critical points are the position of C such that the projection point is an extremal point of the opposite link.

It is clear that for any points on the conics between two critical points either the distance between the links is never the distance associated to the conics or is this distance in which case its value is l. Furthermore on these conics we may have intersection points with the workspace border. By a simple reasoning it can be shown that the critical and intersection points enable to determine the part of the workspace, if any, where there will be no link interference. The vertices of these zones are either critical or intersection points or vertices of $\mathcal{W}$ and their edges either parts of the edges of $\mathcal{W}$ or parts of the safety or point conics. An example of such zone is shown in figure 5.

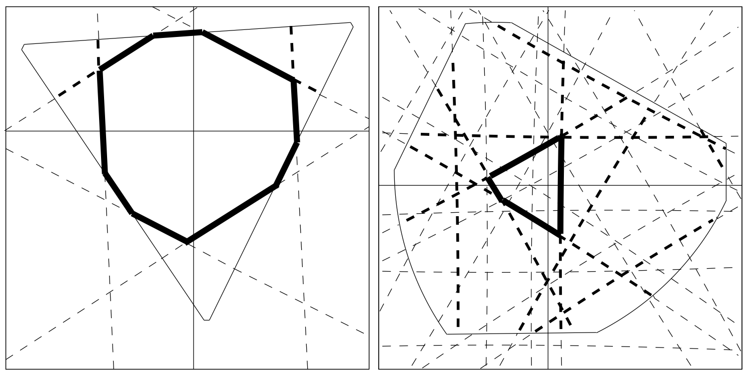

Figure 5: Zone without any links interference. $\mathcal{W}$ is shown in thin line, the safety and points conics are displayed in thin dashed lines. For each pairs of link we may calculate zones in $\mathcal{W}$ where there is no links interference (the border of these zones is displayed in thick dashed line). The intersection of these zones, (in thick line) is the zone where there is no link interference.

We will now consider with more details a particular case in which the safety distance is equal to zero: in that case the safety conic is a line. Let us consider links 1 and 2 : if the links intersect then the projection of A_1B_1 and A_2B_2 on the base plane must intersect. As C moves along the safety conic, i.e. a line D_C, B_1 and B_2 moves along lines D_{B_1}, D_{B_2} parallel to D_C. Let us introduce the half planes P_i defined by the line D_{B_i} and the point A_i: a first condition for the intersection of A_1B_1, A_2B_2 is that the half planes P_1, P_2 have a non-empty intersection. Now we define four points U_1, U_2, U_3, U_4 on line D_{B_1}. U_1 is the intersection of D_{B_1}

with the line going through A_1 which vector is $\mathbf{B_1B_2}$, U_2 is the intersection of D_{B_1} with the line going through A_2 which vector is $\mathbf{B_1B_2}$. U_3 is the intersection of D_{B_1} with the line going through A_1A_2. Let U_{41} be the intersection of the line going through A_1A_2 with D_{B_2}: U_4 is obtained by translating U_{41} by vector $\mathbf{B_2B_1}$. Among the set of points U_i let us consider the two extremal points U_j, U_k. It is possible to show that A_1B_1, A_2B_2 will intersect for any point B_1 lying on the parts of D_{B_1} starting from U_j and going through infinity or starting from U_k and going through infinity. By translating these parts by a vector $\mathbf{B_1C}$ we obtain two parts of D_C for which A_1B_1, A_2B_2 will intersect (figure 6). The intersections of these parts with the workspace, if any,

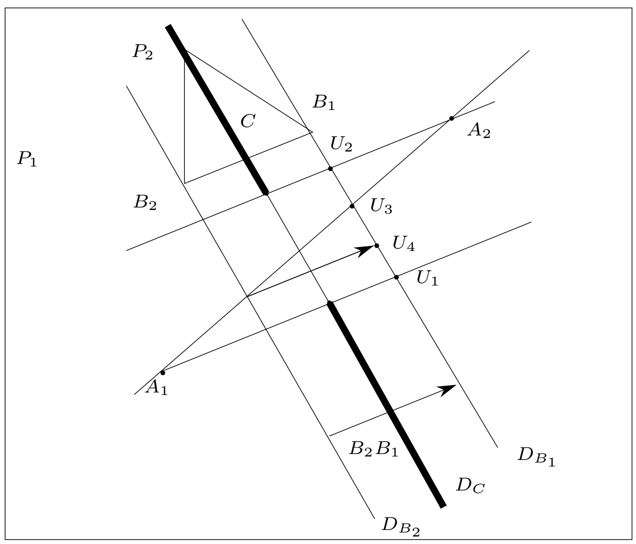

Figure 6: Definition of the points U_1, U_2, U_3, U_4. Here the extremal points are U_1, U_2 and from these points we deduce two componants of D_C (in thick line) for which A_1B_1, A_2B_2 will intersect.

define *forbidden segment* i.e. positions of C such that two links will intersect. These segments can split the workspace in two componants in which case it will not possible to go with a constant orientation from one componants to the other or the segment may define more simply a limit. In that case it will be possible to go from any point in the workspace to any other point but sometimes the direct line will not be possible (figure 7). Note that the forbidden seg-

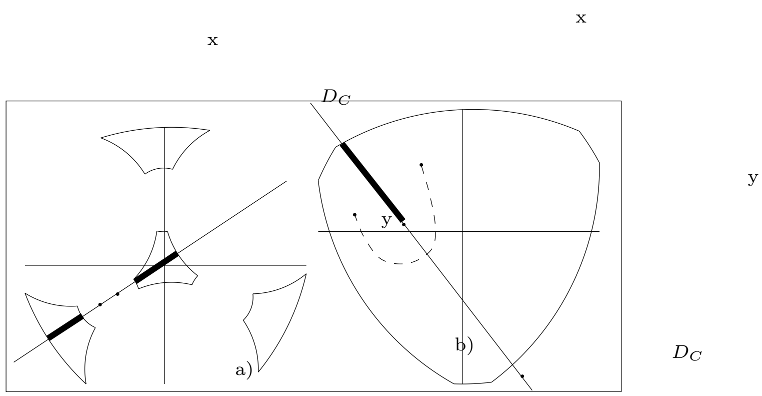

Figure 7: The forbidden segment may split the workspace in two componants (a) or define a limit (b). In the first case it will be not possible to go from one componant to the other with a constant orientation and in the second case it will not always be possible to go from a point to another by the direct path but an alternative path can be found (in dashed line).

ments does not modify the area of a slice. Therefore the area of the slice can be easily calculated according to the method described in [3] and the volume of the workspace can be deduced from the area of the slices.

4 Examples

Let us consider the manipulator described in [1] which has the following dimensions, maximum and minimum links lengths ρ_{max}, ρ_{min}:

	x	y	z
A_1	112.5	-194.856	-25
A_2	-225	0	-25
A_3	112.5	194.856	-25
A_4	67.5	-116.913	-25
A_5	-135	0	-25
A_6	67.5	116.913	-25

	x	y	z	ρ_{max}	ρ_{min}
B_1	95.263	55	-20	757	528
B_2	0	-110	-20	757	528
B_3	-95.263	55	-20	757	528
B_4	-69.282	-40	-20	733	491
B_5	0	80	-20	733	491
B_6	69.282	-40	-20	733	491

Figure 8 shows a perspective view of the workspace obtained by considering only the maximum and minimum links lengths constraints.

Figure 9 shows the constraint regions for C and the resulting slice of the workspace for the same links lengths constraints and with base joint constraints such that the link must lie inside a four-faced pyramid with an angle at center equal to 30 degrees. The influence of the joints constraints are clearly visible. Figure 10 shows a perspective view of the workspace with the same constraints.

Workspace
$\psi = 0.0, \theta = 0.0, \phi = 0.0$

Volume:33035415.272139

z first non empty slice:437.051282, last:705.000000
Constraints:links lengths

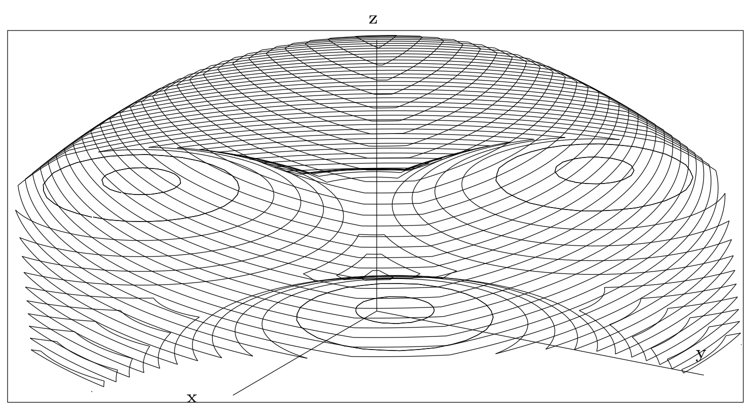

Figure 8: Perspective view of the workspace for orientation angles=0, constraints= links lengths (all dimensions are in mm)

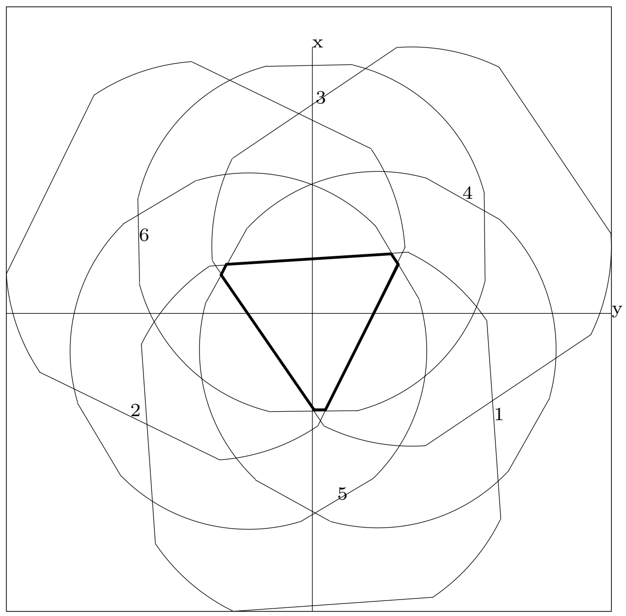

Figure 9: The constraint regions for C and the resulting slice of the workspace (in thick line) for the orientation angles=0, constraints= links lengths and 6 four-faces, 30 degrees angle at center pyramids for the base joints (all dimensions are in mm)

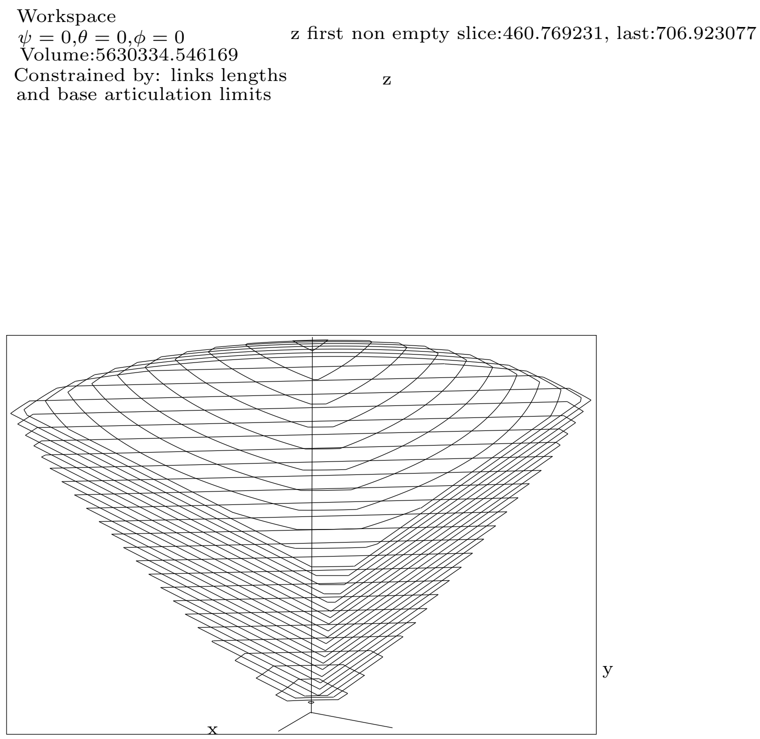

Figure 10: Perspective view of the workspace for the orientation angles=0, constraints= links lengths and 6 four-faces, 30 degrees angle at center pyramids for the base joints (all dimensions are in mm)

5 Conclusion

We have presented a geometric approach for the calculation of the workspace of parallel manipulators for which the orientation is constant. This approach enables to take into account every constraints on the manipulator: links lengths range, mechanical limit of the joints, links interference. To the best of our knowledge this problem was not solved yet except by using a numerical grid method.

Our method is exact and fast and the size of the output is in general small. Furthermore the area and volume of the workspace can be easily calculated. An extension of this method to represent the possible orientations of the mobile plate when the position of its center is fixed is now considered.

Acknowledgment Most of the ideas presented here result from discussions established during a two months stay with Dr. Arai at MEL, Tsukuba, Japan.

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