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TWO−SYSTEMS IN THE SET OF FINITE DISPLACEMENT SCREWS PRODUCED BY A REVOLUTE−DYAD

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ABSTRACT

When a *coupler* is joined to *frame* by a *revolute−dyad, i.e.* by a link with revolute joints to coupler and frame, the displacements available to the coupler, if expressed as *finite displacement screws*, occupy a *3−system of screws*. The complexities of that 3−system can be conveniently analysed in terms of its component 2−systems which, in their own right, are known to be significant in the analysis of certain mechanisms, notably the Bennett and its relations. From the 3−system which sets the larger context, this paper derives new expressions which describe any contained 2−system in the localised terms of a basis of two contained screws.

1. Introduction

Rodrigues's equations [1], when expressed in their dualised form, *viz.*

$$\begin{bmatrix} \cos\hat{\theta} \\ \hat{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \cos\hat{\theta}_1 \cos\hat{\theta}_2 - \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \\ \cos\hat{\theta}_2 \, \hat{\mathbf{S}}_1 + \cos\hat{\theta}_1 \, \hat{\mathbf{S}}_2 - \hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2 \end{bmatrix} . \tag{1.1}$$

specify the sine−form *finite displacement screw* $\hat{\mathbf{S}} = \sin\hat{\theta}\,\hat{\mathbf{s}}$ – representing displacement through dual angle $2\hat{\theta}$ about a *unit screw axis* $\hat{\mathbf{s}}$ – which results from applying two successive finite displacements of a body, similarly specified: first $\hat{\mathbf{S}}_1 = \sin\hat{\theta}_1\,\hat{\mathbf{s}}_1$, and then $\hat{\mathbf{S}}_2 = \sin\hat{\theta}_2\,\hat{\mathbf{s}}_2$. Dividing $\hat{\mathbf{S}}$ by $\cos\hat{\theta}$ yields the tan−form screw $\hat{\mathbf{T}} = \tan\hat{\theta}\,\hat{\mathbf{s}}$ [2] which results from applying two such screws, first $\hat{\mathbf{T}}_1$ and then $\hat{\mathbf{T}}_2$, *viz.*

$$\hat{\mathbf{T}} = \frac{\hat{\mathbf{T}}_1 + \hat{\mathbf{T}}_2 - \hat{\mathbf{T}}_1 \times \hat{\mathbf{T}}_2}{1 - \hat{\mathbf{T}}_1 \cdot \hat{\mathbf{T}}_2} . \tag{1.2}$$

When, in a mechanism, a link Lnk_{12} has screw joints $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ to *coupler* and *frame* respectively, these formulations describe the set of displacements – specified as screws $\hat{\mathbf{S}}$ or $\hat{\mathbf{T}}$ and parameterised by $\hat{\theta}_1$, $\hat{\theta}_2$ –

which are available to the coupler as measured relative to frame.

Generally, these are *dual−linear* sets consisting of linear combinations of screws which, as eqns. (1.1,2) typify, are constructed with dual coefficients [3]. However, under certain kinematic specialisations these sets are found to consist of real−linear combinations of a small number of basis screws [4, 5, 6, 7] and so are very easily interpreted. Notably, Huang [8] has observed that when the axes $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$ are sites of *revolute* joints, so that the angles θ_1, θ_2 are purely real, the screws of eqns. (1.1,2) are real−linear combinations of $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$, and $\hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2$ and so conform to the well−understood geometry of the *3−system* [9, 10].

Huang [11] has gone on to show that in certain mechanisms such as the Bennett, a sub−set of the screws of that 3−system, in the form of the familiar *2−system* [12, 9], is central to understanding of the mechanism. However, to this point in time it has not been possible to write down an expression which, though incorporating parameters of the containing 3−system, represents just the screws of such a 2−system and no others.

This difficulty is solved in the present paper: a parameterisation is discovered (in Section 7) which quite generally allows the 2−system to be expressed as a linear combination of two screws whose real coefficients take simple functional forms.

2. Notation and Basic Geometry

Throughout this paper a *screw* will typically be written as a 3−vector of dual numbers

$$\hat{\mathbf{G}} = |\hat{\mathbf{G}}|(1+\varepsilon p)\hat{\mathbf{g}}\ , \quad \hat{\mathbf{g}} = \mathbf{l}+\varepsilon\mathbf{M}\ , \quad \hat{\mathbf{g}}^2 = \mathbf{l}^2+\varepsilon 2\ \mathbf{l}\cdot\mathbf{M} = 1+\varepsilon 0\ , \quad \mathbf{l}\times\mathbf{M} = \mathbf{R}\ . \quad (2.1)$$

in which ε is a quasi−sacalar satisfying $\varepsilon^2 = 0$ and such that for all real a, b, c, and d, $(a+\varepsilon b = c+\varepsilon d) \Leftrightarrow (a = c) \wedge (b = d)$. Bold letters represent 3−vectors, lower case bold letters indicate *unit* vectors, and the overwritten 'hat' symbol indicates dual quantities. $|\hat{\mathbf{G}}|$ is the *real magnitude* and p is the *pitch* of the screw $\hat{\mathbf{G}}$ which is located spatially by its *normalised line* $\hat{\mathbf{g}}$, of unit magnitude and zero pitch, with direction vector $\mathbf{l} = (l,\ m,\ n)$ and moment vector $\mathbf{M} = (P,\ Q,\ R)$ which together determine its *origin−radius* vector $\mathbf{R}$. The values $l,\ m,\ n,\ P,\ Q,\ R$ are Plücker coordinates of that line.

For any screws $\hat{\mathbf{G}}_1 = \mathbf{G}_1+\varepsilon\mathbf{G}_{p\,1}$ and $\hat{\mathbf{G}}_2 = \mathbf{G}_2+\varepsilon\mathbf{G}_{p\,2}$, their *scalar product* is

$$\hat{\mathbf{G}}_1\cdot\hat{\mathbf{G}}_2 = \mathbf{G}_1\cdot\mathbf{G}_2+\varepsilon\,\hat{\mathbf{G}}_1 @ \hat{\mathbf{G}}_2 \quad \textit{where} \quad \hat{\mathbf{G}}_1 @ \hat{\mathbf{G}}_2 = \mathbf{G}_1\cdot\mathbf{G}_{p\,2}+\mathbf{G}_2\cdot\mathbf{G}_{p\,1}\ ,$$

in which $\hat{\mathbf{G}}_1 @ \hat{\mathbf{G}}_2$ is the *mutual moment* of the screws $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{G}}_2$. Two screws $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{G}}_2$ are said to be *perpendicular* if $\mathbf{G}_1\cdot\mathbf{G}_2 = 0$, to be *reciprocal* if $\hat{\mathbf{G}}_1 @ \hat{\mathbf{G}}_2 = 0$. We shall call them *orthogonal* if both of these are true, *i.e.* if $\hat{\mathbf{G}}_1\cdot\hat{\mathbf{G}}_2 = 0$, which implies that each intersects the other at right angles. Their *cross product* screw $\hat{\mathbf{G}}_1\times\hat{\mathbf{G}}_2$ is sited in the *common perpendicular line* of $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{G}}_2$.

3. Provenance of the Dual−Linear 3−System

Let the successive displacements referred to at eqn. (1.1) take place about unit screw−axes $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ which are spatially separated by the dual angle $2\hat{\phi}_{12} \equiv 2\phi_{12}+\varepsilon 2 d_{12}$ as measured from $\hat{\mathbf{s}}_1$ to $\hat{\mathbf{s}}_2$. For brevity we write

$$c \equiv \cos\phi_{12}\ , \quad \hat{c} \equiv \cos\hat{\phi}_{12} \quad \textit{and} \quad s \equiv \sin\phi_{12}\ , \quad \hat{s} \equiv \sin\hat{\phi}_{12}\ , \qquad (3.1)$$

so that $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{s}}_2 = \cos 2\hat{\phi}_{12} = \hat{c}^2-\hat{s}^2$. In analysing their resultant displacements, it is convenient to identify reference frame axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ which lie on the mid−lines and common perpendicular of $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$, *viz.*

$$\left.\begin{aligned}
\hat{\mathbf{X}} &= \hat{c}\,\hat{\mathbf{x}} &&= c(1+\varepsilon P_X)\,\hat{\mathbf{x}} &&= \frac{\hat{\mathbf{s}}_2+\hat{\mathbf{s}}_1}{2}\,, \quad P_X = -d_{12}\tan\phi_{12}\,, \\
\hat{\mathbf{Y}} &= \hat{s}\,\hat{\mathbf{y}} &&= s(1+\varepsilon P_Y)\,\hat{\mathbf{y}} &&= \frac{\hat{\mathbf{s}}_2-\hat{\mathbf{s}}_1}{2}\,, \quad P_Y = d_{12}\cot\phi_{12}\,, \\
\hat{\mathbf{Z}} &= \hat{\mathbf{X}}\times\hat{\mathbf{Y}} &&= cs(1+\varepsilon[P_X+P_Y])\,\hat{\mathbf{z}} &&= \frac{\hat{\mathbf{s}}_1\times\hat{\mathbf{s}}_2}{2}\,, \quad P_Z = 2d_{12}\cot 2\phi_{12}\,,
\end{aligned}\right\} \qquad (3.2)$$

The axial lines $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are of zero pitch and unit magnitude and intersect at right angles in an origin at the mid−point between the axes $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ on their common perpendicular line $\hat{\mathbf{z}}$, satisfying

$$\left.\begin{aligned}
&\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0\,, \quad \hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2 = \hat{\mathbf{z}}^2 = 1 && \textit{ortho−normality} \\
&\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\,, \quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}\,, \quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}} && \textit{right−handedness}
\end{aligned}\right\}. \qquad (3.3)$$

If we write

$$\hat{\xi} = \frac{\cot\hat{\theta}_2+\cot\hat{\theta}_1}{2}\,, \quad \hat{\eta} = \frac{\cot\hat{\theta}_2-\cot\hat{\theta}_1}{2}\,, \quad \textit{i.e.} \quad \cot\hat{\theta}_1 = \hat{\xi}-\hat{\eta}\,, \quad \cot\hat{\theta}_2 = \hat{\xi}+\hat{\eta}\,, \qquad (3.4)$$

we find that the cosine formulation of eqn. (1.1) yields

$$\cos\hat{\theta} = \cos\hat{\theta}_1\cos\hat{\theta}_2-\sin\hat{\theta}_1\sin\hat{\theta}_2\cos 2\hat{\phi}_{12} = \sin\hat{\theta}_1\sin\hat{\theta}_2\,[(\hat{\xi}^2-\hat{\eta}^2)-(\hat{c}^2-\hat{s}^2)]\,, \qquad (3.5)$$

and, with $\hat{\mathbf{G}}$ defined to be the screw given by

$$\hat{\mathbf{G}} = \hat{\xi}\,\hat{\mathbf{X}} - \hat{\eta}\,\hat{\mathbf{Y}} - \hat{\mathbf{Z}} = \hat{\xi}\,\hat{c}\,\hat{\mathbf{x}} - \hat{\eta}\,\hat{s}\,\hat{\mathbf{y}} - \hat{c}\,\hat{s}\,\hat{\mathbf{z}}\,, \qquad (3.6)$$

the sin−screw resultant of eqn. (1.1) may be written

$$\hat{\mathbf{S}} = \sin\hat{\theta}_1\sin\hat{\theta}_2\,\{\cot\hat{\theta}_2\,\hat{\mathbf{s}}_1 + \cot\hat{\theta}_1\,\hat{\mathbf{s}}_2 - \hat{\mathbf{s}}_1\times\hat{\mathbf{s}}_2\} = 2\sin\hat{\theta}_1\sin\hat{\theta}_2\,\hat{\mathbf{G}}\,; \qquad (3.7)$$

and, on division of this by eqn. (3.5), the tan−screw resultant may be written

$$\hat{\mathbf{T}} = \frac{2}{\cot\hat{\theta}_1\cot\hat{\theta}_2-\cos 2\hat{\phi}_{12}}\,\hat{\mathbf{G}} = \frac{2}{(\hat{\xi}^2-\hat{\eta}^2)-(\hat{c}^2-\hat{s}^2)}\,\hat{\mathbf{G}}\,. \qquad (3.8)$$

In all of these expressions the independent coefficients $\hat{\xi}$ and $\hat{\eta}$ take on all possible dual values under permitted variation of the parameters $\hat{\theta}_1$ and $\hat{\theta}_2$.

We observe that the tan−screw $\hat{\mathbf{T}}$ becomes infinite under the condition

$$(\hat{\xi}^2-\hat{\eta}^2)-(\hat{c}^2-\hat{s}^2) = 0\,, \quad \textit{i.e.} \quad \hat{\xi}^2-\hat{\eta}^2 = \hat{c}^2-\hat{s}^2\,. \qquad (3.9\text{a,b})$$

Equation (3.5) shows this to occur when $\cos\hat{\theta} = 0+\varepsilon 0$ which implies that the resultant screw, $\hat{\mathbf{S}}$ or $\hat{\mathbf{T}}$, then represents a *pure half−turn*; *i.e.* a displacement in which the translation distance is zero, $2\sigma = 0$, and the rotation is a half−turn, $2\theta = \pi$. Since these are of special significance, we shall label a screw with the suffix π, thus $\hat{\mathbf{G}}_\pi$, if its coefficients $\hat{\xi}$, $\hat{\eta}$ satisfy eqn. (3.9), thereby indicating that its site is occupied by a *pure half−turn* screw.

Although much following analysis relates to the identification of such pure half−turn screws, we must observe that the situation of interest in this paper – where the parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ and, consequently, the parameters $\hat{\xi}$ and $\hat{\eta}$ are real, not dual – does not permit pure half−turn screws to exist; eqn. (3.9b), whose right−hand side is generally dual, cannot be satisfied by purely real values of $\hat{\xi}$ and $\hat{\eta}$ on the left−hand side. It follows, therefore, that when (in Section 7) we seek solutions for pure half−turn screws, it will be *purely imaginary* screws that are sought.

When working with subsets of the screws $\hat{\mathbf{T}}$ provided by eqn. (3.8), it difficult at the outset to interpret the leading coefficient of that expression in a way which is specific to the subset of choice. We therefore proceed by the roundabout route of firstly considering dual 2−systems parameterised by $\hat{\xi}$ and $\hat{\eta}$ in the screw $\hat{\mathbf{G}}$ of eqn. (3.5), although this is not a finite displacement screw of recognised definition. Treatment of the complicating magnitude factor which appears with $\hat{\mathbf{T}}$ in eqn. (3.8) is deferred to Section 6.

4. Nodal Line Identification of a Dual−Linear 2−System

Let us define a *nodal line* to be the common perpendicular line of any two given generators

$$\hat{\mathbf{G}}_A = \hat{\xi}_A \hat{c}\,\hat{\mathbf{x}} - \hat{\eta}_A \hat{s}\,\hat{\mathbf{y}} - \hat{c}\hat{s}\,\hat{\mathbf{z}} \quad \textit{and} \quad \hat{\mathbf{G}}_B = \hat{\xi}_B \hat{c}\,\hat{\mathbf{x}} - \hat{\eta}_B \hat{s}\,\hat{\mathbf{y}} - \hat{c}\hat{s}\,\hat{\mathbf{z}}\ , \tag{4.1}$$

specified as in eqn. (3.6). We now identify a general form for *all* generators $\hat{\mathbf{G}}$ which, like $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$, are *orthogonal* to such a nodal line, intersecting it at right angles. The common perpendicular of $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$ is sited in their cross product screw, *viz.*

$$\begin{aligned}\hat{\mathbf{N}} \equiv \hat{\mathbf{G}}_A \times \hat{\mathbf{G}}_B &= (\hat{\eta}_A - \hat{\eta}_B)\hat{c}\hat{s}^2\,\hat{\mathbf{x}} + (\hat{\xi}_A - \hat{\xi}_B)\hat{c}^2\hat{s}\,\hat{\mathbf{y}} + (\hat{\xi}_B\hat{\eta}_A - \hat{\xi}_A\hat{\eta}_B)\hat{c}\hat{s}\,\hat{\mathbf{z}} \\ &= \hat{c}^2\hat{s}^2\{(\hat{\eta}_A - \hat{\eta}_B)\frac{\hat{\mathbf{x}}}{\hat{c}} + (\hat{\xi}_A - \hat{\xi}_B)\frac{\hat{\mathbf{y}}}{\hat{s}} + (\hat{\xi}_B\hat{\eta}_A - \hat{\xi}_A\hat{\eta}_B)\frac{\hat{\mathbf{z}}}{\hat{c}\hat{s}}\}\ ,\end{aligned} \tag{4.2}$$

which expresses $\hat{\mathbf{N}}$ as a dual−linear combination of screws $\hat{\mathbf{x}}/\hat{c}$, $\hat{\mathbf{y}}/\hat{s}$, and $\hat{\mathbf{z}}/\hat{c}\hat{s}$ which are respectively *reciprocal* to the axial screws $\hat{\mathbf{X}} = \hat{c}\,\hat{\mathbf{x}}$, $\hat{\mathbf{Y}} = \hat{s}\,\hat{\mathbf{y}}$, and $\hat{\mathbf{Z}} = \hat{c}\hat{s}\,\hat{\mathbf{z}}$ (*e.g.* $\hat{\mathbf{X}}\cdot\hat{\mathbf{x}}/\hat{c} = 1$, *etc.*). It is convenient to separate the z−coefficient into terms containing the x− and y−coefficients as factors. With sufficient generality we write

$$\hat{\xi}_B\hat{\eta}_A - \hat{\xi}_A\hat{\eta}_B = (\hat{\eta}_A - \hat{\eta}_B)\frac{\hat{\tau}\hat{\xi}_A + \hat{\xi}_B}{\hat{\tau}+1} - (\hat{\xi}_A - \hat{\xi}_B)\frac{\hat{\tau}\hat{\eta}_A + \hat{\eta}_B}{\hat{\tau}+1}$$

for all dual values $\hat{\tau}$ such that the real part of $\hat{\tau}+1$ does not vanish, *i.e.* $\Re(\hat{\tau}) \neq -1$. Then

$$\hat{\mathbf{G}}_A \times \hat{\mathbf{G}}_B = \hat{c}^2\hat{s}^2\{(\hat{\eta}_A - \hat{\eta}_B)\left[\frac{\hat{\mathbf{x}}}{\hat{c}} + \frac{\hat{\tau}\hat{\xi}_A + \hat{\xi}_B}{\hat{\tau}+1}\,\frac{\hat{\mathbf{z}}}{\hat{c}\hat{s}}\right] + (\hat{\xi}_A - \hat{\xi}_B)\left[\frac{\hat{\mathbf{y}}}{\hat{s}} - \frac{\hat{\tau}\hat{\eta}_A + \hat{\eta}_B}{\hat{\tau}+1}\,\frac{\hat{\mathbf{z}}}{\hat{c}\hat{s}}\right]\}\ .$$

Now the requirement that, in place of $\hat{\mathbf{G}}_B$, a general generator $\hat{\mathbf{G}} = \hat{\xi}\,\hat{\mathbf{X}} - \hat{\eta}\,\hat{\mathbf{Y}} - \hat{\mathbf{Z}}$ should, with $\hat{\mathbf{G}}_A$, form a cross product $\hat{\mathbf{G}}_A \times \hat{\mathbf{G}}$ lying on the same line as $\hat{\mathbf{N}}$, implies two conditions to be met. Firstly, that both of the square−braced vectors in this expression – specifically, their $\hat{\mathbf{z}}$−components – should be invariant under replacement of $\hat{\xi}_B$ and $\hat{\eta}_B$ with $\hat{\xi}$ and $\hat{\eta}$ respectively: that is, for all $\hat{\tau}$ and $\hat{\tau}'$,

$$\frac{\hat{\tau}\hat{\xi}_A + \hat{\xi}}{\hat{\tau}+1} = \frac{\hat{\tau}'\hat{\xi}_A + \hat{\xi}_B}{\hat{\tau}'+1} \quad \textit{and} \quad \frac{\hat{\tau}\hat{\eta}_A + \hat{\eta}}{\hat{\tau}+1} = \frac{\hat{\tau}'\hat{\eta}_A + \hat{\eta}_B}{\hat{\tau}'+1}\ ,$$

from which it follows that

$$\hat{\xi} = (\hat{\tau}+1)\frac{\hat{\tau}'\hat{\xi}_A + \hat{\xi}_B}{\hat{\tau}'+1} - \hat{\tau}\hat{\xi}_A = \frac{\{(\hat{\tau}'+1) - (\hat{\tau}+1)\}\hat{\xi}_A + (\hat{\tau}+1)\hat{\xi}_B}{\hat{\tau}'+1}\ ,$$

and correspondingly for $\hat{\eta}$. Equivalently, on introducing an alternative parameter $\hat{\gamma} = (\hat{\tau}+1)/(\hat{\tau}'+1)$,

$$\hat{\xi} = (1-\hat{\gamma})\hat{\xi}_A+\hat{\gamma}\hat{\xi}_B = \hat{\xi}_A+\hat{\gamma}(\hat{\xi}_B-\hat{\xi}_A) \quad and \quad \hat{\eta} = (1-\hat{\gamma})\hat{\eta}_A+\hat{\gamma}\hat{\eta}_B = \hat{\eta}_A+\hat{\gamma}(\hat{\eta}_B-\hat{\eta}_A) \tag{4.3}$$

Secondly, we require that the screws $\hat{\mathbf{G}}_A\times\hat{\mathbf{G}}_B$ and $\hat{\mathbf{G}}_A\times\hat{\mathbf{G}}$, while possibly differing in magnitude and pitch, should not differ in direction or location. That is, the ratio of the leading coefficients

$$\hat{\xi}_A-\hat{\xi} = (1-1+\hat{\gamma})\hat{\xi}_A-\hat{\gamma}\hat{\xi}_B = \hat{\gamma}(\hat{\xi}_A-\hat{\xi}_B) \quad and \quad \hat{\eta}_A-\hat{\eta} = (1-1+\hat{\gamma})\hat{\eta}_A-\hat{\gamma}\hat{\eta}_B = \hat{\gamma}(\hat{\eta}_A-\hat{\eta}_B) \,,$$

must be invariant under respective interchange of $\hat{\xi}_B$ and $\hat{\eta}_B$ with $\hat{\xi}$ and $\hat{\eta}$. But, observably, this condition is already satisfied.

So, for *basis screws* $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$ of eqn. (4.1), the typical generator which is orthogonal to their common perpendicular nodal line is shown by eqns. (4.3) to have, for all $\hat{\gamma}$, the dual−linear form

$$\hat{\mathbf{G}} = (1-\hat{\gamma})\hat{\mathbf{G}}_A+\hat{\gamma}\hat{\mathbf{G}}_B = \{(1-\hat{\gamma})\hat{\xi}_A+\hat{\gamma}\hat{\xi}_B\}\,\hat{\mathbf{X}} - \{(1-\hat{\gamma})\hat{\eta}_A+\hat{\gamma}\hat{\eta}_B\}\,\hat{\mathbf{Y}} - \hat{\mathbf{Z}} \,. \tag{4.4}$$

This result extends the familiar notion of a linear *2−system* in real coefficients [9] to that of a *dual−linear 2−system* in dual coefficients.

It is convenient to introduce the parameter form $\hat{\mu} = 1-\hat{\gamma}$ so that the linear screw combination of eqn. (4.4) may be written more compactly as $\hat{\mathbf{G}} = \hat{\mu}\,\hat{\mathbf{G}}_A+\hat{\gamma}\,\hat{\mathbf{G}}_B$. The following identities then apply:

$$\hat{\mu}+\hat{\gamma} = 1 \,, \quad \hat{\mu}^2+\hat{\gamma}^2 = 1-2\hat{\mu}\hat{\gamma} \,, \quad \hat{\mu}^2-\hat{\gamma}^2 = \hat{\mu}-\hat{\gamma} \,. \tag{4.5}$$

On an original basis $\hat{\mathbf{G}}_A$, $\hat{\mathbf{G}}_B$, to which parameter $\hat{\gamma}$ applies, we may select new basis screws $\hat{\mathbf{G}}_X$, $\hat{\mathbf{G}}_Y$, *viz.*

$$\begin{bmatrix}\hat{\mathbf{G}}_X{}^T\\ \hat{\mathbf{G}}_Y{}^T\end{bmatrix} = \begin{bmatrix}\hat{\mu}_X & \hat{\gamma}_X\\ \hat{\mu}_Y & \hat{\gamma}_Y\end{bmatrix}\begin{bmatrix}\hat{\mathbf{G}}_A{}^T\\ \hat{\mathbf{G}}_B{}^T\end{bmatrix}, \quad i.e. \quad \begin{bmatrix}\hat{\mathbf{G}}_A{}^T\\ \hat{\mathbf{G}}_B{}^T\end{bmatrix} = \frac{1}{\hat{\gamma}_Y-\hat{\gamma}_X}\begin{bmatrix}\hat{\gamma}_Y & -\hat{\gamma}_X\\ -\hat{\mu}_Y & \hat{\mu}_X\end{bmatrix}\begin{bmatrix}\hat{\mathbf{G}}_X{}^T\\ \hat{\mathbf{G}}_Y{}^T\end{bmatrix}, \tag{4.6}$$

for which the determinant $\hat{\mu}_X\hat{\gamma}_Y-\hat{\mu}_Y\hat{\gamma}_X = \hat{\gamma}_Y-\hat{\gamma}_X$ does not vanish. By use of a new parameter $\hat{\gamma}'$, we may then generate the general linear combination screw from the new basis, as

$$\hat{\mathbf{G}} = \hat{\mu}\,\hat{\mathbf{G}}_A+\hat{\gamma}\,\hat{\mathbf{G}}_B = \hat{\mu}\,\frac{\hat{\gamma}_Y\,\hat{\mathbf{G}}_X-\hat{\gamma}_X\,\hat{\mathbf{G}}_Y}{\hat{\gamma}_Y-\hat{\gamma}_X}-\hat{\gamma}\,\frac{\hat{\mu}_Y\,\hat{\mathbf{G}}_X-\hat{\mu}_X\,\hat{\mathbf{G}}_Y}{\hat{\gamma}_Y-\hat{\gamma}_X} = \hat{\mu}'\,\hat{\mathbf{G}}_X+\hat{\gamma}'\,\hat{\mathbf{G}}_Y \,, \tag{4.7}$$

where the parameter $\hat{\gamma}$ applicable to the original basis is related to the parameter $\hat{\gamma}'$ of the new basis by

$$\hat{\gamma} = \hat{\mu}'\hat{\gamma}_X+\hat{\gamma}'\hat{\gamma}_Y \,, \quad \hat{\mu} = \hat{\mu}'\hat{\mu}_X+\hat{\gamma}'\hat{\mu}_Y \quad and \quad \hat{\gamma}' = \frac{\hat{\gamma}-\hat{\gamma}_X}{\hat{\gamma}_Y-\hat{\gamma}_X} \,, \quad \hat{\mu}' = \frac{\hat{\mu}-\hat{\mu}_Y}{\hat{\gamma}_Y-\hat{\gamma}_X} \,. \tag{4.8}$$

5. Solving for Half−Turn Screws

To discover structure among the ∞^2 screws parameterised by $\hat{\gamma}$ in the dual−linear system of eqn. (4.4), let us identify such *pure half−turn* screws as it contains. For typical basis screws $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$ as previously assumed given, we determine those particular values $\hat{\gamma}_\pi$ of parameter $\hat{\gamma}$ for which the quantities $\hat{\xi}$ and $\hat{\eta}$ of eqns. (4.3) satisfy eqn. (3.9), *viz.*

$$\begin{aligned} 0 &= \{\hat{\xi}_A+\hat{\gamma}_\pi(\hat{\xi}_B-\hat{\xi}_A)\}^2-\{\hat{\eta}_A+\hat{\gamma}_\pi(\hat{\eta}_B-\hat{\eta}_A)\}^2-(\hat{c}^2-\hat{s}^2) \\ &= \hat{\mathcal{A}}\hat{\gamma}_\pi{}^2+2\hat{\mathcal{B}}\hat{\gamma}_\pi+\hat{\mathcal{C}} \,, \end{aligned} \tag{5.1}$$

in which it is convenient to define:

$$\left.\begin{aligned}\hat{\mathcal{A}} &= (\hat{\xi}_B-\hat{\xi}_A)^2-(\hat{\eta}_B-\hat{\eta}_A)^2 \ , \\ \hat{\mathcal{B}} &= \hat{\xi}_A(\hat{\xi}_B-\hat{\xi}_A)-\hat{\eta}_A(\hat{\eta}_B-\hat{\eta}_A) \ = \hat{\xi}_A\hat{\xi}_B-\hat{\eta}_A\hat{\eta}_B-(\hat{\xi}_A{}^2-\hat{\eta}_A{}^2) \ , \\ \hat{\mathcal{C}} &= (\hat{\xi}_A{}^2-\hat{\eta}_A{}^2)-(\hat{c}^2-\hat{s}^2) \ . \end{aligned}\right\} \tag{5.2}$$

The quadratic eqn. (5.1) has solutions

$$\hat{\gamma}_\pi = \frac{-\hat{\mathcal{B}}\pm\sqrt{\hat{\Delta}}}{\hat{\mathcal{A}}} \ , \quad \textit{i.e.} \quad \hat{\gamma}_{\pi_A} = \frac{-\hat{\mathcal{B}}-\sqrt{\hat{\Delta}}}{\hat{\mathcal{A}}} \quad \textit{and} \quad \hat{\gamma}_{\pi_B} = \frac{-\hat{\mathcal{B}}+\sqrt{\hat{\Delta}}}{\hat{\mathcal{A}}} \ , \tag{5.3}$$

in which the discriminant of the quadratic − $4\hat{\Delta}$ by standard definition − is specified by

$$\hat{\Delta} = \hat{\mathcal{B}}^2-\hat{\mathcal{A}}\hat{\mathcal{C}} = (\hat{\xi}_B\hat{\eta}_A-\hat{\eta}_B\hat{\xi}_A)^2+\{(\hat{\xi}_B-\hat{\xi}_A)^2-(\hat{\eta}_B-\hat{\eta}_A)^2\}(\hat{c}^2-\hat{s}^2) \ . \tag{5.4}$$

Thus, when real basis screws $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$ are given, the quadratic eqn. (5.1) determines values $\hat{\gamma}_{\pi_A}$ and $\hat{\gamma}_{\pi_B}$ which, according as the discriminant $\hat{\Delta}$ is *positive*, *zero*, or *negative*, are respectively *real and distinct*, *real and coincident*, or mutual *complex conjugates*. Correspondingly, the *pure half−turn* site screws

$$\hat{\mathbf{G}}_{\pi_A} = (1-\hat{\gamma}_{\pi_A})\hat{\mathbf{G}}_A+\hat{\gamma}_{\pi_A}\hat{\mathbf{G}}_B \quad \textit{and} \quad \hat{\mathbf{G}}_{\pi_B} = (1-\hat{\gamma}_{\pi_B})\hat{\mathbf{G}}_A+\hat{\gamma}_{\pi_B}\hat{\mathbf{G}}_B \ , \tag{5.5}$$

obtained by substituting those values into eqn. (4.4), are respectively *real and distinct*, *real and coincident*, or *complex*. In the last of these cases, where they are *complex*, the screws continue to be well defined in mathematical terms although they cannot be realised in a practical kinematic context.

Further to the definitions of eqns. (5.1) it is convenient to define a quantity analogous to $\hat{\mathcal{B}}$, *viz.*

$$\hat{\mathcal{E}} = \hat{\mathcal{A}}+\hat{\mathcal{B}} = -\hat{\xi}_A\hat{\xi}_B+\hat{\eta}_A\hat{\eta}_B+(\hat{\xi}_B{}^2-\hat{\eta}_B{}^2) = \hat{\xi}_B(\hat{\xi}_B-\hat{\xi}_A)-\hat{\eta}_B(\hat{\eta}_B-\hat{\eta}_A) \ . \tag{5.6}$$

6. Conversion to the Finite Displacement Tan−Screw

We now identify the tan−screw $\hat{\mathbf{T}}$ which is sited in the typical generator $\hat{\mathbf{G}}$. For simplicity we assume that two *pure half−turn* screws, identified as in the previous section, are present. We adopt these as basis, writing them in the form of eqn. (3.6), *viz.*

$$\hat{\mathbf{G}}_{\pi_A} = \hat{\xi}_{\pi_A}\hat{\mathbf{X}} - \hat{\eta}_{\pi_A}\hat{\mathbf{Y}} - \hat{\mathbf{Z}} \quad \textit{and} \quad \hat{\mathbf{G}}_{\pi_B} = \hat{\xi}_{\pi_B}\hat{\mathbf{X}} - \hat{\eta}_{\pi_B}\hat{\mathbf{Y}} - \hat{\mathbf{Z}} \ .$$

The typical generator $\hat{\mathbf{G}}$ of the dual−linear 2−system defined by those screws is specified, for all dual values of the parameter $\hat{\gamma}$, by

$$\hat{\mathbf{G}} = \hat{\mu}\hat{\mathbf{G}}_{\pi_A}+\hat{\gamma}\hat{\mathbf{G}}_{\pi_B} \quad \textit{where} \quad \hat{\mu} = (1-\hat{\gamma}) \ ,$$

so we learn that the general specification of $\hat{\mathbf{G}}$ is

$$\hat{\mathbf{G}} = \hat{\xi}\hat{\mathbf{X}} - \hat{\eta}\hat{\mathbf{Y}} - \hat{\mathbf{Z}} \quad \textit{for} \quad \hat{\xi} = \hat{\mu}\hat{\xi}_{\pi_A}+\hat{\gamma}\hat{\xi}_{\pi_B} \ , \quad \hat{\eta} = \hat{\mu}\hat{\eta}_{\pi_A}+\hat{\gamma}\hat{\eta}_{\pi_B} \ .$$

From these expressions for $\hat{\xi}$ and $\hat{\eta}$ we determine that

$$\hat{\xi}^2-\hat{\eta}^2 = \hat{\mu}^2(\hat{\xi}_{\pi_A}{}^2-\hat{\eta}_{\pi_A}{}^2)+\hat{\gamma}^2(\hat{\xi}_{\pi_B}{}^2-\hat{\eta}_{\pi_B}{}^2)+2\hat{\mu}\hat{\gamma}(\hat{\xi}_{\pi_A}\hat{\xi}_{\pi_B}-\hat{\eta}_{\pi_A}\hat{\eta}_{\pi_B}) \ .$$

Now, since the screws $\hat{\mathbf{G}}_{\pi_A}$ and $\hat{\mathbf{G}}_{\pi_B}$ are sites of *pure half−turn* screws, by eqn. (3.10) we have

$$\hat{\xi}_{\pi_A}{}^2 - \hat{\eta}_{\pi_A}{}^2 \;=\; \hat{\xi}_{\pi_B}{}^2 - \hat{\eta}_{\pi_B}{}^2 \;=\; \hat{c}^2 - \hat{s}^2 \;,$$

so it follows, on making the replacement $\hat{\mu}^2 + \hat{\gamma}^2 \;=\; 1 - 2\hat{\mu}\hat{\gamma}$, that

$$\begin{aligned}(\hat{\xi}^2 - \hat{\eta}^2) - (\hat{c}^2 - \hat{s}^2) \;&=\; -\hat{\mu}\hat{\gamma}[\hat{\xi}_{\pi_A}{}^2 - \hat{\eta}_{\pi_A}{}^2 \;+\; \hat{\xi}_{\pi_B}{}^2 - \hat{\eta}_{\pi_B}{}^2 \;-\; 2(\hat{\xi}_{\pi_A}\hat{\xi}_{\pi_B} - \hat{\eta}_{\pi_A}\hat{\eta}_{\pi_B})] \\ &=\; -\hat{\mu}\hat{\gamma}\hat{\mathcal{A}}_{\pi_A\pi_B}\end{aligned}$$

where, on analogy with the definition of $\hat{\mathcal{A}}$ applying to $\hat{\mathbf{G}}_A$ and $\hat{\mathbf{G}}_B$ at eqns. (5.2), we have written

$$\hat{\mathcal{A}}_{\pi_A\pi_B} \;=\; (\hat{\xi}_{\pi_A} - \hat{\xi}_{\pi_B})^2 - (\hat{\eta}_{\pi_A} - \hat{\eta}_{\pi_B})^2 \;, \qquad (6.1)$$

for the quantity which applies correspondingly to the *half−turn screws* $\hat{\mathbf{G}}_{\pi_A}$ and $\hat{\mathbf{G}}_{\pi_B}$. This quantity is a constant of the chosen nodal line. So, using eqn. (3.8), we can convert the typical generator $\hat{\mathbf{G}}$ into the tan−screw $\hat{\mathbf{T}}$ at the same site, *viz.*

$$\hat{\mathbf{T}} \;=\; -\frac{2}{\hat{\mathcal{A}}_{\pi_A\pi_B}}\,\frac{\hat{\mu}\,\hat{\mathbf{G}}_{\pi_A} + \hat{\gamma}\,\hat{\mathbf{G}}_{\pi_B}}{\hat{\mu}\,\hat{\gamma}} \;=\; -\frac{2}{\hat{\mathcal{A}}_{\pi_A\pi_B}}\left(\frac{1}{\hat{\gamma}}\,\hat{\mathbf{G}}_{\pi_A} + \frac{1}{\hat{\mu}}\,\hat{\mathbf{G}}_{\pi_B}\right) . \qquad (6.2a,b)$$

Alternatively, with $\hat{\kappa} \;=\; \kappa + \varepsilon\kappa_0$, $-\infty \;<\; \kappa \;<\; \infty$, $-\infty \;<\; \kappa_0 \;<\; \infty$, defined by

$$\hat{\gamma} \;=\; ½(1+\hat{\kappa}) \quad \textit{so} \quad \hat{\mu} \;=\; 1-\hat{\gamma} \;=\; ½(1-\hat{\kappa}) \quad \textit{and} \quad \hat{\mu}\hat{\gamma} \;=\; \frac{1}{4}(1-\hat{\kappa}^2) \;,$$

we have

$$\hat{\mathbf{T}} \;=\; -\frac{4}{\hat{\mathcal{A}}_{\pi_A\pi_B}}\,\frac{(1-\hat{\kappa})\,\hat{\mathbf{G}}_{\pi_A} + (1+\hat{\kappa})\,\hat{\mathbf{G}}_{\pi_B}}{1-\hat{\kappa}^2} \;=\; -\frac{4}{\hat{\mathcal{A}}_{\pi_A\pi_B}}\,\frac{(\hat{\mathbf{G}}_{\pi_B} + \hat{\mathbf{G}}_{\pi_A}) + \hat{\kappa}(\hat{\mathbf{G}}_{\pi_B} - \hat{\mathbf{G}}_{\pi_A})}{1-\hat{\kappa}^2} \;. \qquad (6.3a,b)$$

If we re−express $\hat{\kappa} \;=\; \tan\hat{\psi}$, in terms of a dual angle parameter $\hat{\psi} \;=\; \psi + \varepsilon d$, $-\pi \;\le\; \psi \;\le\; \pi$, $-\infty \;\le\; d \;\le\; \infty$, we obtain

$$\hat{\mathbf{T}} \;=\; -\frac{4\cos\hat{\psi}}{\hat{\mathcal{A}}_{\pi_A\pi_B}}\,\frac{\cos\hat{\psi}(\hat{\mathbf{G}}_{\pi_B} + \hat{\mathbf{G}}_{\pi_A}) + \sin\hat{\psi}(\hat{\mathbf{G}}_{\pi_B} - \hat{\mathbf{G}}_{\pi_A})}{\cos^2\hat{\psi} - \sin^2\hat{\psi}} \;. \qquad (6.4a,b)$$

We observe that the sum and difference screws $\hat{\mathbf{G}}_{\pi_B} + \hat{\mathbf{G}}_{\pi_A}$ and $\hat{\mathbf{G}}_{\pi_B} - \hat{\mathbf{G}}_{\pi_A}$ are not mutually orthogonal since $\hat{\mathbf{G}}_{\pi_A}$ and $\hat{\mathbf{G}}_{\pi_B}$ have different magnitudes in general. In fact we learn from eqn. (3.11) that

$$(\hat{\mathbf{G}}_{\pi_B} + \hat{\mathbf{G}}_{\pi_A})^{\bullet}(\hat{\mathbf{G}}_{\pi_B} - \hat{\mathbf{G}}_{\pi_A}) \;=\; \hat{\mathbf{G}}_{\pi_B}{}^2 - \hat{\mathbf{G}}_{\pi_A}{}^2 \;=\; ½(\hat{\xi}_{\pi_B}{}^2 + \hat{\eta}_{\pi_B}{}^2 - \hat{\xi}_{\pi_A}{}^2 - \hat{\eta}_{\pi_A}{}^2) \;, \qquad (6.5)$$

which is non−zero in general.

7. Re−Expression of the Finite Displacement Tan−Screw

Motivated by the simplicity of these results, we proceed to re−express them in terms of the general screws $\hat{\mathbf{G}}_A$, $\hat{\mathbf{G}}_B$ which define the nodal line. We consider, particularly, the case when the half−turn screws sited in $\hat{\mathbf{G}}_{\pi_A}$ and $\hat{\mathbf{G}}_{\pi_B}$ are not real, *i.e.* when their constructing coefficients $\hat{\gamma}_{\pi_A}$ and $\hat{\gamma}_{\pi_B}$ are not real. Firstly, we observe of the ξ values (with corresponding remarks applying to the η values), that since

$$\hat{\xi}_{\pi_A} = (1-\hat{\gamma}_{\pi_A})\hat{\xi}_A + \hat{\gamma}_{\pi_A}\hat{\xi}_B \ , \quad \hat{\xi}_{\pi_B} = (1-\hat{\gamma}_{\pi_B})\hat{\xi}_A + \hat{\gamma}_{\pi_B}\hat{\xi}_B \ ,$$

so the differences appearing in the factor at eqn. (6.1) have the form

$$\hat{\xi}_{\pi_A} - \hat{\xi}_{\pi_B} = (\hat{\gamma}_{\pi_B} - \hat{\gamma}_{\pi_A})(\hat{\xi}_A - \hat{\xi}_B) \ ,$$

It follows for the denominator of the leading coefficient in eqns. (6.2,3,4) that

$$(\hat{\xi}_{\pi_A} - \hat{\xi}_{\pi_B})^2 - (\hat{\eta}_{\pi_A} - \hat{\eta}_{\pi_B})^2 = (\hat{\gamma}_{\pi_B} - \hat{\gamma}_{\pi_A})^2 [(\hat{\xi}_A - \hat{\xi}_B)^2 - (\hat{\eta}_A - \hat{\eta}_B)^2] = (\hat{\gamma}_{\pi_B} - \hat{\gamma}_{\pi_A})^2 \hat{\mathcal{A}} \ ,$$

in which $\hat{\mathcal{A}}$, defined by eqn. (5.2), is a constant of the chosen nodal line. Then, since

$$\hat{\mathbf{G}}_{\pi_A} = (1-\hat{\gamma}_{\pi_A})\hat{\mathbf{G}}_A + \hat{\gamma}_{\pi_A}\hat{\mathbf{G}}_B \quad \textit{and} \quad \hat{\mathbf{G}}_{\pi_B} = (1-\hat{\gamma}_{\pi_B})\hat{\mathbf{G}}_A + \hat{\gamma}_{\pi_B}\hat{\mathbf{G}}_B \ , \tag{7.1}$$

we obtain

$$\hat{\mathbf{G}}_{\pi_B} + \hat{\mathbf{G}}_{\pi_A} = 2\hat{\mathbf{G}}_A + (\hat{\gamma}_{\pi_B} + \hat{\gamma}_{\pi_A})(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A) \quad \textit{and} \quad \hat{\mathbf{G}}_{\pi_B} - \hat{\mathbf{G}}_{\pi_A} = (\hat{\gamma}_{\pi_B} - \hat{\gamma}_{\pi_A})(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A) \ , \tag{7.2}$$

in which, from eqns. (5.3),

$$\hat{\gamma}_{\pi_B} + \hat{\gamma}_{\pi_A} = -2\frac{\hat{\mathcal{B}}}{\hat{\mathcal{A}}} \quad \textit{and} \quad \hat{\gamma}_{\pi_B} - \hat{\gamma}_{\pi_A} = 2\frac{\sqrt{\hat{\Delta}}}{\hat{\mathcal{A}}} \ . \tag{7.3a,b}$$

So, on substituting for these expressions in eqn. (6.3),

$$\hat{\mathbf{T}} = -\frac{2}{\hat{\Delta}}\,\frac{\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B + \hat{\kappa}\sqrt{\hat{\Delta}}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)}{1-\hat{\kappa}^2} \ , \tag{7.4}$$

in which the root of the discriminant, $\sqrt{\hat{\Delta}}$, is uniquely associated with the first power of parameter $\hat{\kappa}$.

To discover those values of $\hat{\kappa}$ for which $\hat{\mathbf{T}}$ is a purely real screw, consider $\hat{\kappa}$ to be complex, of form $\hat{\kappa} = \hat{\rho} + i\hat{\tau}$ where $i^2 = 1$ and where $\hat{\rho}$ and $\hat{\tau}$ are real duals. Then $1-\hat{\kappa}^2 = (1-\hat{\rho}^2+\hat{\tau}^2) - i2\hat{\rho}\hat{\tau}$, and

$$\hat{\mathbf{T}} = -\frac{2}{\hat{\Delta}}\left\{\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B + (\hat{\rho}+i\hat{\tau})\sqrt{\hat{\Delta}}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)\right\}\frac{(1-\hat{\rho}^2+\hat{\tau}^2)+i2\hat{\rho}\hat{\tau}}{(1-\hat{\rho}^2+\hat{\tau}^2)^2+4\hat{\rho}^2\hat{\tau}^2} \ . \tag{7.5}$$

Thus, for *negative* discriminant, $\hat{\Delta} < 0$, the screw $\hat{\mathbf{T}}$ contains an imaginary part proportional to

$$2\hat{\rho}\hat{\tau}[\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B - \hat{\tau}\sqrt{|\hat{\Delta}|}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)] + (1-\hat{\rho}^2+\hat{\tau}^2)\hat{\rho}\sqrt{|\hat{\Delta}|}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)$$
$$= 2\hat{\rho}\hat{\tau}[\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B] + (1-\hat{\rho}^2-\hat{\tau}^2)\hat{\rho}\sqrt{|\hat{\Delta}|}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A) \ ,$$

which for $\hat{\mathbf{G}}_A$, $\hat{\mathbf{G}}_B$ being linearly independent, vanishes only for $\hat{\rho} = 0$, and shows that real $\hat{\mathbf{T}}$ are specified by purely imaginary $\hat{\kappa}$. Thus, on re−expressing that imaginary component in terms of the real parameter $\hat{\tau} = \tau + \varepsilon\tau_0$, $-\infty < \tau < \infty$, $-\infty < \tau_0 < \infty$, we have

$$\hat{\mathbf{T}} = -\frac{2}{\hat{\Delta}}\,\frac{\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B - \hat{\tau}\sqrt{|\hat{\Delta}|}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)}{1+\hat{\tau}^2} \ , \quad \hat{\Delta} < 0 \ . \tag{7.6}$$

If, in terms of a dual angle parameter $\hat{\psi} = \psi + \varepsilon d$, $-\pi \le \psi \le \pi$, $-\infty \le d \le \infty$, we write $\hat{\tau} = \tan\hat{\psi}$ with $1+\hat{\tau}^2 = 1/\cos^2\hat{\psi}$, we obtain

$$\hat{\mathbf{T}} = \frac{2}{|\hat{\Delta}|}\cos\hat{\psi}\{\cos\hat{\psi}(\hat{\mathcal{E}}\hat{\mathbf{G}}_A - \hat{\mathcal{B}}\hat{\mathbf{G}}_B) - \sin\hat{\psi}\sqrt{|\hat{\Delta}|}\,(\hat{\mathbf{G}}_B - \hat{\mathbf{G}}_A)\}\ , \quad \hat{\Delta} < 0\ . \tag{7.7}$$

We can, if we choose, re−write this expression in terms of tan−screws alone. For, by suitable choice of angle, we may eliminate each of the basis screws, thus:

$$\left.\begin{aligned} -\cos\hat{\psi}\,\hat{\mathcal{B}}\hat{\mathbf{G}}_B - \sin\hat{\psi}\sqrt{|\hat{\Delta}|}\,\hat{\mathbf{G}}_B &= 0 \quad \mathit{for} \quad \tan\hat{\psi} = -\frac{\hat{\mathcal{B}}}{\sqrt{|\hat{\Delta}|}}\,, \\ \cos\hat{\psi}\,\hat{\mathcal{E}}\hat{\mathbf{G}}_A + \sin\hat{\psi}\sqrt{|\hat{\Delta}|}\,\hat{\mathbf{G}}_A &= 0 \quad \mathit{for} \quad \tan\hat{\psi} = -\frac{\hat{\mathcal{E}}}{\sqrt{|\hat{\Delta}|}}\,, \end{aligned}\right\} \tag{7.8}$$

from which

$$\left.\begin{aligned} \hat{\mathbf{T}}_A &= \frac{2}{|\hat{\Delta}|}\,\frac{|\hat{\Delta}|}{\hat{\mathcal{B}}^2 + |\hat{\Delta}|}\,(\hat{\mathcal{E}} - \hat{\mathcal{B}})\,\hat{\mathbf{G}}_A = \frac{2\hat{\mathcal{A}}}{\hat{\mathcal{B}}^2 + |\hat{\Delta}|}\,\hat{\mathbf{G}}_A\ , \\ \hat{\mathbf{T}}_B &= \frac{2}{|\hat{\Delta}|}\,\frac{|\hat{\Delta}|}{\hat{\mathcal{E}}^2 + |\hat{\Delta}|}\,(\hat{\mathcal{E}} - \hat{\mathcal{B}})\,\hat{\mathbf{G}}_B = \frac{2\hat{\mathcal{A}}}{\hat{\mathcal{E}}^2 + |\hat{\Delta}|}\,\hat{\mathbf{G}}_B\ . \end{aligned}\right\} \tag{7.9}$$

8. Conclusion

A compact representation has been derived for the typical screw of any real 2−system which exists as a subset of the 3−system of finite displacement screws associated with a revolute dyad. It is expected that this representation will allow such studies as that of Huang [11] – of the Bennett mechanism – to be carried further in elucidating detailed properties.

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References

1. Bottema, O., Roth, B., 1990, *Theoretical Kinematics.* Dover, New York.

2. Yang, A.T., 1974, Calculus of screws. *Basic Questions of Design Theory* (ed. W.R. Spillers), Elsevier, New York, pp. 265−281.

3. Parkin, I.A., 1997, Dual systems of finite displacement screws in the screw triangle. *Mechanism and Machine Theory* **32** (8), pp. 993−1003.

4. Parkin, I.A., 1992, A third conformation with the screw systems: finite twist displacements of a directed line and point. *Mechanism and Machine Theory* **27** (2), pp. 177−188.

5. Huang, C., Roth, B., 1994, Analytic expressions for the finite screw systems. *Mechanism and Machine Theory* **29**, pp. 207−222.

6. Hunt, K.H., Parkin, I.A., 1995, Finite displacements of points, planes and lines *via* screw theory. *Mechanism and Machine Theory* **30**, pp. 177−192.

7. Parkin, I.A., 1998, Linear systems of tan−screws for finite displacement of a rigid body with symmetries. *Proc. Sixth International Workshop on Advances in Robot Kinematics*, Strobl, Austria, June−July, (eds. M. Husty and J. Lenarcic), Kluwer Academic Publishers, pp. 317−326.

8. Huang, C., 1993, On the finite screw system of the third order associated with a revolute−revolute chain. *Advances in Design Automation*, *Proc 1993 ASME Design Technical Conf.*, Albuquerque, New Mexico, Sept. 19−22, **2**, pp. 81−89.

9. Hunt, K.H., 1990, *Kinematic Geometry of Mechanisms.* Clarendon Press, Oxford.

10. Phillips, J., 1990, *Freedom in machinery.*, **1** and **2**, Cambridge University Press.

11. Huang, C., 1997, The cylindroid associated with finite motions of the Bennett mechanism. *Journal of Mechanical Design*, *Trans. ASME*, **119**, pp. 521−524.

12. Phillips, J.R., Hunt, K.H., 1964, On the theorem of three axes in the spatial motion of three bodies. *Australian Journal of Applied Science* **15**, pp. 267−287.