The (true) Stewart Platform has 12 configurations

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Abstract

We consider a Stewart platform and show that it forward kinematics has at most 12 solutions. A first geometrical demonstration is provided which uses the concept of circularity and in a second proof we show that this problem is equivalent to find a system of two planar parallel manipulators with each 6 solutions to the forward kinematic problem. A geometrical construction is provided to construct such a system and a Stewart Platform with 12 configurations is exhibited.

1 Introduction

In 1965 Stewart [16] describes a mechanism intended to be used as a flight simulator. This mechanism (figure 1) consists in a triangular mobile plate connected to the ground through three identical mechanisms. This mechanism is composed of a fixed vertical beam F_i on which two articulated beams are connected. These beams are in turn linked to each other and one of the beam is connected to the mobile plate by a ball-and-socket joint at the point B_i. In each of these beams a linear actuator enables to change the beam length. For a given posture of the mobile plate there is a unique length for each of the 6 beams. In the litterature most of the time the name "Stewart Platform" refers to two rigid bodies connected by 6 variable length links but most of Stewart's paper deals with the presented mechanism (hence denoted the "true" Stewart platform).

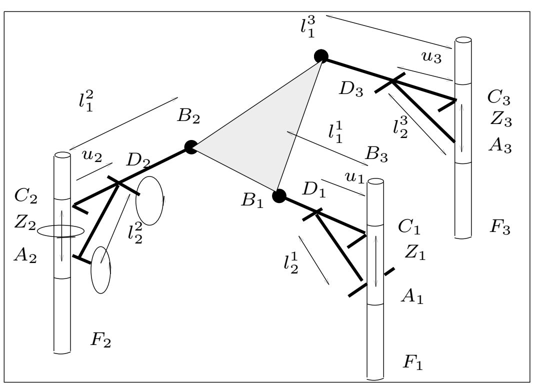

Figure 1: The Stewart Platform

The forward kinematic problem is to find the postures of the mobile plate for a fixed set of beam lengths. This problem has recently been the subject of many papers especially for the manipulator which is usually called the "Stewart Platform" (two bodies connected by 6 links with a variable length, in fact it was first proposed by Gough as it can be seen in Stewart paper). In that case it has been shown that the problem has at most 16 solutions if the mobile plate is a triangle as it can be reduced to solving a sixteen order polynomial [1], which may have effectively 16 real roots [11]. If the articulation points have different location, there will be at most 40 solutions when either the fixed or the mobile plate is planar [9] and also for the most general manipulator [8],[15]. The forward kinematic problem has been solved for many others parallel manipulators [2],[4],[6],[7],[10],[13],[14],[17]. We have shown in [12] that the analysis of Innocenti on the RRR-3S mechanism [5] can be used to find an upper-bound of the number of solutions and a polynomial for many different architectures of parallel manipulators. It was shown that for the Stewart Platform this upper-bound was 16. We will show in this paper that the real upper-bound is 12 except in the degenerate case where there are an infinity of solutions.

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2 Maximum number of solutions

2.1 First approach

Let denote B_1, B_2, B_3 the three centers of the ball-and-socket joints on the mobile plate and l_1^i, l_2^i the lengths of the beams for the mechanism with B_i as extremal point. As these lengths are fixed B_i must lie on an horizontal circle $\mathcal{C}_i$ whose center belongs to F_i. Therefore we may substitute the mechanism by a single link which can rotate around the vertical axis. From the forward kinematic viewpoint the Stewart Platform is therefore equivalent to the mechanism presented in figure 2. Let A_i, C_i denote the articulation points of the beams on F_i, Z_i the distance between these points, D_i the connection point between the beams, and u_i the fixed distance between D_i, C_i. Let z_{A_i} denotes the coordinates of A_i along the vertical axis, z_i the coordinate of the center of $\mathcal{C}_i$ along this axis and r_i the radius of $\mathcal{C}_i$. We get:

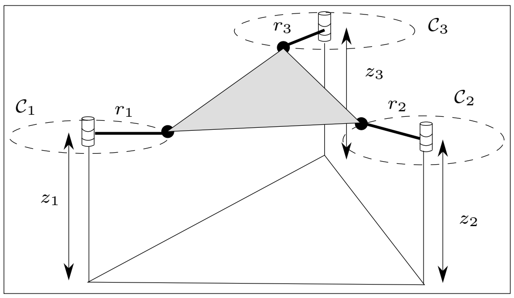

Figure 2: The equivalent mechanism of the Stewart Platform

$$z_i = z_{A_i} + Z_i + l_1^i \frac{(l_2^i)^2 - Z_i^2 - u_i^2}{2Z_i u_i} \tag{1}$$

$$r_i^2 = (l_1^i)^2 \frac{(Z_i + u_i)^2 - (l_2^i)^2}{2Z_i u_i} \tag{2}$$

Now assume that we disconnect B_1. This point will be now the coupler point of a RR-2S mechanism. For any valid solution of the forward kinematic B_1 will therefore be an intersection point of the coupler surface of the RR-2S mechanism and the circle C_1.

We use now a theorem of Cayley [3]:

Two points C, D a fixed distance apart on a movable line are constrained to lie respectively on two planar algebraic curve of order n_c, n_d and circularity p_c, p_d that lie on parallel planes. Then the line generates a ruled surface of degree $2n_c(n_d - p_d) + 2n_d(n_c - p_c) - 2p_c p_d$.

For our RR-2S mechanism we apply this theorem on the line going through B_2, B_3. We have $n_c = n_d = 2, p_c = p_d = 1$ and therefore B_2, B_3 lie on a ruled surface of order 6. Therefore B_1 lie on a surface of order 12, circularity 6 which has at most 12 real intersection points with a circle. Therefore the forward kinematic problem has at most 12 solutions except in the degenerate case where there are an infinity of solutions.

2.2 Second approach

Let us denote Π_1, Π_2, Π_3 the three planes containing the circles $\mathcal{C}_i$. Let us consider U_3 the projection of B_3 on Π_1 and U_2 the projection of B_2 on Π_1 (figure 3). As B_3 moves on $\mathcal{C}_3$ U_3 will move on a similar circle in Π_1. Let us consider now the triangle $B_1B_3U_3$: the length of its edge B_1B_3 is constant and the length of its edge U_3B_3 is also constant and equal to the distance between the plane Π_1, Π_2. As the edges U_3B_3, B_1U_3 form a right angle we deduce that the length $||U_3B_1||$ is also fixed. In a similar manner for the triangle $B_2B_1U_2$ the length $||U_2B_1||$ is fixed and the length $||U_2U_3||$ is constant. Therefore we get in the plane Π_1 a triangle $U_3U_2B_1$ whose edges have a constant length and whose vertices are connected to fixed points through three links of length r_1, r_2, r_3 which can rotate around the normal to the plane: we get a planar parallel manipulator. Consequently any solution of the forward kinematic for this manipulator yield to a solution for the forward kinematic of the Stewart Platform.

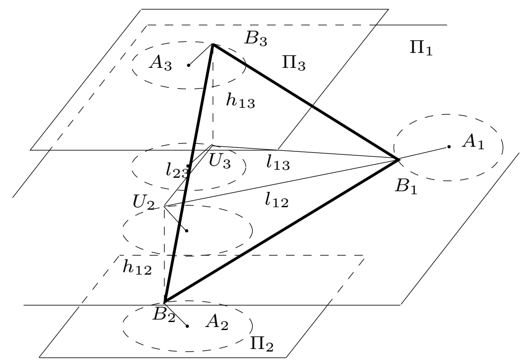

Figure 3: The planar parallel manipulator equivalent to the Stewart Platform

Let h_{1i} denotes the distance the plane Π_i and the plane Π_1, l_{ij} the length of the edge U_iU_j (with $U_1 = B_1$) of the triangle $U_3U_2U_1$ and d_{ij} the distance between the points B_iB_j. We have:

$$l_{13}^2 = d_{13}^2 - h_{13}^2 \tag{3}$$

$$l_{12}^2 = d_{12}^2 - h_{12}^2 \quad (4)$$
$$l_{23}^2 = d_{23}^2 - (h_{12} + h_{13})^2 \quad (5)$$

Let us notice now that we have shown that the edges of the mobile plate of the planar parallel manipulator have a fixed length. But there exist two different triangles which fulfill this condition : the original triangle and its mirror image. Therefore we get two different planar mechanisms and for each forward kinematic solution of one of the mechanism we get a solution for the Stewart Platform. Each of this mechanism is a planar parallel manipulators and we will call the manipulator whose mobile plate is the mirror image of the initial one its *mirror manipulator*. A *system of planar parallel manipulators* will denote a planar parallel manipulator and its mirror manipulator.

It is well known that the forward kinematic of a planar parallel manipulator may have up to 6 solutions [2]. Consequently there will be up to 12 solutions for the forward kinematic of the Stewart Platform.

Corollary: A Stewart Platform with 12 solutions will be obtained if and only if we exhibit a system of planar parallel manipulators with 12 solutions. Each of the manipulator in the system will have therefore a maximum of solutions except in the degenerate case where there are an infinity of solutions.

3 Computing the solution

Innocenti has shown that the direct kinematic problem of any RRR-3S mechanism can be reduced to the analysis of a sixteen order polynomial [5]. As the equivalent mechanism of a Stewart Platform is an RRR-3S mechanism its analysis can be applied here. This approach has been implemented and an intensive numerical investigation has enabled us to find Stewart Platforms with a maximum of 8 solutions (therefore not the expected maximum number).

In fact using the planar correspondence developed in the previous section it is possible to determine a 12th order polynomial. Let us consider a planar parallel manipulator (figure 4). The coordinates of the fixed articulation points A, C, F are:

$$A : (0, 0) \qquad C : (c_2, 0) \qquad F : (c_3, d_3)$$

The inverse kinematic equations are:

$$\rho_1^2 = x^2 + y^2 \quad (6)$$
$$\rho_2^2 = (x + l_2 \cos\Phi - c_2)^2 + (y + l_2 \sin\Phi)^2 \quad (7)$$
$$\rho_3^2 = (x + l_3 \cos(\Phi+\theta) - c_3)^2 + (y + l_3 \sin(\Phi+\theta) - d_3)^2 \quad (8)$$

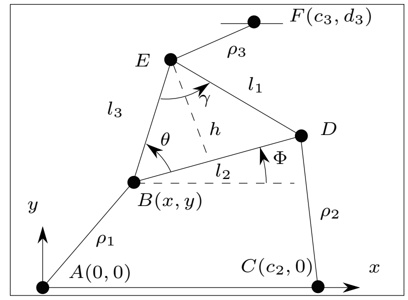

Figure 4: A planar parallel manipulator and its notation

These equations can be written as:

$$\rho_1^2 = x^2 + y^2 \quad (9)$$
$$\rho_2^2 - \rho_1^2 = Rx + Sy + Q \quad (10)$$
$$\rho_3^2 - \rho_1^2 = Ux + Vy + W \quad (11)$$

Equations (10-11) are linear in x, y. By solving this linear system and using the value of x, y in equation (9) we get an equation in the unknown $\cos\Phi, \sin\Phi$. We use then the classical change of variable :

$$T = \tan(\frac{\Phi}{2}) \qquad \cos(\Phi) = \frac{1 - T^2}{1 + T^2} \qquad \sin(\Phi) = \frac{2T}{1 + T^2}$$

to transform this equation into a sixth order polynomial P in T. In our problem only the lengths of the edges of the mobile plate are fixed. We have:

$$\cos(\theta) = \frac{l_2^2 + l_3^2 - l_1^2}{2 l_2 l_3} \quad (12)$$

Using this equation P can be written as:

$$P = a\sin(\theta) + b = 0 \quad (13)$$

where a, b are sixth order polynomials in T which does not contain any term in θ. Therefore the polynomial for the mirror parallel manipulator can be written as:

$$P_m = -a\sin(\theta) + b = 0 \quad (14)$$

and the polynomial P_s for the Stewart platform is defined by the product of P, P_m:

$$P_s = b^2 - a^2\sin^2(\theta) = 0 \quad (15)$$

which is a 12th order polynomial in T, obtained as the product of two sixth order polynomials whose coefficients are rational functions of the parameters. The roots of this polynomial define the solutions of the forward kinematic of the planar parallel manipulator and its mirror manipulator and therefore the solution of the forward kinematic of the corresponding Stewart Platform. Consequently there will be at most 12 solutions for the direct kinematic of the Stewart Platform.

4 A system of planar manipulators with 12 solutions

Various numerical investigation on the polynomial P_s has failed to produce more than 8 real solutions. To determine if a Stewart Platform may have 12 real different postures we have then decided to investigate the system of planar parallel manipulators. We have shown that the coupler curve described by E for the four-bar mechanism $ABDC$ is symmetric with respect to the line AC to the coupler curve described by E for the mirror four-bar mechanism. We have then be able to discover a geometrical construction which yield to systems of manipulators which admit 12 real solutions (this is confirmed by the fact that for this manipulators the polynomial P_s has 12 real solutions).

We consider the planar manipulator such that:

$$A = (-a, 0) \qquad C = (a, 0)$$

The mobile plate is an isoscele triangle with:

$$|BE| = |DE| = u \qquad |BD| = 2a$$

with the value of its height h small. The length of the links will be such that:

$$|AB| = |CD| = r$$

The coupler curve described by point E for the four bar mechanism and the mirror mechanism is composed of a circle of radius r whose center is located at $C_1^s = (0, \pm h)$ and a fourth degree curve C_4. Indeed the coupler curve equation can be written as:

$$\begin{aligned}&\left(y^2 + h^2 - r^2 \mp 2hy + x^2\right)(h^4 + 3h^4C^2 \mp 8h^3yC^2 \\ &+6y^2h^2C^2 - h^2r^2 - 2h^2y^2 + r^2h^2C^2 + 2x^2C^2h^2 \\ &-2x^2h^2 \mp 2r^2hyC^2 \pm 2r^2hy + y^4 - x^4C^2 - y^4C^2 \\ &+2x^2y^2 - r^2y^2 - 2x^2C^2y^2 + x^4 + r^2y^2C^2 \\ &-x^2r^2 + x^2C^2r^2) = 0 \end{aligned} \tag{16}$$

with $C = \cos(\gamma)$. If E is located at $(0, \pm h)$, there are two positions for the bar BD, showing that the center D_d of C_1 is a double point of C_4, the only one if h is small enough. Furthermore C_4 passes through the points $A_1 = (-r, \pm h)$, $A_2 = (r, \pm h)$ which are the points of C_1 which have extremal coordinates on the x axis. Therefore C_4 looks like an ∞ sign. Now consider a circle $\mathcal{C}_i$ centered at $F = (0, c)$ ($c \geq 0$, by symmetry) and of radius r_3. If $r_3 > c + h$ and $r_3^2 < r^2 + (h - c)^2$ (this implies $hc < r^2/4$), then $\mathcal{C}_i$ contains D_d and does not contains A_1 and A_2. It follows that $\mathcal{C}_i$ has four intersection points with C_4 and two with C_1 if $r_3 > r - |c - h|$. A similar reasoning for the coupler curve of the mirror mechanism can be made. Therefore we have discovered a system with 12 intersection points, from which we may deduce a Stewart Platform with 12 configurations.

Let us give an example. We have chosen:

$$a = 10 \qquad r = 8 \quad u = 12 \quad \cos(\gamma) = 0.92128466$$

$$r_3 = 9 \quad c = 0.5$$

This system is presented in figure 5 where the coupler curves of the four-bar mechanisms are shown together with the circle whose center is F and the radius r_3. It may be seen that we get effectively 12 intersection points. We are able to deduce from this system a Stewart Platform with 12 configurations. First we define the circles $\mathcal{C}_i$ by the coordinates of their centers C_i^c and their radii r_i:

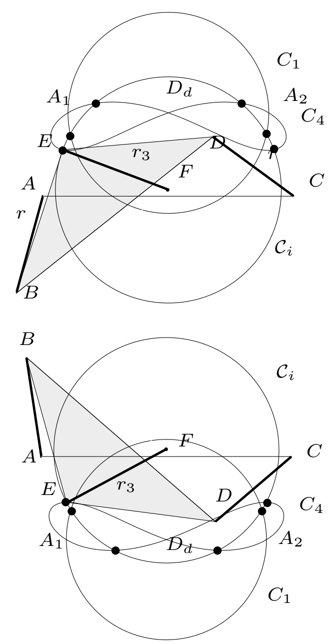

Figure 5: A system of planar parallel manipulators with 12 solutions. The intersection points of the coupler curves with the circle $\mathcal{C}_i$ are the small black circles.

$$C_1^c = (-10, 0, 2) \quad C_2^c = (10, 0, 2.5) \quad C_3^c = (0, 0.5, 3)$$

from which we get $z_1 = 2, z_2 = 2.5, z_3 = 3$.

$$r_1 = r_2 = 8 \quad r_3 = 9$$

The coordinates of A_i, C_i are:

$$A_1 = (-10, 0, 0) \;\; A_2 = (10, 0, 0) \;\; A_3 = (0, 0.5, 0)$$

$$C_1 = (-10, 0, 2) \;\; C_2 = (10, 0, 2) \;\; C_3 = (0, 0.5, 2)$$

from which we may deduce the Z_i's. The values of u_i are $u_1 = u_2 = u_3 = 2$. From equations (1)(2) we are able to deduce l_1^i, l_2^i for which we will get 12 solutions:

$$l_1^1 = 8 \quad l_2^1 = 2.236068 \;\; l_1^2 = 8.015610$$

$$l_2^2 = 2.291182 \;\; l_1^3 = 9.055385 \;\; l_2^3 = 2.332751$$

The distances between the points B_i are:

$$d_{12} = 20.006249 \;\; d_{13} = 12.041595 \;\; d_{23} = 12.093387$$

Among the two solutions for the locations of the B_i we have used:

$$B_1 = (-10, -2.25, 0) \qquad B_2 = (10.013, -2.25, 0)$$

$$B_3 = (-0.0208, 4.45, 0)$$

The corresponding Stewart Platform is shown in figure 6.

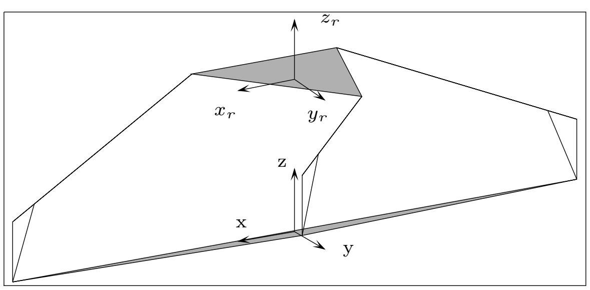

Figure 6: A Stewart Platform with 12 solutions

Using the polynomial (15) we have been able to find the 12 postures defined in Table 1. The position of the Stewart Platform is defined by the location of the barycenter of the B_i's and its orientation is defined by the three Euler's angles ψ, θ, ϕ. The various postures are shown in figures 7,8.

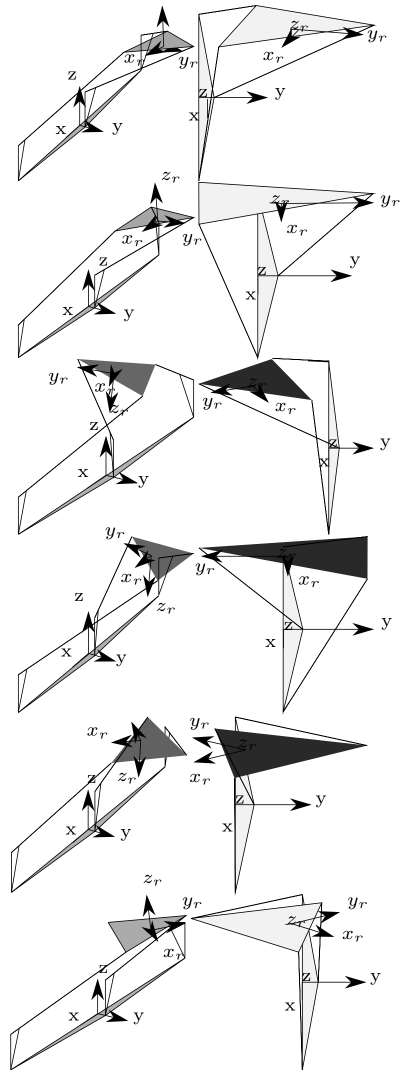

Figure 7: 6 solutions for the direct kinematic of the Stewart Platform (perspective and top view).

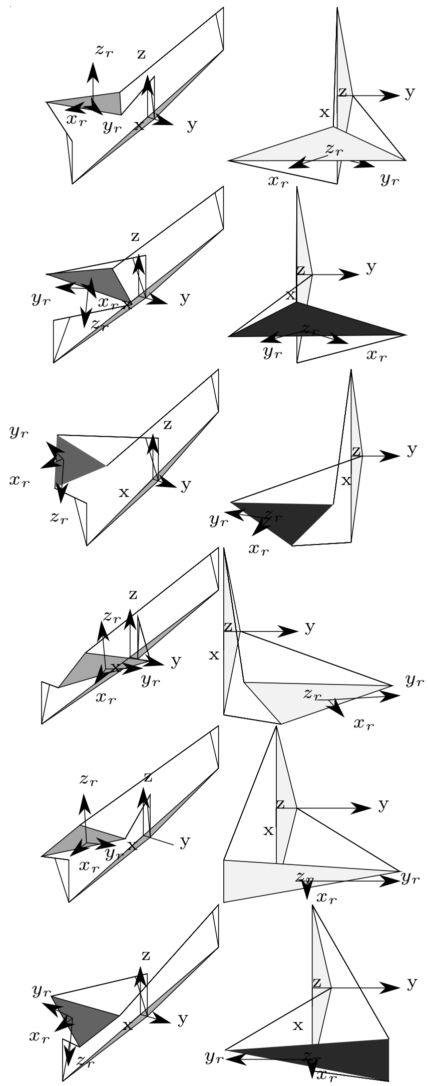

Figure 8: 6 solutions for the direct kinematic of the Stewart Platform (perspective and top view).

x	y	z	ψ	θ	ϕ
-8.0959	3.2199	2.558	-26.5534	7.247	12.875
-8.8645	0.5703	2.558	-12.7200	7.247	12.875
-7.2081	-3.6272	2.558	34.5548	172.753	12.875
-8.7042	0.1256	2.558	12.7368	172.753	12.875
-6.2860	0.2857	2.558	-30.7496	172.753	12.875
-6.6882	-0.3071	2.558	28.6821	7.247	12.875
6.7438	-0.2812	2.558	-53.7772	7.247	12.875
6.3237	0.2640	2.558	55.9921	172.753	12.875
7.2918	-3.5705	2.558	-8.0739	172.753	12.875
8.1625	3.1372	2.558	-0.0932	7.247	12.875
8.8553	0.7365	2.558	-12.7702	7.247	12.875
8.7024	-0.0124	2.558	12.7716	172.753	12.875

Table 1: The 12 postures for the Stewart Platform

5 Conclusion

We have presented in this paper various methods to show that the forward kinematics of the Stewart Platform have a maximum of 12 solutions. A polynomial of order 12 enabling to compute the solutions has been exhibited. We have demonstrated that the study of the forward kinematics can be reduced to the study of the forward kinematics of a system of two planar parallel manipulators and have presented an example with 12 solutions.

References

[1] Charentus S., Diaz C, and Renaud M. Modular serial parallel redundant robot. In *IMACS*, Cetraro, Italie, September, 18-21, 1988.

[2] Gosselin C., Sefrioui J., and Richard M.J. Solution polynomiale au problème de la cinématique directe des manipulateurs parallèles plans à 3 degrés de liberté. *Mechanism and Machine Theory*, 27(2):107–119, March 1992.

[3] Hunt K.H. *Kinematic geometry of mechanisms.* Clarendon Press, 1978.

[4] Hunt K.H. and Primrose E.J.F. Assembly configurations of some in-parallel actuated manipulators. *Mechanism and Machine Theory*, 28(1):31–42, January 1993.

[5] Innocenti C. and Parenti-Castelli V. Direct position analysis of the Stewart platform mechanism. *Mechanism and Machine Theory*, 25(6):611–621, 1990.

[6] Innocenti C. and Parenti-Castelli V. A novel numerical approach to the closure of the 6-6 Stewart

platform mechanism. In *ICAR*, pages 851–855, Pise, June, 19-22, 1991.

[7] Innocenti C. and Parenti-Castelli V. Echelon form solution of direct kinematics for the general fully-parallel spherical wrist. *Mechanism and Machine Theory*, 28(4):553–561, July 1993.

[8] Lazard D. On the representation of rigid-body motions and its application to generalized platform manipulators. In J. Angeles P. Kovacs, G. Hommel, editor, *Computational Kinematics*, pages 175–182. Kluwer, 1993.

[9] Lazard D. Stewart platform and Gröbner basis. In *ARK*, pages 136–142, Ferrare, September, 7-9, 1992.

[10] Lin W., Duffy J., and Griffis M. Forward displacement analysis of the 4-4 Stewart platform. In *ASME Proc. of the the 21th Biennial Mechanisms Conf.*, pages 263–269, Chicago, September, 16-19, 1990.

[11] Merlet J-P. Manipulateurs parallèles, 4eme partie : mode d'assemblage et cinématique directe sous forme polynomiale. Research Report 1135, INRIA, December 1989.

[12] Merlet J-P. Direct kinematics and assembly modes of parallel manipulators. *International Journal of Robotics Research*, 11(2):150–162, April 1992.

[13] Nair P. On the kinematics geometry of parallel robot manipulators. Master's thesis, Université du Maryland, College Park, 1992.

[14] Parenti-Castelli V. and Innocenti C. Forward displacement analysis of parallel mechanisms: closed-form solution of PRR-3S and PPR-3S structures. *ASME J. of Mechanical Design*, 114:68–73, March 1992.

[15] Ronga F. and Vust T. Stewart platforms without computer?, 1992. Preprint.

[16] Stewart D. A platform with 6 degrees of freedom. *Proc. of the Institution of mechanical engineers*, 180(Part 1, 15):371–386, 1965.

[17] Zhang C-D and Song S.M. Forward kinematics of a class of parallel (Stewart) platforms with closed-form solutions. In *IEEE Int. Conf. on Robotics and Automation*, pages 2676–2681, Sacramento, April, 11-14, 1991.