Geometrico-static Analysis of Under-constrained Cable-driven Parallel Robots

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Abstract. This paper studies the kinematics and statics of cable-driven parallel robots with less than six cables, in crane configuration. A geometrico-static model is provided, together with a general procedure aimed at effectively solving, in analytical form, the inverse and direct position problems. The stability of equilibrium is assessed within the framework of a constrained optimization problem, for which a purely algebraic formulation is provided. A spatial robot with three cables is studied as an application example.

Key words: Under-constrained cable-driven parallel robots, geometric analysis, static analysis.

1 Introduction

Cable-driven parallel robots (CDPRs) are ordinarily referred to as fully-constrained or under-constrained, depending on whether all six degrees of freedom (dofs) of the moving platform are controlled or not [1]. It is well known that, in the general case, fully-constrained CDPRs require at least seven cables, and only six in crane configuration, namely when gravity acts as an additional cable. The distinction between the two aforementioned categories of robots is somewhat fictitious, since a theoretically fully-constrained CDPR is, in considerable parts of its workspace, actually under-constrained, namely when a full restraint would require a negative tension in some cable. Permanently under-constrained CDPRs with less than six cables are furthermore used in a number of applications, such as rescue operations, in which a limitation of dexterity is acceptable in order to decrease complexity, set-up time, likelihood of cable interference, etc.

The above considerations motivate a careful study of under-constrained robots. However, while fully-constrained CDPRs have been extensively investigated [2], few studies have been dedicated to under-constrained ones [3, 4, 5]. A major challenge in the study of these robots consists in the intrinsic coupling between kinematics and statics (or dynamics). Indeed, while in a fully-constrained CDPR the platform posture is determined in a purely geometric way by assigning the cable lengths (provided that all cables are under tension), in an under-constrained CDPR the pose depends on both cable lengths and equilibrium equations. Consequently, kinematics and statics must be dealt with simultaneously. Furthermore, as the platform posture depends on the applied load, it may change due to external disturbances. Hence, it is of great importance to investigate equilibrium stability [5, 6].

This paper presents a kinematic and static study of under-constrained *nn*-CDPRs, in crane configuration. An under-constrained *nn*-CDPR is controlled by n cables and it exhibits n distinct anchor points on both the base and the platform, with $n < 6$. A geometrico-static model is presented, together with a general procedure aimed at effectively solving, in analytical form, the inverse and direct position problems. These consist in determining the overall robot configuration (and cable tensions) once n configuration variables (e.g., platform posture coordinates or cable lengths) are given. The problem of equilibrium stability is formulated as a constrained optimization problem, and a purely algebraic method, which rules out the need of differentiation, is provided. The geometrico-static study of a general 33-CDPR is outlined as an application example.

2 Geometrico-static model

Let a mobile platform be connected to a fixed base by n cables, with $2 \leq n \leq 5$. A_i and B_i are, respectively, the anchor points of the ith cable on the frame and the platform, and $\mathbf{s}_i = B_i - A_i$ (Fig. 1a). The set $\mathscr{C}$ of theoretical geometrical constraints imposed on the platform comprises the relations

$$|\mathbf{s}_i| = \sqrt{\mathbf{s}_i \cdot \mathbf{s}_i} = \rho_i, \quad i = 1 \ldots n, \tag{1}$$

where ρ_i is the length of the ith-cable, which is assumed, for apparent practical reasons, strictly positive (so that, as a consequence, $B_i \not\equiv A_i$).

Since only n geometrical constraints are enforced, the platform preserves $6-n$ degrees of freedom, with its posture being determined by equilibrium laws. If $Q\mathbf{\$}_e$, with $Q > 0$, is an arbitrary external wrench acting on the platform (including inertia forces, in case of dynamic conditions) and $(\tau_i/\rho_i)\mathbf{\$}_i$ is the force exerted by the ith cable ($\mathbf{\$}_e$ and $\mathbf{\$}_i/\rho_i$ are assumed to be unit screws), then

$$\sum_{i=1}^{n} \frac{\tau_i}{\rho_i}\mathbf{\$}_i + Q\mathbf{\$}_e = \mathbf{0}, \tag{2}$$

with

$$\tau_i \geq 0, \quad i = 1 \ldots n. \tag{3}$$

Equations (1)-(2) amount to $6+n$ scalar relations involving $6+2n$ variables, namely, the cable tensions and lengths, and the variables parameterizing the platform posture. In general, a finite set of system configurations may be determined if n of such variables are assigned.

In this paper, only *static* equilibrium is considered and $Q\mathbf{\$}_e$ is assumed to be a *constant* force applied on a point G of the platform (e.g., the platform weight acting through its center of mass). Hence, Eqs. (1)-(2) are algebraic, or may be easily rendered so. If Eq. (2) is written as

$$\underbrace{\begin{bmatrix} \$_1 \cdots \$_n \ \$_e \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} (\tau_1/\rho_1) \\ \vdots \\ (\tau_n/\rho_n) \\ Q \end{bmatrix} = \mathbf{0}, \tag{4}$$

$\mathbf{M}$ is a $6 \times (n+1)$ matrix only depending on the platform posture and equilibrium is possible only if

$$\mathrm{rank}(\mathbf{M}) \leq n, \tag{5}$$

namely, if the cables and the line of action of $\$_e$ span the same n-dimensional system of lines. Within the domain of rigid-body mechanics, the problem is *statically determinate* if the equality holds, *indeterminate* otherwise. In the former case, it is always possible to replace Eq. (4) with $6-n$ scalar relations that do not contain the unknowns τ_i, $i = 1 \dots n$. In fact, the linear dependence of $\$_1, \dots, \$_n$ and $\$_e$ is a purely geometrical condition. A most straightforward strategy consists in computing cable tensions by way of n linearly independent relations chosen within Eq. (4), then substituting them back into the remaining ones. The resulting equations, however, exhibit a remarkable complexity. A more convenient strategy consists in setting all $(n+1) \times (n+1)$ minors of $\mathbf{M}$ equal to zero, which amounts to $\binom{6}{n+1}$ scalar relations, among which $6-n$ linearly independent ones may be suitably chosen. By such an approach, the resulting equations are significantly simpler. Furthermore, since they do *not* comprise cable lengths, they lead to a partial decoupling of the system equations, with cable lengths only appearing in Eq. (1). Such an approach may also be applied when the problem is statically indeterminate.

Depending on the variables designated as input, one may tackle an inverse geometric problem (IGP), if n variables concerning the platform posture are assigned, or a direct one (DGP), if cable lengths are given. The IGP takes particular advantage of the partial decoupling of system equations, since the platform configuration may be computed by simply solving the $6-n$ relations emerging from Eq. (5). Cable lengths and tensions may be subsequently (and straightforwardly) computed by Eq. (1) and a suitable set of linear independent relations chosen within Eq. (4). The set of *admissible* solutions consists of all those for which cable tensions are non-negative (cf. Eq. (3)) and the platform equilibrium is stable (cf. §3). The DGP is remarkably more complex, since in this case the platform configuration must be determined by simultaneously solving both the $6-n$ relations emerging from Eq. (5) and the n relations in Eq. (1).

It must be said that Eq. (1) represents a set of *theoretical* constraints, since the *actual* constraint imposed by a generic cable consists in that

$$|\mathbf{s}_i| = \sqrt{\mathbf{s}_i \cdot \mathbf{s}_i} \leq \rho_i. \tag{6}$$

The above refinement causes no concern when the IGP is dealt with, for in this case the *theoretical* values of cable lengths are conveniently computed by Eq. (1), after the platform posture has been established. Conversely, when the DGP is tackled, cable lengths are assigned as inputs, and *a priori* nothing assures that *all* cables are called upon to sustain the load. Indeed, if a subset $\mathscr{W}$ of cable indexes exists such that card$(\mathscr{W}) < n$ and $\$_e \in \text{span}\left\{\$_j, j \in \mathscr{W}\right\}$, equilibrium configurations possibly exist such that $|\mathbf{s}_k| < \rho_k$, for all $k \notin \mathscr{W}$, and thus $\tau_k = 0$. These are legitimate solutions of the problem at hand. It follows that the overall solution set is obtained by solving the DGP for *all possible* constraint sets $\{|\mathbf{s}_j| = \rho_j, j \in \mathscr{W}\}$, with $\mathscr{W} \subseteq \{1\ldots n\}$ and card$(\mathscr{W}) \leq n$, and by retaining, for each corresponding solution set, the solutions for which $|\mathbf{s}_k| < \rho_k$, $k \notin \mathscr{W}$. In general, this amounts to solving $\sum_{h=0}^{n-1}\binom{n}{n-h}$ DGPs.

A caveat is worth to be mentioned. Equation (5) is only a *necessary* condition for equilibrium. In very special conditions, it may happen that equilibrium is not possible, in spite of Eq. (5) being fulfilled and irrespective of the sign of cable tensions. In particular, this occurs if $\mathbf{M}$ loses its full rank because a subset of its n first columns becomes linearly dependent, i.e. if the rank loss is 'concentrated' among the set of screws associated with the cable lines. In this case, the rank of the block[1] $\mathbf{M}_{1\ldots6,1\ldots n}$ is at most equal to $n-1$ and Eq. (2) may be satisfied only if rank$(\mathbf{M}) \leq n-1$. Cases like the one described here, however, are sufficiently unlikely to occur not to be, in practice, of particular concern. Nonetheless, a check of the rank of $\mathbf{M}_{1\ldots6,1\ldots n}$ is advisable before attempting to solve for cable tensions.

Throughout the text, the following notation is adopted (Fig. 1a). *Oxyz* is a Cartesian coordinate frame fixed to the base, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ being unit vectors along the coordinate axes. $Gx'y'z'$ is a Cartesian frame attached to the platform. $\mathbf{e}$ is a unit vector directed as $\$_e$, $\mathbf{x} = G - O$, $\mathbf{a}_i = A_i - O$, $\mathbf{r}_i = B_i - G$, $\mathbf{s}_i = B_i - A_i = \mathbf{x} + \mathbf{r}_i - \mathbf{a}_i$, $\mathbf{u}_i = (A_i - B_i)/\rho_i = -\mathbf{s}_i/\rho_i$ and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, with $i, j = 1\ldots n$, $i \neq j$. Without loss of generality, O is chosen to coincide with A_1 (so that $\mathbf{a}_1 = \mathbf{0}$) and $\mathbf{k} = \mathbf{e}$. If $\mathbf{b}_i$ is the projection of $B_i - G$ on $Gx'y'z'$, Φ is the array grouping the variables parameterizing the platform orientation with respect to the fixed frame and $\mathbf{R}(\Phi)$ is the corresponding rotation matrix, then $\mathbf{r}_i = \mathbf{R}(\Phi)\,\mathbf{b}_i$. The platform posture is described by the array $\mathbf{X} = (\mathbf{x}; \Phi)$, with the components of $\mathbf{x}$ in *Oxyz* being denoted, for the sake of brevity, as x, y and z. If O is chosen as the reduction pole of moments, $\$_i$ and $\$_e$ may be respectively expressed, in axis coordinates, as $\$_i = [\mathbf{s}_i;\ \mathbf{a}_i \times \mathbf{s}_i]$ and $\$_e = [\mathbf{e};\ \mathbf{x} \times \mathbf{e}]$. Accordingly, $\mathbf{M}$ becomes

$$\mathbf{M} = \begin{bmatrix} \mathbf{x}+\mathbf{r}_1 & \cdots & \mathbf{x}+\mathbf{r}_i-\mathbf{a}_i & \cdots & \mathbf{e} \\ \mathbf{0} & \cdots & \mathbf{a}_i \times (\mathbf{x}+\mathbf{r}_i) & \cdots & \mathbf{x} \times \mathbf{e} \end{bmatrix}, \tag{7}$$

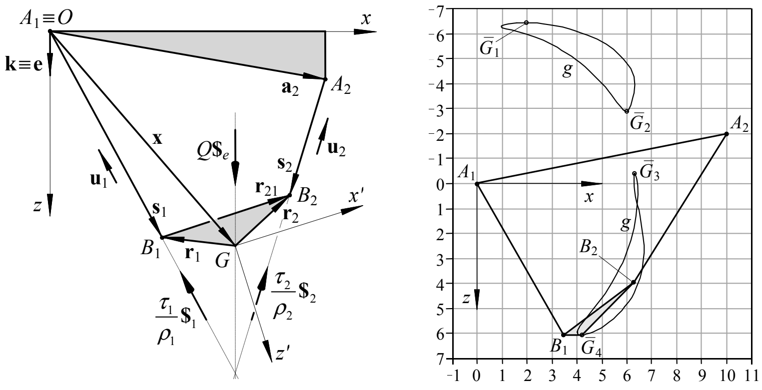
or, equivalently, after subtracting the first column from the columns 2 through n,

$$\mathbf{M}' = \begin{bmatrix} \mathbf{x}+\mathbf{r}_1 & \cdots & \mathbf{r}_{i1}-\mathbf{a}_i & \cdots & \mathbf{e} \\ \mathbf{0} & \cdots & \mathbf{a}_i \times (\mathbf{x}+\mathbf{r}_i) & \cdots & \mathbf{x} \times \mathbf{e} \end{bmatrix}. \tag{8}$$

3 The stability of equilibrium

Let an equilibrium configuration $\left(\bar{\mathbf{X}}, \bar{\rho}_1 \ldots \bar{\rho}_m\right)$ be considered, with m being the number of *active* constraints (i.e. the number of cables contributing to supporting

[1] The notation $\mathbf{M}_{hij,klm}$ denotes the block matrix obtained from rows h, i and j, and columns k, l and m of $\mathbf{M}$. When all columns of $\mathbf{M}$ are used, the corresponding subscripts are omitted.



(a) (b)

Fig. 1: 22-CDPR: (a) general model; (b) equilibrium configurations in the case $\mathbf{a}_2 = (10,0,-2)$, $\mathbf{b}_1 = (-0.5,0,-0.5)$, $\mathbf{b}_2 = (3,0,0)$, and $\rho_1 = \rho_2 = 7$.

the platform). By a convenient reordering of indexes, taut cables may be assumed to be the first m, with $m \leq n$. Since the platform preserves $6-m$ dofs, it may displace under the effect of a change in the external force acting on it, while cable lengths remain unvaried (for the sake of simplicity, it is assumed that the number of cables in tension does not change because of the perturbation, which is reasonable, but not necessarily true). The problem of assessing equilibrium stability is, thus, in order. In particular, G may generally move within a closed region of $\mathbb{R}^3$ (in some cases, a surface or a curve). If g is the frontier of this region, the equilibrium is stable any time the potential energy U associated with the external force $Q\$_e$, namely $-Q\mathbf{e}\cdot\mathbf{x}$, is at a minimum on g. Loosely speaking, the platform is at rest in all points $\bar{G}$ of g in which the variety tangent to g is perpendicular to $\mathbf{e}$, with the equilibrium being stable if and only if a neighborhood $W_{\bar{G}}$ of $\bar{G}$ exists such that $(P-\bar{G})\cdot\mathbf{e} < 0$, for all $P \in (g \cap W_{\bar{G}})$. In such a condition, when the platform displaces under the effect of a perturbation, the original configuration is restored if the perturbation ceases. Figure 1b helps to depict this concept. The figure shows the locus g of the positions that G may assume for an exemplifying 22-CDPR, under the constraints (1) and with $m = n = 2$. If the platform is thought of as the coupler of a four-bar linkage whose grounded links are the cables (with assigned lengths), g is the coupler curve of G, namely a bicursal sextic. The stationary configurations of G are the points of g in which the tangent line is perpendicular to $\mathbf{e}$, with U being at a minimum in $\bar{G}_2$ and $\bar{G}_4$. These are the stable equilibrium poses (of course, since cable tensions must be negative in the configurations lying above the base, and positive otherwise, $\bar{G}_4$ is *de facto* the only feasible configuration for the example at hand). Finding the minima of a constrained function is a classical issue in optimization theory. An efficient algorithmic formalization is presented in the following.

At equilibrium, the variation of the *global* potential energy of the platform due to a virtual displacement of it must be zero. Such a variation is the opposite of the work carried out by all forces acting on the platform, namely

$$\delta L = -\sum_{i=1}^{m} \tau_i \mathbf{u}_i \cdot \delta B_i - Q\mathbf{e} \cdot \delta G = 0. \tag{9}$$

If $\delta\mathbf{x}$ and $\delta\Theta$ are, respectively, the virtual displacement of G and the virtual rotation of the platform, then $\delta G = \delta\mathbf{x}$ and $\delta B_i = \delta\mathbf{s}_i = \delta\mathbf{x} + \delta\Theta \times \mathbf{r}_i$, so that

$$\delta L = -\left(\sum_{i=1}^{m} \tau_i \mathbf{u}_i + Q\mathbf{e}\right) \cdot \delta\mathbf{x} - \left(\sum_{i=1}^{m} \tau_i \mathbf{r}_i \times \mathbf{u}_i\right) \cdot \delta\Theta = \mathbf{f} \cdot \delta\mathbf{x} + \mathbf{m} \cdot \delta\Theta = 0. \tag{10}$$

Equation (10), from which $\mathbf{f}$ and $\mathbf{m}$ are inferred to be zero, is clearly equivalent to Eq. (2), by letting $n = m$. Since, for $\rho_i = \overline{\rho_i}$, $\delta\left(|\mathbf{s}_i| - \rho_i\right) = \delta|\mathbf{s}_i|$ and

$$\delta|\mathbf{s}_i| = \frac{\mathbf{s}_i \cdot \delta\mathbf{s}_i}{\rho_i} = \frac{\mathbf{s}_i \cdot \delta\mathbf{x} + \mathbf{r}_i \times \mathbf{s}_i \cdot \delta\Theta}{\rho_i} = -\left(\mathbf{u}_i \cdot \delta\mathbf{x} + \mathbf{r}_i \times \mathbf{u}_i \cdot \delta\Theta\right), \tag{11}$$

δL may be written as

$$\delta L = -Q\mathbf{e} \cdot \delta\mathbf{x} + \sum_{i=1}^{m} \tau_i \delta\left(|\mathbf{s}_i| - \rho_i\right), \tag{12}$$

i.e., as the virtual variation of the Lagrange function

$$L = -Q\mathbf{e} \cdot \mathbf{x} + \sum_{i=1}^{m} \tau_i \left(|\mathbf{s}_i| - \rho_i\right), \tag{13}$$

with Lagrange multipliers coinciding with the cable tensions, namely, with the forces necessary to impose the geometrical constraints [7][2]. Such an observation is useful, since it allows the stability characteristics of the equilibrium to be assessed by evaluating the definiteness of the reduced Hessian $\mathbf{H}_r$ of L, i.e. the Hessian of L taken with respect to the configuration variables, further projected on the tangent space of the constraints $\mathscr{C}$ [7]. An algebraic expression of $\mathbf{H}_r$ is derived hereafter.

The second-order variation of δL is given by

$$\delta^2 L = -Q\mathbf{e} \cdot \delta^2\mathbf{x} + \sum_{i=1}^{m} \tau_i \frac{\delta\mathbf{s}_i \cdot \delta\mathbf{s}_i}{\rho_i} + \sum_{i=1}^{m} \tau_i \frac{\mathbf{s}_i \cdot \delta^2\mathbf{s}_i}{\rho_i}, \tag{14}$$

with $\delta^2\mathbf{s}_i = \delta^2\mathbf{x} + \delta^2\Theta \times \mathbf{r}_i + \delta\Theta \times \left(\delta\Theta \times \mathbf{r}_i\right)$. Enforcing $\mathbf{f} = \mathbf{m} = \mathbf{0}$ in Eq. (14) yields

$$\delta^2 L = \sum_{i=1}^{m} \frac{\tau_i}{\rho_i} \left\{\delta\mathbf{x} \cdot \delta\mathbf{x} - 2\delta\mathbf{x} \cdot (\mathbf{r}_i \times \delta\Theta) - (\mathbf{r}_i \times \delta\Theta) \cdot [(\mathbf{x} - \mathbf{a}_i) \times \delta\Theta]\right\} \tag{15}$$

[2] Equation (9) plus the relations $\{\tau_i > 0, |\mathbf{s}_i| = \rho_i\}$ for $i = 1 \ldots m$ and $\{\tau_i = 0, |\mathbf{s}_i| < \rho_i\}$ for $i = m \ldots n$ are equivalent to the Karush-Kuhn-Tucker conditions for the minimization of L under the constraints (6), provided that $\$_1, \ldots, \$_m$ are linearly independent.

or, in matrix notation,

$$\delta^2 L = \sum_{i=1}^{m} \frac{\tau_i}{\rho_i} \left[\delta \mathbf{x}^T \delta \mathbf{x} - 2 \delta \mathbf{x}^T \tilde{\mathbf{r}}_i \delta \Theta + \delta \Theta^T \tilde{\mathbf{r}}_i \left(\tilde{\mathbf{x}} - \tilde{\mathbf{a}}_i \right) \delta \Theta \right], \tag{16}$$

where $\tilde{\mathbf{n}}$ denotes, for a generic vector $\mathbf{n}$, the skew-symmetric matrix expressing the operator $\mathbf{n}\times$. $\delta^2 L$ is a bilinear form in the twist space of the platform. If the platform virtual displacement is expressed, in ray coordinates, as $\delta \mathbf{t} = [\delta \mathbf{x}; \delta \Theta]$, and $\mathbf{I}_3$ denotes the 3×3 identity matrix, the symmetric matrix associated with this form is

$$\mathbf{H}_p = \sum_{i=1}^{m} \frac{\tau_i}{\rho_i} \begin{bmatrix} \mathbf{I}_3 & -\tilde{\mathbf{r}}_i \\ \tilde{\mathbf{r}}_i & \frac{1}{2} \left(\tilde{\mathbf{r}}_i \tilde{\mathbf{x}} - \tilde{\mathbf{r}}_i \tilde{\mathbf{a}}_i + \tilde{\mathbf{x}} \tilde{\mathbf{r}}_i - \tilde{\mathbf{a}}_i \tilde{\mathbf{r}}_i \right) \end{bmatrix}, \tag{17}$$

which represents the pseudo-Hessian of L ($\mathbf{H}_p$ is not a true and proper Hessian, since $\delta \Theta$ is not generally integrable).

The tangent space of $\mathscr{C}$ is obtained by setting Eq. (11) equal to zero for all values of i. In matrix notation, this amounts to

$$\mathbf{J}_p \delta \mathbf{t} = \begin{bmatrix} \mathbf{s}_1^T & (\mathbf{r}_1 \times \mathbf{s}_1)^T \\ \vdots & \vdots \\ \mathbf{s}_m^T & (\mathbf{r}_m \times \mathbf{s}_m)^T \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \Theta \end{bmatrix} = \mathbf{0}, \tag{18}$$

where the ith row of $\mathbf{J}_p$ coincides with $\$_i$, expressed in axis coordinates and assuming G as the moment pole. $\mathbf{J}_p$ is the pseudo-Jacobian of the constraint equations.

If $\mathbf{N}_p$ is any $6 \times (6-m)$ matrix whose columns generate the null space of $\mathbf{J}_p$, the reduced Hessian of $\mathscr{C}$ is the following $(6-m) \times (6-m)$ matrix:

$$\mathbf{H}_r = \mathbf{N}_p^T \mathbf{H}_p \mathbf{N}_p. \tag{19}$$

A sufficient condition for the equilibrium to be stable consists in $\mathbf{H}_r$ being positive definite.

If the method described above is applied to the example portrayed in Fig. 1b, results in agreement with those expected are obtained. The equilibrium configurations are the *real* solutions of the DGP of the robot[3], i.e. $\bar{G}_1 = (1.94, -6.43)$, $\bar{G}_2 = (5.98, -2.89)$, $\bar{G}_3 = (6.31, -0.42)$ and $\bar{G}_4 = (4.18, 6.07)$. Since the problem is planar, $\mathbf{H}_p$ and $\mathbf{J}_p$ are, respectively, 3×3 and 2×3 matrices, so that the reduced Hessian is a scalar. H_r is positive in $\bar{G}_2$ and $\bar{G}_4$ and negative in $\bar{G}_1$ and $\bar{G}_3$, namely

$$H_r|_{\bar{G}_1} = -29589,\ H_r|_{\bar{G}_2} = 18709,\ H_r|_{\bar{G}_3} = -22875,\ H_r|_{\bar{G}_4} = 61650. \tag{20}$$

If $\tau = (\tau_1, \tau_2)$, corresponding cable tensions are

$$\tau_{\bar{G}_1} = -(8.6, 2.3),\ \tau_{\bar{G}_2} = -(24.2, 22.7),\ \tau_{\bar{G}_3} = (22.1, 24.5),\ \tau_{\bar{G}_4} = (6.0, 5.6). \tag{21}$$

[3] The analytical solution of the DGP of the general 22-CDPR will be reported in a future paper. As the class of a generic coupler curve is 12 [8], there are 12 lines tangent to it passing through the point at infinity perpendicular to **e**, so that the DGP admits up to 12 complex solutions [5].

4 Application example: the 33-CDPR

Due to space limitations, only a brief outline of the IGP and the DGP of the 33-CDPR is sketched hereafter. Technical details and convenient discussions about empty and nonzero-dimensional solution sets will be provided in future papers.

When $n=3$, Eq. (5) is satisfied and $\text{rank}(\mathbf{M})=3$ only if $\$_1$, $\$_2$, $\$_3$ and $\$_e$ belong to the same tridimensional subspace of lines [8]. Letting all 4×4 minors of $\mathbf{M}'$ vanish leads to 15 polynomial equations in $\mathbf{x}$ and Φ of the form $p_j = 0$. We look for the variety V of the ideal generated by such equations. If three configuration variables are known (as in the IGP), any three p_j, say p_l, p_h, p_k, may be chosen and a corresponding (generally zero-dimensional) variety V_{lhk} is obtained. V is the intersection of the five varieties that may be generated this way. Clearly, a primary objective of the solving algorithm consists in limiting the number of varieties to be computed to the lowest possible value, possibly to just one.

It is useful observing that letting $B_i \equiv A_i$ causes the ith column of $\mathbf{M}$ to vanish (since $\mathbf{s}_i = \mathbf{a}_i \times \mathbf{s}_i = \mathbf{0}$) and, hence, it causes all 4×4 minors of $\mathbf{M}$ (and thus of $\mathbf{M}'$) to be zero. It follows that a configuration for which $B_i \equiv A_i$ always belongs to V: we call it a *trivial* solution and we need to discard it (cf. §2). This observation is particularly important for the IGP with assigned orientation. In this case, in fact, it is always possible to displace the platform (with a given orientation) so as to superimpose B_i onto A_i. Consequently, all varieties V_{lhk} necessarily contain the trivial solutions corresponding to $B_i \equiv A_i$, namely $\bar{\mathbf{x}}_i = \mathbf{a}_i - \mathbf{r}_i$, $i = 1\ldots 3$.

Inverse geometric problem. When the orientation is assigned, all vectors $\mathbf{r}_i$, $i = 1\ldots 3$, are known. If the equations

$$p_1 \coloneqq \det \mathbf{M}'_{1236} = 0, \quad p_2 \coloneqq \det \mathbf{M}'_{1235} = 0, \quad p_3 \coloneqq \det \mathbf{M}'_{1234} = 0 \qquad (22)$$

are considered, it may be proven that $V \equiv V_{123}$. Such equations comprise the lowest-degree polynomials among all minors of $\mathbf{M}'$. In particular, p_1 is quadratic in x and y, whereas p_2 and p_3 are quadratic in x, y and z. By eliminating z and y from Eq. (22), a 4th-degree polynomial equation in x may be obtained, i.e. $p_{123} = 0$. Since three roots of p_{123} necessarily correspond to trivial solutions, the fourth root is real and it may be computed by Vieta's formulas in closed form. The problem admits, thus, a single solution. Of course, it is admissible only if the corresponding cable tensions are nonnegative and the equilibrium is stable.

When the position $\mathbf{x}$ is assigned, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are unknown. If the Rodrigues parameters e_1, e_2 and e_3 are chosen to describe the platform orientation, the relations in Eq. (22) assume a particularly favorable structure. Indeed, after letting $\mathbf{r}_i = \mathbf{R}\mathbf{b}_i$, $i = 1\ldots 3$, p_1, p_2 and p_3 become quartic polynomials in e_1, e_2 and e_3. The minors of $\mathbf{M}'$ other than $\mathbf{M}'_{1234}$, $\mathbf{M}'_{1235}$ and $\mathbf{M}'_{1236}$ yield, instead, sextic equations in the Rodrigues parameters. Another useful quartic is obtained as follows. By setting the minors $\mathbf{M}'_{j456}$, $j = 1\ldots 3$, equal to zero, one has that $(\mathbf{x} + \mathbf{r}_1)\det \mathbf{M}'_{456,234} = \mathbf{0}$. The variety defined by the above equation comprises the trivial solution $\mathbf{x} = -\mathbf{r}_1$ and the set of all configurations for which

$$p_8 := \det \mathbf{M}'_{456,234} = 0. \tag{23}$$

Eq. (23) is, indeed, of degree four in e_1, e_2 and e_3 (and it is quadratic in x, y, z). It is well known that three polynomial equations of the same total degree always admit a Sylvester-type resultant free from extraneous polynomial factors [9]. For the case of three quartics, such a resultant is a univariate 64th-degree polynomial in one of the unknowns, say e_3. However, if the resultant of p_1, p_2 and p_3 is attempted to be computed, it appears to be identically nought. Conversely, if any one among p_1, p_2 and p_3 is replaced by p_8, the expected 64th-degree polynomial is obtained. The problem admits, thus, at the most 64 solutions.

Direct geometric problem. When cable lengths are assigned, the platform configuration has to be determined. Equation (5) provides up to 15 (non-independent) polynomial equations in the platform posture variables. Among them, Eqs. (22) and (23) are of degree four in e_1, e_2 and e_3 and degree two in x, y, z.

Equation (1) provides three further relations in the platform posture variables. In particular, one may conveniently consider

$$q_1 := |\mathbf{s}_1|^2 - \rho_1^2 = 0, \tag{24a}$$

$$q_2 := |\mathbf{s}_2|^2 - \rho_2^2 - |\mathbf{s}_1|^2 + \rho_1^2 = 0, \tag{24b}$$

$$q_3 := |\mathbf{s}_3|^2 - \rho_3^2 - |\mathbf{s}_1|^2 + \rho_1^2 = 0, \tag{24c}$$

which, after clearing the denominator $1 + e_1^2 + e_2^2 + e_3^2$, are quadratic in e_1, e_2 and e_3. q_1 is also quadratic in the elements of $\mathbf{x}$, whereas q_2 and q_3 are linear in these variables.

The point-to-point distance relations in Eq. (24) represent the typical constraints governing the forward kinematics of parallel manipulators with telescoping legs connected to the base and the platform by ball-and-socket joints. In particular, the DGP of the general Gough-Stewart manipulator depends on six equations of this sort, one of which is equivalent to Eq. (24a) and five more to Eqs. (24b)-(24c). This problem is known to be very difficult and it has attracted the interest of researchers for several years [10, 11]. The DGP of the 33-CDPR appears to be even more complex. In fact, three equations analogous to Eqs. (24b)-(24c), namely *linear* in the components of $\mathbf{x}$ and *quadratic* in the components of $\boldsymbol{\Phi}$, are replaced by relationships that are, at least, *quadratic* in the components of $\mathbf{x}$ and *quartic* in the components of $\boldsymbol{\Phi}$ (cf. Eqs. (22) and (23)). Possible simplifications may, indeed, arise from the fact that some power-products of x, y and z actually miss in the equations emerging from Eq. (5). The redundancy of such equations may also play a role. The problem appears to be a daunting task and it has not been solved yet.

5 Conclusions

This paper studied the kinematics and statics of under-constrained cable-driven parallel robots with less than six cables, in crane configuration. In these robots, kinematics and statics are intrinsically coupled and they must be dealt with simultaneously. This poses major challenges.

A geometrico-static model was presented, together with an original and general procedure aimed at effectively solving the inverse and direct position problems in analytical form. A spatial robot with three cables was considered as a case study, in order to show the feasibility of the presented approach. It was shown that the position problems that arise gain remarkable complexity with respect to those of analogous rigid-link robots, such as the Gough-Stewart manipulator. The inverse analysis may lead up to 64 solutions, when the platform position is assigned and the its orientation and the cable lengths must be determined. The direct problem, with the platform posture being unknown and the cable lengths being given, appears to be a much more difficult task and it will be the subject of future research.

A purely algebraic method, based on a constrained optimization formulation, was provided for the assessment of equilibrium stability. The method proposed in [5] differs from the one presented here in that it determines the stability of equilibrium by looking at the Hessian of an *unconstrained* potential, explicitly expressed as a function of a number of independent coordinates equal to the number of tensioned cables. Such a mapping is, generally, very difficult to obtain (indeed, Michael *et al.* [5] apply important simplifications on the geometry of the robot) and extensive differential symbolic computation is needed. The advantage of the method described here consists in that it relies on a reduced Hessian of which a purely algebraic formulation is provided, and it may be very simply applied to the most general cases, with no need to perform any differentiation.

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