# Kochen-Specker Vectors

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**Abstract.** We give a constructive and exhaustive definition of Kochen-Specker (KS) vectors in a Hilbert space of any dimension as well as of all the remaining vectors of the space. KS vectors are elements of any set of orthonormal states, i.e., vectors in $n$-dim Hilbert space, $\mathcal{H}^n$, $n \geq 3$ to which it is impossible to assign 1s and 0s in such a way that no two mutually orthogonal vectors from the set are both assigned 1 and that not all mutually orthogonal vectors are assigned 0. Our constructive definition of such KS vectors is based on algorithms that generate MMP diagrams corresponding to blocks of orthogonal vectors in $\mathbb{R}^n$, on algorithms that single out those diagrams on which algebraic `0-1` states cannot be defined, and on algorithms that solve nonlinear equations describing the orthogonalities of the vectors by means of statistically polynomially complex interval analysis and self-teaching programs. The algorithms are limited neither by the number of dimensions nor by the number of vectors. To demonstrate the power of the algorithms, all 4-dim KS vector systems containing up to 24 vectors were generated and described, all 3-dim vector systems containing up to 30 vectors were scanned, and several general properties of KS vectors were found.

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## 1. Introduction

Recently proposed experimental tests of the Kochen-Specker (KS) theorem [1, 2], skepticism on the feasibility of such experiments [3 - 7], positive experiments recently carried out [8], and recent theoretical elaborations on the theorem [9 - 22] prompted a renewed interest in the KS theorem.

The KS theorem proves that there is a set of measurements that can be carried out on a finite dimensional quantum system in such a way that if one assumed that the values of measured observables are completely independent of all other observables that can be measured on the same system, then one would run into a contradiction. Hence, a quantum system cannot posses a definite value of a measurable property prior to measurement, and quantum measurements (essentially detector clicks) carried out on quantum systems cannot be ascribed predetermined values (say 0 and 1). To arrive at the claim, one considers an orthonormal set of states $\{\psi_1, \ldots, \psi_n\}$, i.e., vectors in $n$-dim Hilbert space, $\mathcal{H}^n$, $n \geq 3$. Projectors onto these states satisfy: $\sum_{i=1}^n P_i = I$, where $P_i = \psi_i\psi_i^\dagger$. Now, Kochen and Specker proved [23] that there is no function $f : \mathcal{H} \to \mathbb{R}$ satisfying the Sum Rule $\sum_{i=1}^n f(P_i) = f(\sum_{i=1}^n P_i) = f(I)$ for all sets of projectors $P_i$. Hence, there is at least one set of projectors $\{P_i, P_i', \ldots\}$ and the corresponding set of vectors $\{\psi_i, \psi_i', \ldots\}$ for which the Sum Rule is not satisfied. Choosing $f(P_i) \in \{0, 1\}$ ($f(I) = 1$), the theorem amounts to the following claim: In $\mathcal{H}^n$, $n \geq 3$, it is impossible to assign 1s and 0s to all vectors from such a set—which we call a *KS set*—in such a way that [24]:

(i) No two orthogonal vectors are both assigned the value 1;

(ii) In any subset of $n$ mutually orthogonal vectors, not all of the vectors are assigned the value 0.

All the vectors from a KS set, as defined above, we call *KS vectors*. KS vectors in each KS set form subsets of $n$ mutually orthogonal vectors. We arrive at one subset from another by a series of rotation in 2-dim planes around ($n$–2)-dim subspaces as explained in Sec. 4. Thus, any two subsets share at least one vector which is orthogonal to all other vectors in both subsets and in an $n$-dim space, two subsets can share up to $n$–2 vectors. The KS vectors correspond to the directions of the quantisation axes of the measured eigenstates within experiments which have no classical counterparts, and when we speak of finding KS vectors we mean finding these directions. We stress here that it is not our aim to give yet another proof of the KS theorem but to determine the class of all KS vectors from an arbitrary $\mathcal{H}^n$ as well as the class of all *non-KS vectors*, i.e., vectors from the remaining sets of vectors from $\mathcal{H}^n$. By the class of non-KS vectors we mean vectors that allow `0-1` states and that correspond to the directions of the quantisation axes of the measured eigenstates within experiments which do have classical counterparts and when we speak of finding non-KS vectors we mean finding the latter directions.

The original KS theorem [23] made use of 192 (claimed 117) 3-dim vectors. Subsequent attempts to reduce the number of vectors gave the following minimal results (usually called *records*): Bub's system contains 49 vectors (claimed 33) [25], Conway-Kochen's has 51 (claimed 31) [26, p. 114], and Peres' system has 57 (claimed 33) [27] 3-dim vectors † ; Kernaghan's system contains 20 4-dim vectors with the smallest loops (see the definition below) of size two [29]; Cabello's system has 18 4-dim vectors with the smallest loops of size three [30], etc. Reducing the number of vectors is important for devising experimental setups [16], especially so as recently a single qubit KS scheme was formulated [9] by means of auxiliary quantum systems (ancillas) of the measuring apparatus and subsequently connected with the original KS formulation [10]. On the other hand, knowing the class of all KS vectors is important for better theoretical insight into quantum theory and possibly designing quantum computers. However, no general method for constructing sets of KS vectors has been proposed so far and the aim of this paper is to give one. In doing so we will follow the ideas put forward in [31, 32, 33].

So far, KS vectors have been constructed either by means of partial Boolean algebras and orthomodular lattices [23, 34, 38, 39], by direct experimental proposals [1, 2, 16], or by combining rays in $\mathbb{R}^n$ [25, 27, 29]. These approaches have two disadvantages: first, they depend on human ingenuity to find ever new examples and "records," and second, their complexity grows exponentially with increasing numbers of dimensions and vectors. For example, lattices of orthogonal $n$-tuples have $2^n$ elements (Hasse diagrams) [40] and, on the other hand, the complexity of nonlinear equations describing combinations of orthogonalities also grows exponentially.

As opposed to this, we are able to give algorithms for generation of all the equations that have KS vectors as their solutions and to effectively solve them (up to a reasonably chosen number of vectors and dimensions—limited only by the speed of today's computers) in a way that is essentially of a statistically polynomial complexity. We first recognise that a description of a discrete observable measurement (e.g., spin) in $\mathcal{H}^n$ can be rendered as a `0-1` measurement of the corresponding projector along the vector in $\mathbb{R}^n$ onto which the projector projects. Hence, we deal with orthogonal triples in $\mathbb{R}^3$, quadruples in $\mathbb{R}^4$, etc., which correspond to possible experimental designs, and to find KS vectors means finding such $n$-tuples in $\mathbb{R}^n$.

The orthogonalities of vectors within these $n$-tuples can be described by nonlinear equations of type given in Eq. (1) that have solutions. There are however billions of such nonlinear equations that have no solutions even for the smallest KS sets. And their number grows exponentially with the increase of both the number of KS vectors and the dimension of their space. So, we established a one-to-one correspondence between nonlinear equations and graphs (MMP diagrams). We can handle graphs exponentially faster than nonlinear equations but there are nevertheless billions of them. Therefore, we designed a self-teaching generation algorithm for MMP diagrams: graphs containing subgraphs that correspond to equations that cannot have a solution are not generated. This reduces the generation complexity to a statistically polynomial one

† The reasons why Kochen-Specker's, Bub's, Conway-Kochen's, and Peres' systems should be considered as 192, 49, 51, and 57 and not as 117, 33, 31, and 33 vector systems, respectively, are given in Sec. 5-(xi) in accordance with the results independently obtained by J.-Å. Larsson. [28]

and the time required for obtaining the MMP diagrams corresponding to systems of nonlinear equations with solutions from billions of years to hours and days.

To switch back from the MMP diagrams to nonlinear equations to solve them at this stage would again take quite some time. Therefore we defined the notion of an *algebraic dispersion-free state* (`0-1` state) on MMP diagrams. It turns out that only a small percentage of the obtained MMP diagrams cannot have `0-1` states. Their direct verification is again of exponential complexity. So, we developed algorithms with backtracking that discard MMP diagrams with `0-1` states and whose complexity turns out to be statistically polynomial.

The diagrams finally obtained correspond to candidate sets of nonlinear equations that contain KS sets provided the equations have real solutions. Algorithms for solving nonlinear equations, such as Gröbner basis and homotopy, are also mostly of at least exponential complexity and have been tested without success on representative systems. Therefore we designed new ones based on interval analysis and Ritt's characteristic set calculations and were able to reduce their complexity to a statistically polynomial one. This rounds up the constructive and exhaustive definition of KS sets and vectors and makes their generation feasible for reasonably chosen numbers of vectors and dimensions.

The paper is organised as follows. In Sec. 2, MMP diagrams are defined and the algorithms as well as the programs for their generation are presented. In Sec. 3, we give the algorithm and program for finding whether MMP diagrams can be assigned a set of dispersion-free `0-1` states and determining the latter sets when there is at least one set of `0-1` states together with the smallest MMP diagrams that do not allow `0-1` states. In Sec. 4, we establish a link between MMP diagrams that do not allow `0-1` states and systems of nonlinear equations whose solutions are the KS vectors. We then give the algorithms and methods for solving the equations in a statistically polynomial time. In Sec. 5, we present the new results we obtained.

## 2. MMP diagrams

We start by describing vectors as vertices (points) and orthogonalities between them as edges (lines connecting vertices), thus obtaining MMP diagrams [31, 33, 35] which are defined as follows:

1. Every vertex belongs to at least one edge;
2. Every edge contains at least 3 vertices;
3. Edges that intersect each other in $n-2$ vertices contain at least $n$ vertices;

Isomorphism-free generation of MMP diagrams follows the general principles established by [36], which we now recount briefly.

Deleting an edge from an MMP diagram, together with any vertices that lie only on that edge, yields another MPP diagram (perhaps the vacuous one with no vertices). Consequently, every MMP diagram can be constructed by starting with the vacuous diagram and adding one edge at a time, at each stage having an MMP diagram.

We can represent this process as a rooted tree whose vertices correspond to MMP diagrams whose vertices and edges have unique labels. The vacuous diagram is at the root of the tree, and for any other diagram its parent node is the diagram formed by deleting the edge with the highest label. The isomorph rejection problem is to prune this tree until it contains just one representative of each isomorphism class of diagram. This can be achieved by the application of two *rules*.

Given a diagram $D$, we can identify the valid positions to add a new edge such that Conditions 3–4 are enforced. According to the symmetries of $D$, some of these positions are equivalent. The first rule is that exactly one position in each equivalence class of positions is used; a node in the tree formed by adding an edge in any other position is deleted together with all its descendants.

To understand the second rule, consider a diagram $D'$ with at least one edge. We label the edges of $D'$ in a canonical order, which is an order independent of any previous labelling. Then we define the *major class* of edges as those that are equivalent under the symmetries of $D'$ to the edge that is last in canonical order. The second rule is: when $D'$ is constructed by adding an edge $e$ to a smaller diagram, delete $D'$ (and all its descendants) unless $e$ is in the major class of edges of $D'$.

According to the theory in [36], application of both rules together is sufficient: exactly one diagram from each isomorphism class remains in the tree. Our implementation used `nauty` [37] for computing symmetries and canonical orderings. The method allows for very efficient parallelisation of the computation. A generation tree for MMP diagrams with 9 vertices and the smallest loop of size 5 is shown in the Fig. 1.

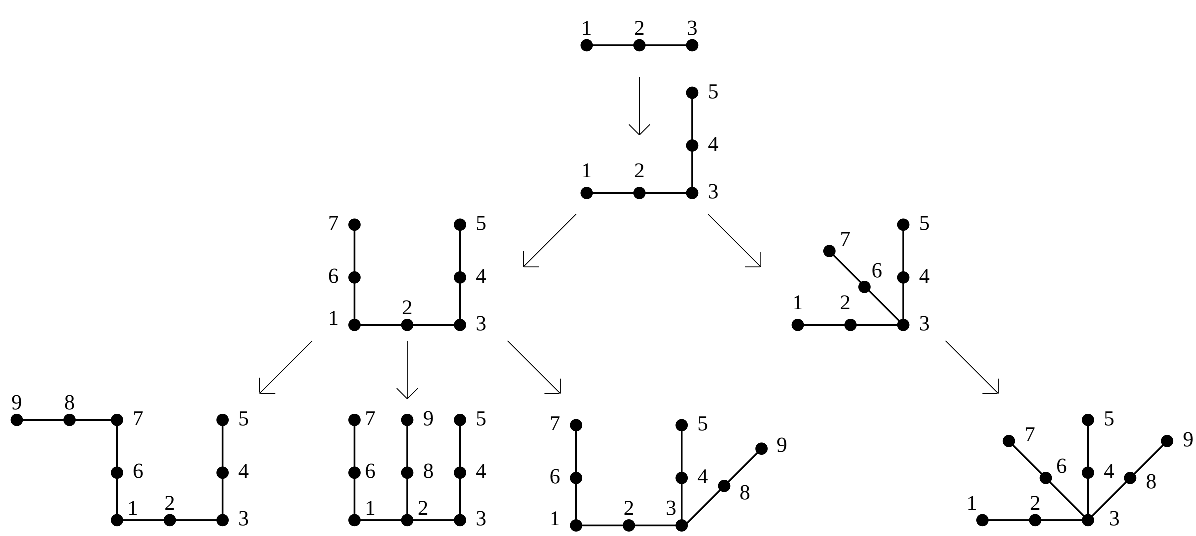

**Figure 1.** An example of a generation tree for connected MMP diagrams: 9 vertices and the smallest loop of size 5 (for 9 vertices a loop cannot be formed; the first loop appears with 10 vertices: `123,345,567,789,9A1`). Cf. [33, 35]

MMP diagrams with three vertices per edge and with smallest loops (edge polygons) of size five graphically resemble Greechie diagrams [38]. Greechie diagrams are a handy way to draw Hasse diagrams that represent orthomodular lattices. The complexity of Hasse diagrams grows exponentially with increasing dimensions and the smallest loops of the corresponding are of size 5, while MMP diagrams allow loops of size 4, 3, and 2 (2 edges share at least 2 vertices). Besides, it would be quite a challenge to find a direct lattice representation of KS vectors.

We denote vertices of MMP diagrams by `1,2,..,A,B,..a,b,..` By the above algorithm we generate MMP diagrams with chosen numbers of vertices and edges and a chosen minimal loop size. E.g., in the examples given in Fig. 2 we generate diagrams with 4 vertices within an edge and minimal loops of size 2 and 3. Our programs handle diagrams with up to 90 vertices, but this limit could easily be extended.

## 3. Algebraic states on MMP diagrams

To find diagrams that cannot be ascribed `0-1` values we apply an algorithm which we call `states01`. The algorithm is an exhaustive search of MMP diagrams with backtracking. The criterion for assigning `0-1` (dispersion-free) states is that each edge must contain exactly one vertex assigned to 1, with the others assigned to 0. As soon as a vertex on an edge is assigned a 1, all other vertices on that edge become constrained to 0, and so on. The algorithm scans the vertices in some order, trying 0 then 1, skipping vertices constrained by an earlier assignment. When no assignment becomes possible, the algorithm backtracks until all possible assignments are exhausted (no solution) or a valid assignment is found. In principle the algorithm is exponential, but because the diagrams of interest are tightly coupled, constraints build up quickly. For the range of diagram size in our study, we found that the average time per diagram appeared to grow polynomially with the diagram size.

To implement the algorithm we wrote a program that selects MMP diagrams with 3 and 4 vertices per edge on which `0-1` states cannot be defined. The smallest such diagrams are given in Fig. 2.

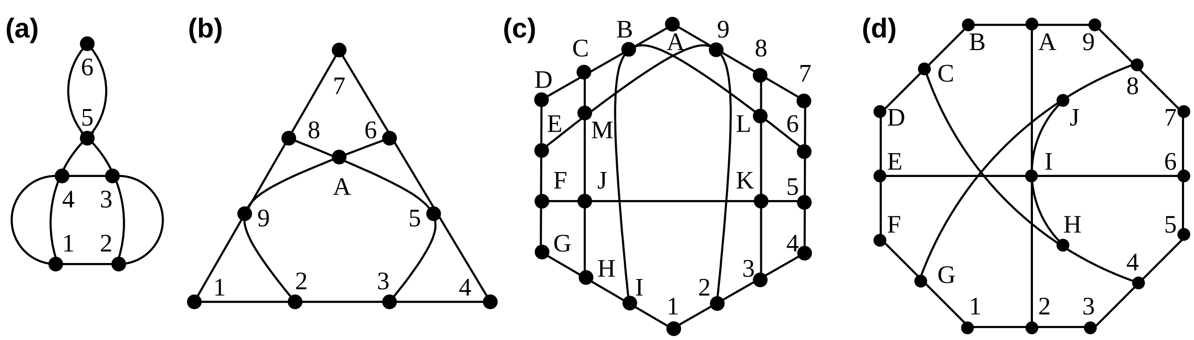


**Figure 2.** Smallest MMP diagrams without `0-1` states: (1) 4 vertices per edge: (a) loops of size 2: 6 vertices—3 edges; (b) loops of size 3: 10-5; (c) loops of size 4: 22-11; (2) 3 vertices per edge: (d) loops of size 5: one of two 19-13; the other is shown in Fig. 2 (b) of [33].

- *3 vertices per edge*

  7 vertices—5 edges (smallest loops of size 3): `123,345,561,275,476` (triangle);

  15-11 (4): `123,345,567,789,9AB,BC1,CD6,2DA,2E8,4FA,CEF` (hexagon),
  `123,345,567,789,9AB,BCD,DE1,4AE,28C,2FA,6FD` (heptagon);

  19-13 (5): `123,345,567,789,9AB,BCD,DEF,FG1,2IA,6IE,4HC,8JG,HIJ` Fig. 2 (d),
  `123,345,567,789,9AB,BCD,DE1,EI7,2F9,4GB,IJG,FJH,CH6` (heptagon);

- *4 vertices per edge*

  6-3 (smallest loops of size 2): `1234,2356,1456` Fig. 2 (a);

  10-5 (smallest loops of size 3): `1234,4567,7891,35A8,29A6` Fig. 2 (b);

  22-11 (smallest loops of size 4):
  `1234,4567,789A,ABCD,DEFG,GHI1,FJK5,HJMC,3KL8,IBL6,29ME.` Fig. 2 (c),
  `1234,4567,789A,ABCD,DEF1,FGH5,EMJ6,2GLC,3IJ8,HIKB,MLK9` (pentagon);

  38-19 (5): `1234,1567,289A,5BCD,8BEF,3GHI,6JKL,GJMN,CHOP,EMQR,OQST,RUVW,`
  `4UXY,9SZa,FIbc,KTXb,7VZc,ALPW,DNYa` (dodecagon).

Further details of the algorithm and the program `states01` will be given elsewhere. [42]

## 4. Kochen-Specker vectors

To find KS vectors we follow the idea put forward in [31, 33] and proceed so as to require that their number, i.e. the number of vertices within edges, corresponds to the dimension of $\mathbb{R}^n$ and that edges correspond to $n(n-1)/2$ equations resulting from inner products of vectors being equal to zero which means orthogonality. So, e.g., an edge of length 4, `BCDE`, represents the following 6 equations:

$$\begin{aligned}
\mathbf{a}_B \cdot \mathbf{a}_C &= a_{B1}a_{C1} + a_{B2}a_{C2} + a_{B3}a_{C3} + a_{B4}a_{C4} = 0,\\
\mathbf{a}_B \cdot \mathbf{a}_D &= a_{B1}a_{D1} + a_{B2}a_{D2} + a_{B3}a_{D3} + a_{B4}a_{D4} = 0,\\
\mathbf{a}_B \cdot \mathbf{a}_E &= a_{B1}a_{E1} + a_{B2}a_{E2} + a_{B3}a_{E3} + a_{B4}a_{E4} = 0,\\
\mathbf{a}_C \cdot \mathbf{a}_D &= a_{C1}a_{D1} + a_{C2}a_{D2} + a_{C3}a_{D3} + a_{C4}a_{D4} = 0,\\
\mathbf{a}_C \cdot \mathbf{a}_E &= a_{C1}a_{E1} + a_{C2}a_{E2} + a_{C3}a_{E3} + a_{C4}a_{E4} = 0,\\
\mathbf{a}_D \cdot \mathbf{a}_E &= a_{D1}a_{E1} + a_{D2}a_{E2} + a_{D3}a_{E3} + a_{D4}a_{E4} = 0.
\end{aligned} \tag{1}$$

Each possible combination of edges for a chosen number of vertices corresponds to a system of such nonlinear equations. A solution to systems which correspond to MMP diagrams without `0-1` states is a set of components of KS vectors we want to find. Thus the main clue to finding *all* KS vectors is the exhaustive generation of all MMP diagrams as given in Sec. 2, then picking out all those diagrams that cannot have `0-1` states as presented in Sec. 3, establishing the correspondence between the latter diagrams and the equations for the vectors as shown in Eq. (1), and finally solving the systems of the so obtained equations.

In practice, we actually merge these four stages so as to avoid generating those diagrams that cannot have a solution. ♯ For systems of equations of type given by Eq. (1) that do have solutions that do not allow `0-1` states, such solutions are KS vectors that correspond to vertices of MMP diagrams. Mutually orthogonal vectors correspond to edges, and connected edges, i.e., MMP diagrams themselves correspond to the systems of equations. For instance, in the connected edges `1234,4567` vectors `1,2,3,4` and `4,5,6,7`, are mutually orthogonal and `4567` is obtained from `1234` by 4-dim rotations (`1234` and `4567` are connected by `4`). A general 2-dim rotation is a rotation by an angle around a fixed point in the same plane. A general 3-dim rotation is a rotation in a 2-dim plane by an angle around a fixed axis perpendicular to this plane. So, we define a general 4-dim rotation as a rotation in a 2-dim plane by an angle around a fixed 2-dim plane. What is common to all these rotations is that they always take place in a 2-dim plane. Hence, we define an $n$-dim rotation as a rotation in a 2-dim plane by an angle around a fixed $(n-2)$-dim subspace [41]. This also explains the case of a smallest loop of size 2 in the 4-dim case. E.g., we arrive at `4561` from `1234` by a rotation in the 2-dim plane determined by the vectors `2,3` (and also by the vectors `5,6`) around the plane determined by the vectors `1,4`.

Finding KS vectors is not a well-posed problem in terms of solving, though. Indeed if $\mathcal{V}$ is a KS vector then $\lambda\mathcal{V}$ is also a KS vector for any non-zero scalar $\lambda$. Furthermore, if $\mathcal{S}$ is a set of KS vectors, then $\mathcal{RS}$ is also such a set for any arbitrary rotation matrix $\mathcal{R}$. We may simplify the problem by considering only unit vectors (i.e. vectors whose Euclidean norm is 1 and hence vectors whose components have a value in the range [-1,1]). To avoid the rotation problem, we may assume that one $n$-tuple is the orthonormal basis of $\mathbb{R}^n$. Under these assumptions, some of the orthogonality equations simplify. E.g., if `1234` is the basis of $\mathbb{R}^4$ with `1`$=[0,0,0,1]$ then `1567` indicates that the fourth components of `5,6,7` are 0. The non-collinearity constraints also plays an important role. E.g., `1235` is not a possible $n$-tuple as three components of `5` would be 0 and hence `5` would be collinear with `4`.

This has prompted us to develop a *preliminary pass*, which allows elimination of $n$-tuples that cannot lead to a solution. Consider a system of $m$ 4-dim vectors. The preliminary pass makes use of an $m \times 4$ table $\mathcal{T}$, called the *0-table*, with an entry set to 1 when a vector component cannot be 0. For example, if vector `j` has components $[a_{j1}, 0, 0, a_{j4}]$, then neither $a_{j1}$ nor $a_{j4}$ can be 0 (otherwise the vector will be collinear with one of the vectors of the basis) and $\mathcal{T}[j,1] = \mathcal{T}[j,4] = 1$. The preliminary pass selects a set of four 4-dim vectors as the basis of $\mathbb{R}^4$. It then applies a set of simplification rules on the the orthogonality equations. For example, if the equation is $a_{jk}\, a_{ik} = 0$ and $\mathcal{T}[j,k] = 1$ then $a_{ik}$ is set to 0. Each time a vector component value is determined the 0-table is updated and the preliminary pass is restarted. The process will stop when no further simplification may be performed or when a constraint violation occurs (e.g., one equation implies that $a_{jk}$ should be 0 while the 0-table indicates that this component

♯ This merging is crucial. Without it we would not be able to reduce the exponential complexity of the problem to the statistically polynomial one.

cannot be 0) in which case the system cannot have a solution. The simplification rules used in the preliminary pass depend on the space dimension.

The preliminary pass has been implemented as a C program that has been added as a filter in the generation program. For avoiding the exponential growth of the number of generated MMP diagrams it is essential that the candidate KS-sets should be generated incrementally i.e. that the program generates sequentially all systems starting with a given $m$ $n$-tuples before modifying the $m$th $n$-tuple. By using this incremental generation during the preliminary pass determines that an initial set of $m$ $n$-tuples has no solution and that no further systems starting with this set will be generated. E.g., for 18 vectors and 12 quadruples, without such a filter we would generate $> 2.9 \cdot 10^{16}$ systems—what would require more than 30 million years on a 2 GHz CPU—while the filter reduces the generation to 100220 systems (obtainable within $< 30$ mins on a 2 GHz CPU). Thereafter `states01` gives us 26800 systems without `0-1` states in $< 5$ secs.

For the remaining systems, two solvers have been developed. One is based on a specific implementation of Ritt characteristic set calculation [43]. Assume that a vector `V`$=[a_{V1}, 0, 0, a_{V4}]$ (this implies that $a_{V1}, a_{V4}$ cannot both be 0) is orthogonal to `W`$=[a_{W1}, 0, a_{W2}, a_{W4}]$. From the orthogonality condition $a_{W1}\ a_{V1} + a_{W4}\ a_{V4} = 0$, we deduce that $a_{W1} = -a_{W4}\ a_{V4}/a_{V1}$ (as $a_{V1}$ cannot be 0) and that $a_{W4}$ cannot be 0. This information is propagated to the other equations and, as for the preliminary pass, simplification rules are applied to the equations, allowing us to determine further unknowns and to update the 0-table. The process is repeated until no further unknowns can be determined or until a constraint violation occurs. If no violation occurs, we will usually get a set of remaining equations that is quite simple and that allows us to determine all solutions. This solver has been implemented using the symbolic computation software Maple. E.g., for system (a) in Fig. 3: `1234,4567,789A,ABCD,DEFG,GHI1,35CE,29BI,68FH`, we get (in $< 10$ secs on a 2 GHz CPU) the remaining set of 10 equations: $2a_{62}^2 = 2a_{C1}^2 = 2a_{G3}^2 = 2a_{54}^2 = 4a_{E4}^2 = 1$, $2a_{53}^2 = 2a_{I2}^2 = 2a_{94}^2 = 4a_{A2}^2 = 2a_{I1}^2 = 1$ from the roots of which we may deduce the other vector components (e.g., we get 6 as $a_{62}[0, 1, -2a_{54}a_{94}a_{53}, a_{94}]$). The drawback of this approach is that it does not always allow us to completely solve the equational systems: we may end up with a system with fewer equations for which no further constraints can be propagated.

Our second solver is based on interval analysis. An interval evaluation of an equation $f(x_1, \ldots, x_m) = 0$ is a range $F = [a, b]$ such that if all the unknowns $x_1, \ldots, x_m$ are restricted to lie within given ranges, then whatever is the values of the unknowns in their range we have $a \leq f(x_1, \ldots, x_n) \leq b$. A simple way to calculate an interval evaluation is to use interval arithmetic that simply replaces all mathematical operators by an interval equivalent. E.g., the interval evaluation of the orthogonality condition $x_1y_1 + x_2y_2$ with $x_1, x_2 \in [0.5, 1]$ and $y_1 \in [0.1, 0.2]$, $y_2 \in [0.2, 1]$ is calculated as $[0.5, 1][0.1, 0.2] + [0.5, 1][0.2, 1] = [0.05, 0.2] + [0.01, 1] = [0.06, 1.2]$. Note that if the interval evaluation of an equation does not include 0 then there is no value of the unknowns in their range that can cancel the equation. A *box* will be a set of ranges, one

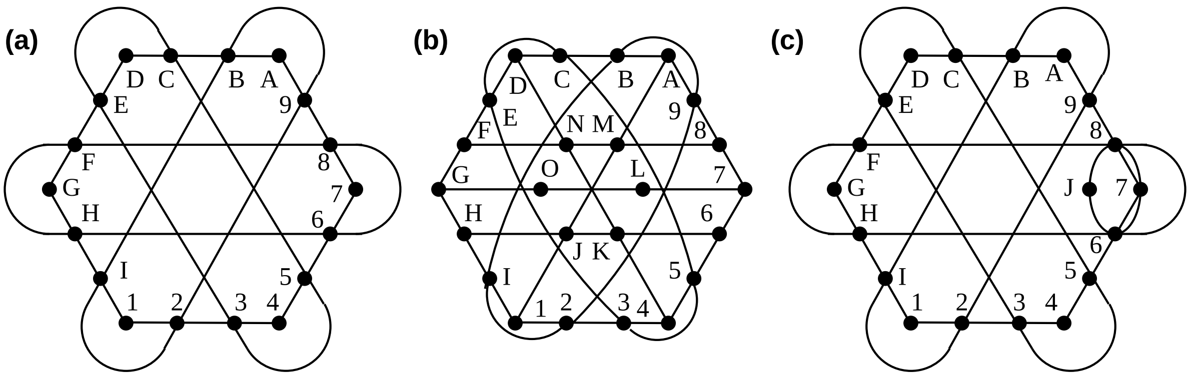


**Figure 3.** Smallest 4-dim KS systems with: (1) loops of size 3: (a) 18-9 (isomorphic to Cabello et al. [30]); (b) 24(22)-13 not containing system (a), with values $\notin \{-1, 0, 1\}$; (2) loops of size 2: (c) 19(18)-10.

for each unknowns.

Solving a KS-system is an appropriate problem for interval analysis, since all the unknowns are in the range [-1,1]. The set of these unknowns is the box $\mathcal{B}_0$. The system of equations to be solved consists of the equations derived from the orthogonality conditions between the vectors and of the *unitary equations* that describe that each vector is a unit vector.

A basic solver uses a list of boxes $\mathcal{L}$ that initially has element $\mathcal{B}_0$. At step $i$, the algorithm processes box $\mathcal{B}_i$ of $\mathcal{L}$ and calculates the interval evaluation of the orthogonality and unitary equations: if the interval evaluation of one of these equations does not include 0, then the algorithm will process the next box in the list. Otherwise two new boxes will be generated from $\mathcal{B}_i$ by bisecting the range of the box. These boxes will be added to the list, and the next box in the list will be processed. The algorithm will stop either when all the boxes have been processed (meaning the system has no solution) or when the width of all the ranges in a box is less than a small value while the interval evaluations of all the equations still include 0 (meaning a solution is obtained). Note that the method is mostly sensitive to the number of unknowns (which explains why the vectors that appear only once in the KS-system should be eliminated) and not so much on the number of equations. On the contrary, additional equations may even reduce the computation time. For example, consider a triplet $\mathbf{X_i X_j X_k}$ in 3D: using the orthogonality condition, $\mathbf{X_k}$ is obtained as $\mathbf{X_k} = \pm\mathbf{X_i} \times \mathbf{X_j}$. We get therefore two possible solutions for $\mathbf{X_k}$, and additional equations will be obtained by writing that the square of each component of $\mathbf{X_k}$ should be equal to the square of the same component of $\mathbf{X_i} \times \mathbf{X_j}$,

Numerous methods may be used to improve the efficiency of the basic solver (especially to prove that indeed a system has a solution). We use the interval analysis library `ALIAS` ‡ to deal efficiently with the KS systems.

‡ www.inria.fr/coprin/logiciels/ALIAS/ALIAS.html

Interval analysis has in principle an exponential complexity, due to the bisection process. But it has been experimentally shown that in some cases, the practical complexity is only polynomial. According to our tests (over 400 billion systems have been checked) it appears that the solving of the KS systems has indeed only a statistically polynomial complexity. It must also be noted that the solver may be used during the generation of the MMP diagrams as a complement to the preliminary pass to avoid the exponential growth of the number of generated MMP diagrams. Indeed, for a diagram that has not been rejected by the preliminary pass, we may run the solver to further check if the diagram has a solution. But since the solver may be relatively computer intensive, we have to use an adaptive version in which the number of allowed bisections is limited. For example this number may be large for relatively small sub-graphs because determining that they don't have a solution allows us to avoid the generation of a large number of diagrams. On the other hand, the number of allowed bisections will be small for sub-graphs whose size is close to the maximum (and consequently from which few diagrams will be deduced), thus avoiding increased generation time.

We also developed a checking program that finds solutions from assumed sets, say $\{-1, 0, 1\}$, even faster ($<$ 1 sec on a 2 GHz CPU) by precomputing all possible scalar products. The main algorithm scans the vertices and tries to assign unique vectors to them so that all vectors assigned to a given edge are orthogonal. In case of a conflict the algorithm backtracks, until either all possible assignments have been exhausted or a solution is found. We match its exponential behaviour by scanning next those vertices most tightly coupled to those already scanned, helping to force conflicts to show up early on so that backtracking can take care of them more quickly.

Further details of the algorithms and programs presented in this section will be given elsewhere. [42]

## 5. New results and conclusions

In this paper we presented algorithms that generate and those that solve sets of arbitrary many Kochen-Specker (KS) vectors that are of polynomial complexity or at least of statistically polynomial complexity. The algorithms merge generation of MMP diagrams corresponding to blocks of orthogonal vectors in $\mathbb{R}^n$ (Sec. 2), singling out MMP diagrams on which `0-1` states cannot be defined (Sec. 3), and solving nonlinear equations describing the orthogonalities of the vectors by means of interval analysis (Sec. 4), so as to eventually generate KS vectors in a statistically polynomially complex way. Using the algorithms we obtained the following results:

(i) A general feature we found to hold for all MMP diagrams without `0-1` states we tested is that the number of edges, $b$ and the number of vertices that share more than one edge, $a^*$ satisfy the following inequality: $nb \geq 2a^*$, where $n$ is the number of vertices per edge. Hence, there are no KS vectors that share at least 2 of $b$ $n$-tuples in their KS set whose number $a^* > \frac{nb}{2}$. In $\mathbb{R}^n$ this means that we cannot arrive at systems with more unknowns than equations when we disregard the unknowns that appear in only two equations. To prove the feature for an arbitrary $n$ remains an open problem.

(ii) For MMP diagrams without `0-1` states with 3 vertices per edge and $a < 30$ as well as with 4 vertices per edge and $a < 23$ the stronger inequality holds: $nb \geq 2a$. The only exception to this rule we have found is the original Kochen-Specker system with 192 vertices [see (xi) and Fig. 6]. At what $a$ for a chosen $n$ this inequality ceases to hold is an open problem.

(iii) None of the systems corresponding to the smallest diagrams without `0-1` states given in Sec. 3 and Fig. 2 has a solution. ‖ The smallest KS vectors that we found to have real solutions are presented in Figs. 3 and 4.

(iv) Between the 4-dim system shown in Fig. 3 (a) and the one shown in Fig. 3 (a) (both with smallest loops of size 3) there are 62 systems with loops of size 3, all containing the system (a), 37 of which do not have solutions from $\{-1, 0, 1\}$. System (b) is the first system with loops of size 3 not containing (a): `1234,4567,789A,ABCD,DEFG,GHI1,FNM8,GOL7,HJK6,DNK4,AMJ1,35CE,B29J`. One of its solutions is: `12...NO` = {1,0,1,1}{1,0,-2,1}{1,0,0,-1}{0,1,0,0}{0,0,1,0}{0,0,0,1}{1,0,0,0} {0,2,2,1} {0,2,-1,-2} {0,1,-2,2} {3,2,2,1} {1,-2,0,1} {-1,0,1,1} {1,1,0,1} {1,-1,1,0} {0,1,1,-1} {1,1,-1,0}{1,-1,0,-1}{1,-2,-1,0}{1,0,1,0}{0,0,1,1}{3,2,-1,-2}{1,0,-1,2}{0,2,-1,1} (which can, of course, easily be normalised. The system does not have a solution from $\{-1, 0, 1\}$.

(v) The smallest 4-dim system with the smallest loop of size 2 is the following 19-10 one: `1234,4567,789A,ABCD,DEFG,GHI1,35CE,29BI,68FH,678I` shown in Fig. 3 (c). It contains system (a) of Fig. 3 and it is the only MMP system with 19 vertices which has a solution from $\{-1, 0, 1\}$ for the corresponding vectors.

(vi) The two smallest 4-dim systems with the smallest loops of size 2 that do not contain system (a) of Fig. 3 are the following 20-10 ones: `1234,4567,789A,ABCD,DEFG,GHI1,68FH,12JI,1J9B,345K,4KEC` and `1234,4567,789A,ABCD,DEFG,GHI1,68FH,2IAK,345J,4JEC,9ABK` shown in Figs. 4 (a) and (b). The latter system is isomorphic to Kernaghan's system [29]. A solution to the former one is: `12...JK` = {0,0,0,1}{1,0,0,0}{0,1,1,0}{0,1,-1,0}{1,0,0,-1}{1,1,1,1}{1,-1,-1,1}{1,1,-1,-1} {1,0,1,0} {0,1,0,1} {1,0,-1,0} {1,1,1,-1} {1,-1,1,1} {1,-1,-1,-1} {0,0,1,-1} {1,1,0,0} {1,-1,0,0} {0,0,1,0}{0,1,0,0}{1,0,0,1}.

(vii) All 4-dim systems with up to 22 vectors and 12 edges with the smallest loops of size 2 which do have solutions from $\{-1, 0, 1\}$ contain at least one of the systems (a) and (b) of Fig. 4 and in many cases also (a) of Fig. 3. The two smallest 4-dim systems with the smallest loops of size 2 that contain neither of the latter three systems are 22-13 systems (c) and (d) of Fig. 4: `1234,4567,789A,ABCD,DEFG,GHI1,2ILA,345J,4JEC,678K,7KMG,9ABL,FGHM` and `1234,4567,789A,ABCD,DEFG,GHI1,12IJ,345K,678L,GML7,1J9B,4KEC,FGHM`. Their

‖ Still, they might be significant for other fields. E.g., the two diagrams 19-13(5) given in Sec. 3 [one of them is shown in Fig. 2 (b) of [33] and the other in Fig. 2 (d)] are equivalent to the Greechie diagrams with 19 *atoms* and 13 *blocks* and to our knowledge, the smallest Greechie diagram with 3 atoms per edge without `0-1` states known so far was the one given by Greechie [44, 45], with 27 atoms and 18 blocks. The system 38-19(5) from Sec. 3 yields the smallest Greechie diagram with 4 atoms per block.

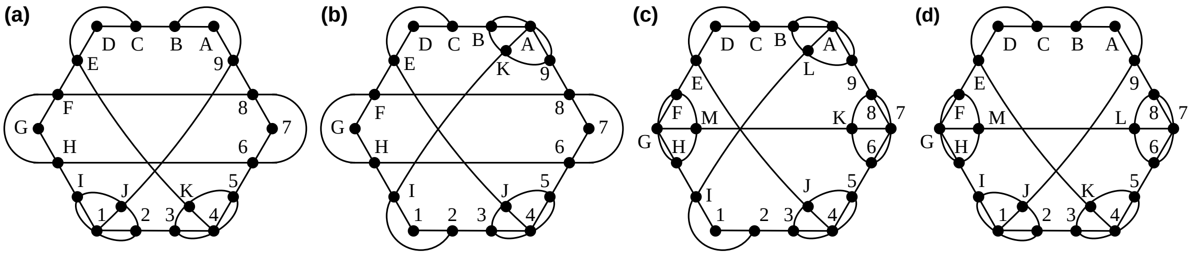


**Figure 4.** Smallest 4-dim KS systems with loops of size 2: (1) — not containing system (a) of Fig. 3: (a) 20-11; (b) 20-11 isomorphic to Kernaghan [29]; (2) — containing neither system (a) of Fig. 3 nor systems (a) and (b) off this figure: (c) 22-13; (d) 22-13.

solutions are: `12...M`={1,1,0,0}{1,-1,0,0}{0,0,1,0}{0,0,0,1}{1,0,0,0}{0,1,1,0}{0,1,-1,0}{1,0,0,1}{1,-1,-1,-1}{1,1,1,-1}{1,-1,1,1}{1,0,-1,0}{0,1,0,1}{1,0,1,0}{1,1,-1,-1}{1,-1,-1,1}{1,-1,1,-1}{0,0,1,1}{0,1,0,0}{1,0,0,-1}{1,1,-1,1}{1,1,1,1} and `12...M`={0,0,0,1}{1,0,0,0}{0,1,1,0} {0,1,-1,0} {1,0,0,-1} {1,1,1,1} {1,-1,-1,1} {1,-1,1,-1} {1,1,0,0} {0,0,1,1} {1,-1,0,0}{1,1,1,-1} {1,1,-1,1} {1,-1,-1,-1} {0,1,0,-1} {1,0,1,0} {1,0,-1,0} {0,1,0,0} {0,0,1,0} {1,0,0,1}{1,1,-1,-1}{0,1,0,1}.

(viii) As shown in [33], Peres' 4-dim vectors [27] build a hexagon with 24 vertices and 24 edges and not with 22 edges as presented by Tkadlec in [39]. (One can easily verify that the edges {1,-1,1,-1}{1,1,-1,-1}{1,-1,-1,1}{1,1,1,1} and {1,-1,1,1}{1,-1,-1,-1}{1,1,1,-1}{1,1,-1,1} are missing in the middle of Fig. 1 in [39].) This Peres' 4-dim 24-24 KS system contains systems (a) and (c) from Fig. 3 and all the systems from Fig. 4.

(ix) 4-dim systems with more than 41 vectors cannot have solutions from $\{-1,0,1\}$, and there are no such solutions to systems without `0-1` states with minimal loops of size 5 up to 41 vectors [there are altogether two such systems: 38-19(5) given in Sec. 3 and a 40-20(5) system], what brings the Hasse (Greechie) diagram approach to the KS problem [38, 39] into question.

(x) It can easily be shown that a 3-dim system of equations representing diagrams containing loops of size 3 and 4 cannot have a real solution. For loops of size 3, e.g. `123,345,561` the proof runs as follows. Let us choose `1`={1,0,0}, `2`={0,1,0}, `3`={0,0,1}, and `i`=$\{a_{i1},a_{i2},a_{i3}\}$, $i=4,5,6$ and consider block `345`. Using `3`·`5`=0 we get $a_{53}=0$. Let us next consider group `561`. Using `5`·`1`=0 we get $a_{51}=0$ Hence, `5`={0,$a_{52}$,0} and is therefore collinear with `2`. Thus, the system cannot have a solution. The proof for loops of size 4 is similar, only a little longer.

(xi) The smallest 3-dim systems without a `0-1` valuation have a minimal loop of size 5, 19 vertices and 13 edges [Sec. 3, Fig. 2 (d)], but they do not have real solutions. We scanned all systems with up to 30 vectors and 20 orthogonal triads and there are no KS vectors among them. This does not mean that Conway-Kochen's system (CK) [26, p. 114] is the smallest KS system, though. It turns out that we cannot drop vectors that belong to only one edge from orthogonal triads because (a) there are cases where a solution to a full system allows `0-1` valuation while one to a system with dropped vectors does not and (b) there are cases where the full system does not allow `0-1` valuation but has no solution. So, CK is actually not a 31 but a 51 vector system: `123,145,267,2AB,3CD,CEF,CGm,DIn,DKL,6EM,6KN,7IO,7GP,4GQ,4Ko,5Ep,5IS,ALW,AFX,BSY,BQZ,3cf,3de,cOh,dMT,cN9,dP8,eSl,fQg,iR1,jk1,iFa,jLb,kOU,kMV,RPH,RNJ` with 37 edges. (Tkadlec's claims [39] that CK can have 55 and 56 vectors and 54 edges are wrong.) A solution to CK is `12...op`={0,0,1}{1,0,0}{0,1,0}{1,-1,0}{1,1,0} {0,1,-1} {0,1,1} {2,5,1} {2,5,-1} {0,1,2} {0,2,-1} {1,0,1} {1,0,-1} {1,-1,-1} {1,2,-1} {1,1,-1} {2,-1,-5} {1,-1,1} {2,-1,5} {1,1,1} {1,-2,1} {2,1,1} {2,-1,-1} {2,1,-1} {2,-1,1} {1,1,2} {1,2,0} {1,-1,-2} {2,-5,1} {2,1,5} {2,1,-5} {5,2,-1} {5,-2,1} {5,1,2} {5,-1,-2} {1,2,5} {1,-2,-5} {1,0,2} {1,0,-2} {2,0,1} {2,0,-1} {1,-5,2} {2,-5,-1} {2,-1,0} {2,1,0} {1,-2,0} {1,5,-2} {1,-2,-1} {1,2,1} {1,1,-2}{1,-1,2}. Thus, when all the vectors are taken into account, Bub's system [25] with 49 vectors and 36 edges: `123,345,167,AB6,AC4,DEG,DFH,F9O,E8V,5JI,7MN,GIa,HNh,7LT,5KR,DAe,UTS,PRS,1GP,3HU,3Vj,Pgh,Uba,1Oi,VZg,OYb,6Xk,4Wn,Sde,dci,dfj,imn,jlk,akQ,hnQ,eQ2` is so far the smallest.

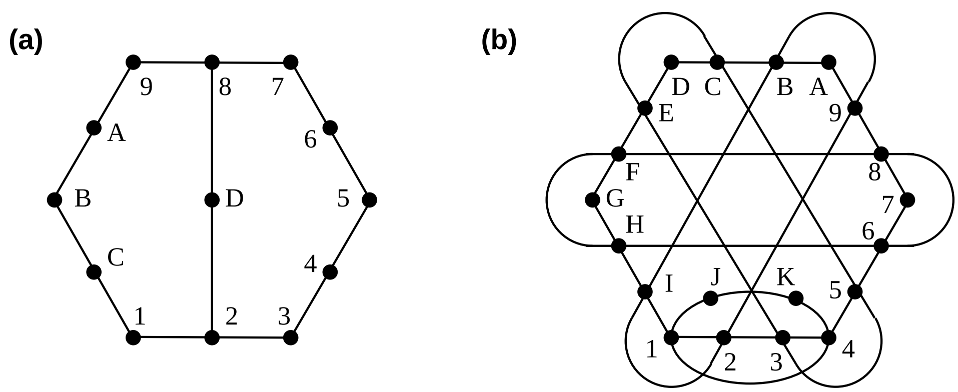


**Figure 5.** (a) System with dropped vectors that belong to only one edge (`4,6,A,C,D`) does not have any set of `0-1` states. Taking any of these vectors into account results with systems with at least one set of `0-1` states; (b) Neither the system itself nor the systems obtained by dropping vectors (`J` or `K` or both) allow `0-1` states. The latter systems have solutions while the original system does not have any.

Let us see why we cannot drop vectors that belong to only one edge in detail. First, as mentioned above, if we drop all vectors that belong to only one edge Fig. 5 (a) we get: `123,35,567,789,9B,B1,28`. This system has no `0-1` states. But if we add back a single such vector, say `123,345,567,789,9B,B1,28` or `123,35,567,789,9B,B1,2D8`, the system has at least one set of `0-1` states. All these systems do have solutions. The opposite situation is given by Fig. 5 (b). The system does not admit `0-1` states but has no solution. If we dropped `J` or `K` or both we would have a system with no `0-1` states and the systems would have solutions. E.g., the system with dropped `K` has the following solution: `12...IJ` = {0,0,0,1}{1,0,0,0}{0,1,1,0}{0,1,-1,0}{1,0,0,-1}{1,-1,-1,1}{1,1,1,1} {1,-1,1,-1} {0,1,0,-1} {1,0,-1,0} {0,1,0,1} {1,-1,1,1} {1,1,1,-1} {1,1,-1,1} {0,0,1,1} {1,-1,0,0} {1,1,0,0}{0,0,1,0}{1,-1,-1,0}. Second, in any KS diagram and therefore in Kochen-Specker (see Fig. 6), Bub, Peres, and Conway-Kochen's ones in particular, only all vectors together make a complete description of their KS sets. Recall that one arrives from an $n$-tuple to an adjoining one by rotation around an $(n-2)$-dim subspace. E.g., in Fig. 6 (*ii*) one starts with `123` and by rotations around `3`, `5`, and `7` one arrives at `789`. So, `4` and `6` are indispensable for the construction and cannot be dropped as also shown by Larsson [28].

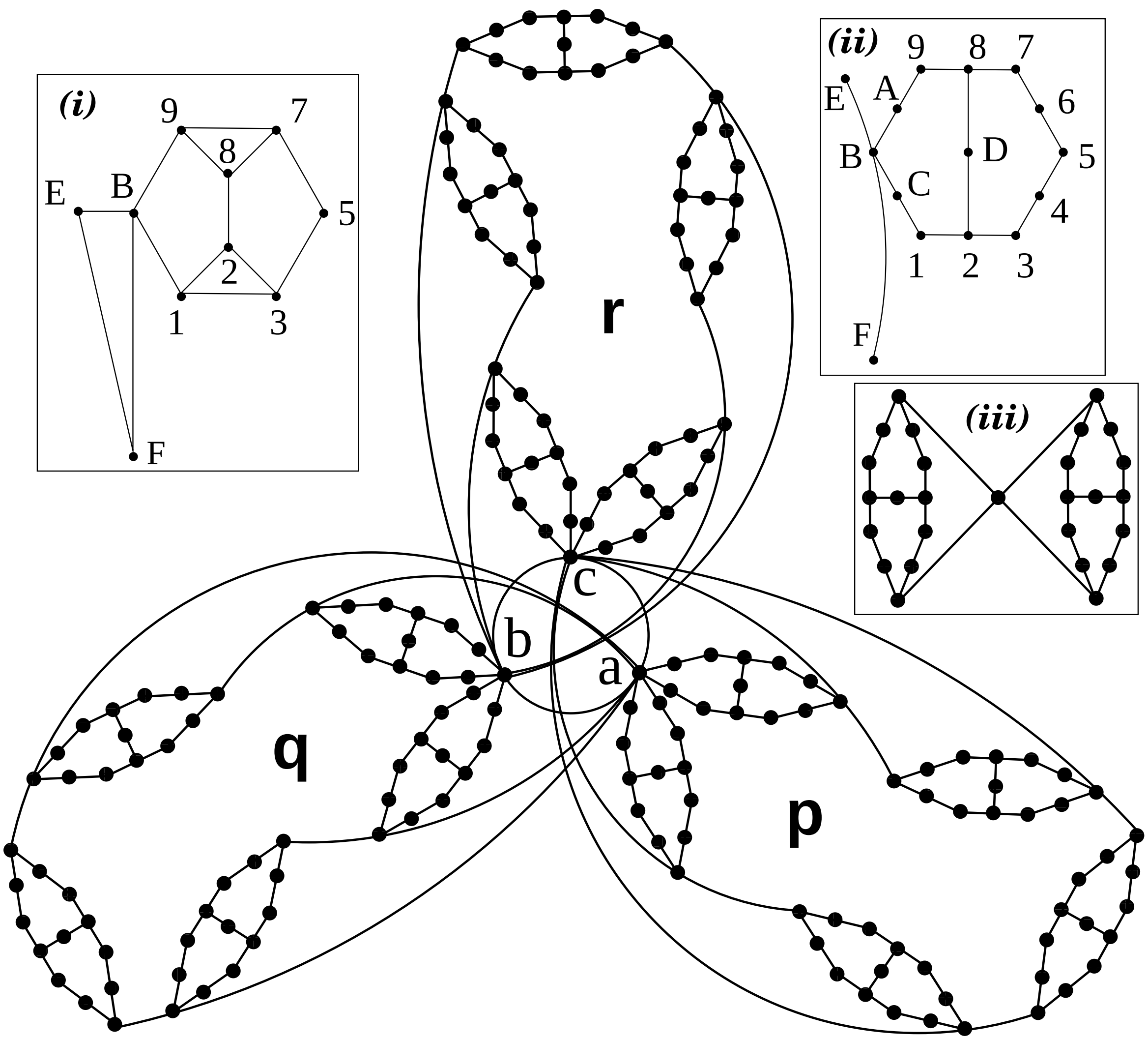


**Figure 6.** Historical Kochen-Specker 192 (117) graph [23, Lemma 2, p. 68, Fig. p. 69] in the MMP diagram notation. Inset *(i)* shows the hexagon with the adjoining triangle from [23, Lemma 1, p. 68]. Inset *(ii)* shows the same graph in our MMP diagram notation: triangles translate as edges and everything else stays the same except that Kochen and Specker drop the vectors that do not share edges, in particular, vectors 4, 6, A, C and D. In [23, Fig. p. 69] $a = p_0$, $b = q_0$, and $c = r_0$ hold. Here we glue these points together graphically. The groups of hexagons $p$, $q$, and $r$ here represent the hexagons containing $p_i$, $q_i$, and $r_i$, $i = 0, \ldots, 4$ in [23, Fig. p. 69]. Inset *(iii)* represents the 27(17)-point graph from [23, Fig. p. 70] in the MMP diagram notation.

Special attention is deserved by the first KS graph ever, given by Kochen and Specker themselves [23]. We translated it into the MMP diagram notation in Fig. 6, where we also explain the correspondence between the two notations. Vectors of the KS graph are all contained in the three groups of five hexagons of the type shown in the inset *(ii)* of the figure. Since each such hexagon contains 13 vectors and since two hexagons in each group $p$, $q$, and $r$ share a vector (vectors $a$, $b$, and $c$, respectively) this makes 192 vectors (vertices) and 118 edges. By dropping vectors that do not share edges Kochen and Specker obtained 117 vectors.

Let us just mention here that Kochen and Specker's 27(17)-point graph [Fig. 6-*(iii)*] provides a partial Boolean sub-algebra (characterising the operations of commensurable observables in both quantum and classical mechanics) that cannot be embedded into a Boolean algebra (i.e., not all classical tautologies from the Boolean algebra correspond to equalities in the partial Boolean algebra). The graph does allow `0-1` states. It has properties similar to those of its hexagons [cf. Fig. 5-(a)] since it represents a vector system `123,345,567,789,9AB,BC1,4DA,EFG,GHI,IJK,KLM,MNO,OPE,HQN,1RK,7RE` with the following components: `12...QR` = {{0,1,-2}{5,2,1}{1,-2,-1}{1,0,1}{1,1,-1}{2,-1,1}{0,1,1}{2,1,-1}{1,-1,1}{1,0,-1}{1,2,1}{5,-2,-1}{0,1,0}{0,1,-1}{2,1,1}{1,-1,-1}{1,1,0}{1,-1,2}{5,1,-2}{0,2,1}{5,-1,2}{1,1,-2}{1,-1,0}{1,1,1}{2,-1,-1}{0,0,1}{1,0,0}} which does have a set of `0-1` states when all vectors are taken into account and does not have it when the vectors that do not share edges are dropped. However, we obviously cannot dispense with vectors that build the system and therefore we cannot use this system for proving the KS theorem. Of course, Kochen and Specker were aware of this fact too and this is why they designed the aforementioned 15-hexagon 192-vector system.

(xii) The concept of *KS dual diagrams* [39, 46, 47] is apparently either a misnomer or insufficiently defined. Tkadlec claims that one arrives from a standard 3-dim KS diagram to its dual so as to "replace the role of points [vertices] and smooth curves [edges]: points [vertices of the dual diagram] represent blocks [edges of the standard diagram] and maximal smooth blocks [maximal (?) edges of the dual diagram] represent atoms [vertices of the standard diagram]." [39] The instructions for such a construction are ambiguous, but two figures are given in [39] and [46] and we have tested them. One is a dual diagram to Peres' 57 (33) diagram ¶ and it reads: `123,345,567,869,9AH,8C2,7DG,HG1,4BA,CBD,6gE,BhE,3IJ,2RO,1VU,VPN,UML,JKN,OKL,IQM,RQP,jSK,jiQ,UWX,Veb,Gfa,HCZ,ZYb,XYa,WdC,edf,TFd,TcY,1kE,1lj,1mT` and the other: `123,145,16C,768,7HK,4FB,GEC,89A,5IJ,HGI,EF9,KBD,JAD,CDV,KLM,BON,DgS,VUT,SP2,QRS,MQU,NPT,c6R,GPd,VWX,X3Y,3Ze,EZQ,abJ,YfA,cde,ehD,acW` is dual to Conway-Kochen's diagram. Of these two diagrams only the latter does not allow `0-1` state. The former has at least one set of `0-1` states. Then our solvers prove that the equation system corresponding to the latter diagram does not have a solution. Hence, neither of the two diagrams is a KS set, and we would expect a KS dual to be a KS set.

(xiii) We obtain the class of all remaining (non-KS) vectors from $\mathcal{H}^n$ by first filtering

¶ `123,39R,89A,47D,56E,DRE,EFG,CBD,NML,LKE,DJQ,QST,PJI,HKO,RVX,RUW,14Y,1Z5,4aA,5b8,8gB,AhF,7cH,6dI,CiO,GjP,7eM,6fS,ClN,GkT,NqX,PsV,OrU,MmU,SnV,HoX,IpW,TtW,2uB,2vF` the MMP diagrams so as to keep only those that allow `0-1` states. Out of these, a second filter then keeps only those diagrams whose corresponding equations have solutions. Vectors corresponding to these solutions are the wanted non-KS vectors.

(xiv) The presented algorithms can easily be generalised beyond the KS theorem. One can use MMP diagrams to generate Hilbert lattice counterexamples, partial Boolean algebras, and general quantum algebras which could eventually serve as an algebra for quantum computers. [48] One can also treat any condition imposed upon inner products in $\mathbb{R}^n$ to find solutions not by directly solving all nonlinear equations but also by first filtering the corresponding diagrams and solving only those equations that pass the filters.

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