Kinematics of a Two-Legged Manipulator with Actuated Spherical Joints

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Abstract:- Kinematic analysis of a two-legged -FF-(S)-P-U-EE-U-P-(S)-FF- closed chain, wherein all six degrees of freedom in the two spherical joints are actuated, requires nothing more elaborate than a quadratic equation. This result was obtained by observing that the -U-EE-U- line traces a ruled surface with simple quadratic parametrization if one actuates only *two* of each S-joint's actuators. A selection among the ruling lines is easily made by noting the constraint imposed by the two remaining S-joint actuators upon the perpendicular to compatible -U-EE-U- lines. -EE- stands for end effector, -FF- for fixed frame while -P-, -S- and -U- indicate prismatic, spherical and universal (Cardan or Hooke) joints, respectively. () states that all axes of the enclosed joint are actuated.

1 Introduction

Recently, an article [8] dealing with inverse and direct position and direct velocity analysis of an interesting two legged parallel manipulator was submitted to *Journal of Robotic Systems* for review. This manipulator is interesting because of its peculiar architecture and because the direct kinematics are easily amenable to a closed form solution if one chooses a geometric approach which apparently eluded the authors. Each identical leg contains a base(-FF-) mounted spherical joint wherein all three axes are actuated. Furthermore the leg contains a passive prismatic joint and is connected to the end effector(-EE-) with a passive universal joint. Consider the configuration of this manipulator whose layout and actuation arrangement is shown in Fig. 1 and whose design parameters and joint variables are described in Fig. 2.

The actuated S-joints attached to -FF- position the P-joint leg lines via the joint angles θ_i and ϕ_i, $i = 1, 2$. Although the two free P-joints cannot locate the U-joint centres, C and Q, the third S-joint actuators ψ_1 and ψ_2 fully define the orientations $\mathbf{v}$ and $\mathbf{w}$ of the universal joint cross legs supported on bearings attached to the P-joint pistons. There is no loss in generality if one chooses U-joint cross legs, carried in bearings on -EE-, to have common axial orientation $\mathbf{u}$. Since the legs on the same cross are designed to be perpendicular therefore $\mathbf{v} \times \mathbf{w} = \mathbf{u}$. Furthermore if $(\mathbf{q} - \mathbf{c})$ is
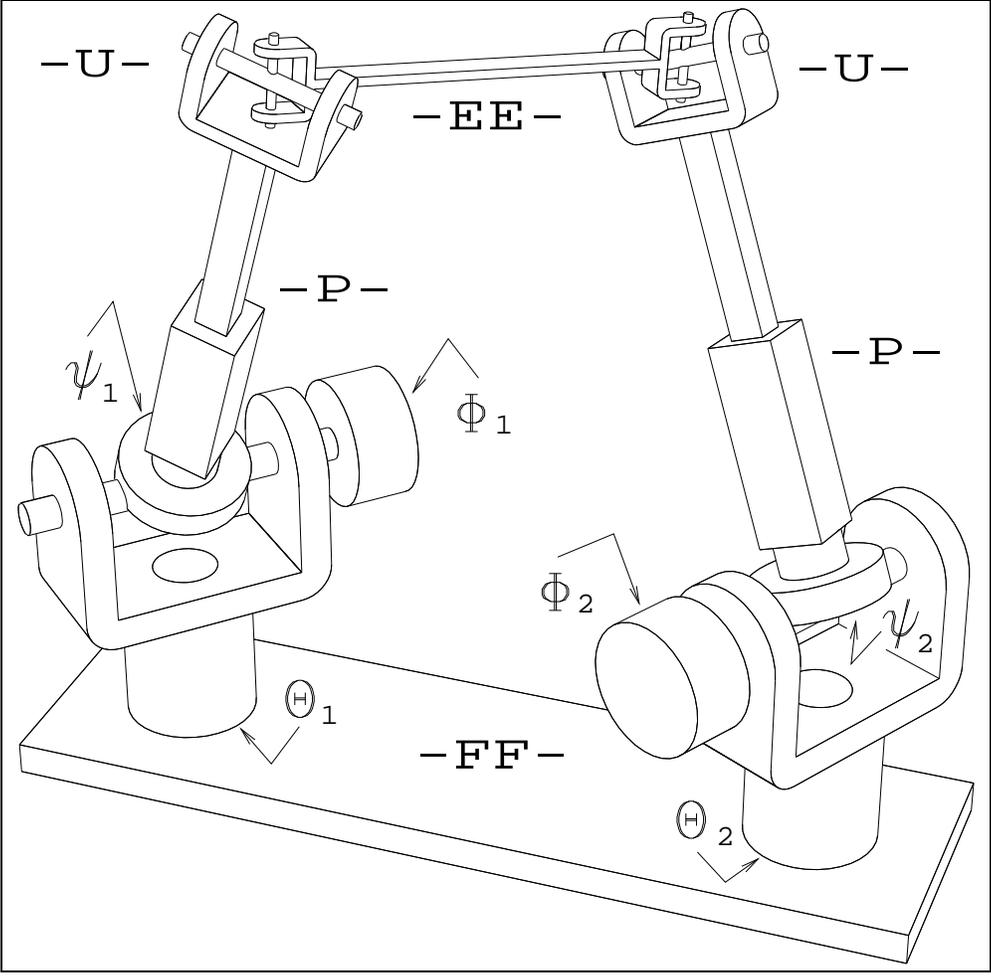
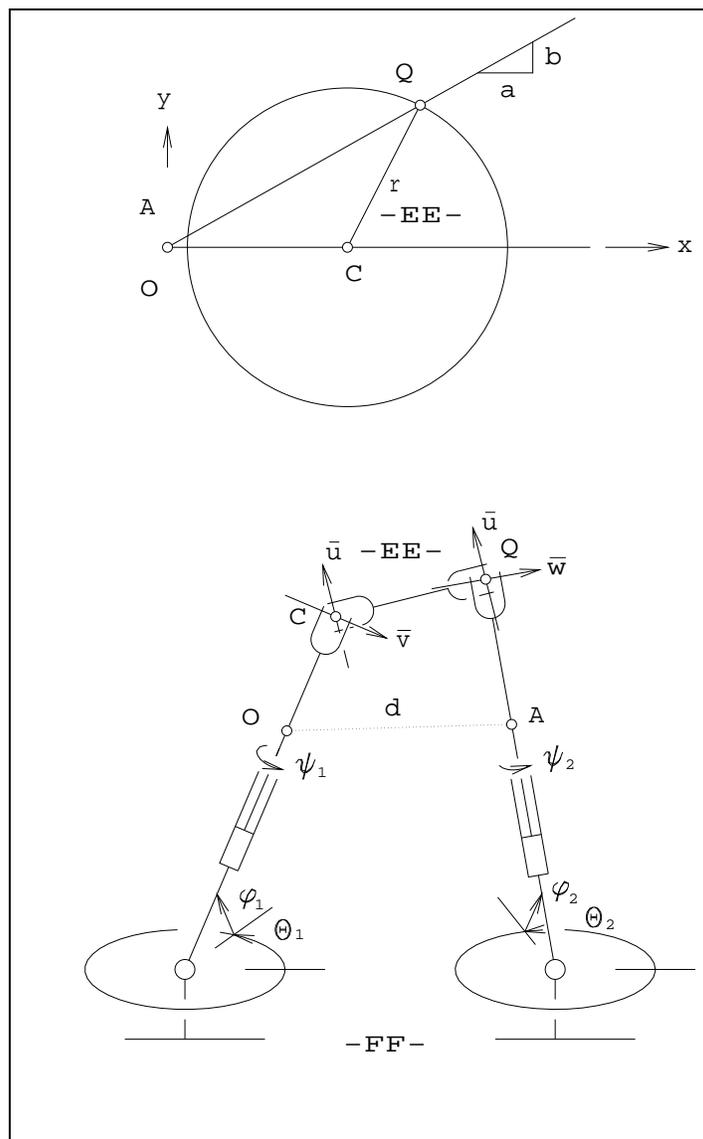Figure 1: The two-legged (S)-P-U manipulator
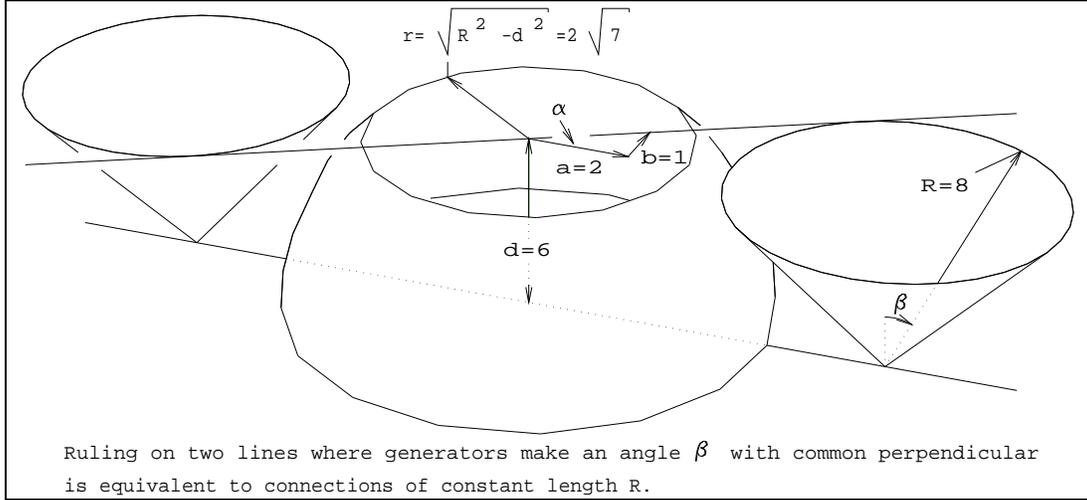
Figure 2: Manipulator geometry

Figure 3: Solution model geometry

the difference of the position vectors of Q and C then $(\mathbf{q} - \mathbf{c}) \cdot \mathbf{u} = 0$. All that remains is to describe the direct positioning of C and Q. This will be carried out algebraically first, then by means of descriptive geometry. If solutions are to exist, then $\mathbf{u}$ cannot vanish and $OC \cap AQ$ cannot exit.

2 Direct Kinematics

Origin O is chosen on the leg line actuated by θ_1, ϕ_1, ψ_1. O is at the foot of the common perpendicular on that line while A is at the foot on the other leg line. The common perpendicular OA and its length d is found using the known positions of the S-joint centres and the distance separating them on -FF-. After setting up θ_1, ϕ_1 the unknown displacement, c_x, of the P-joint on the leg containing O, locates the point C which becomes the centre of a sphere radius R which is the length of -EE-, *i.e.*, the distance between C and Q. The sphere contains all possible locations of Q but then so does the leg line on A whose unknown P-joint displacement locates Q. Due to fortunate coordinate frame choice one may write the position of Q in terms of the free parameter c_x by intersecting the circle of radius $r = \sqrt{R^2 - d^2}$ with the line AQ. r may be nondimensionalized to $\rho = r/d$. The circle is the sphere section formed by a plane on AQ and parallel to OC. Alternately, the circle is a right section of a cone of revolution, apex coincident with the sphere centre as shown in Fig. 3, and with an apex half-angle of $\beta = \cos^{-1}(d/R)$. This makes $a_z = q_z = d$; always. Intersections, *e.g.*, like the one on Q, of this circle and the line AQ, shown in the upper drawing on Fig. 2, are expressed by the two following equations.

$$(q_x - c_x)^2 + q_y^2 - r^2 = 0, \qquad a^2 q_y^2 - b^2 q_x^2 = 0$$

The univariate in q_x is

$$(a^2 + b^2)q_x^2 - 2a^2 c_x q_x + a^2(c_x^2 - r^2) = 0$$

which simplifies to

$$q_x^2 - 2C^2 c_x q_x + C^2(c_x^2 - r^2) = 0$$

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Consider that $\{a : b : 0\}$ are direction numbers of the line $A \rightarrow Q$.

$$\tan \alpha \equiv b/a \equiv T, \ \ \cos^2 \alpha \equiv a^2/(a^2 + b^2) \equiv C^2, \ \ \sin^2 \alpha \equiv b^2/(a^2 + b^2) \equiv S^2$$

Position vectors $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{q}$ of A, C and Q with respect to O, x-axis along OC and z-axis along OA, can now be written.

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} C(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) \\ S(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) \\ d \end{bmatrix} \tag{1}$$

The cosines C in Eq. 1 are not to be confused with the *point* C. Now the scalar product $(\mathbf{q} - \mathbf{c}) \cdot \mathbf{u} = 0$ is formed and solved for c_x.

$$[C(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) - c_x]u_x + S(Cc_x \pm \sqrt{r^2 - S^2 c_x^2})u_y + du_z = 0$$

$$S^2[(Cu_y - Su_x)^2 + (Cu_x + Su_y)^2]c_x^2$$

$$+2Sd(Cu_y - Su_x)u_z c_x - [r^2(Cu_x + Su_y)^2 - d^2 u_z^2] = 0$$

$$c_x = d\frac{(Su_x - Cu_y)u_z}{S(u_x^2 + u_y^2)}$$

$$\pm d\frac{\sqrt{(Su_x - Cu_y)^2 u_z^2 + [\rho^2(Cu_x + Su_y)^2 - u_z^2](u_x^2 + u_y^2)}}{S(u_x^2 + u_y^2)} \tag{2}$$

Finally, c_x computed with Eq. 2 can be used in Eq. 1 to produce the direct pose.

3 The Ruled Surface CQ

The two solutions of Eq. 2 produce two regulii of a surface ruled by CQ. Fig. 4 shows half of the real part of each regulus. The half in the upper right quadrant is produced with $c_x \geq 0$ and selection of the positive discriminant in Eq. 1. The other half in the lower left quadrant is the result when $c_x \leq 0$, still with a positive discriminant $\sqrt{r^2 - S^2 c_x^2}$. Notice that the entire real surface can be generated with only the positive discriminant while allowing c_x to range between tangent condition limits $c_{xT} = \pm r/S$, then negating all coordinates of C and Q . Of course the entire surface extends above and below the two leg lines but only the segments between them are shown. These are of interest because they span the length of EE between the U-joints. One may ask about the nature of this surface, at least to know its implicit form $f(x, y, z) = 0$. This can be obtained by invoking a second parameter t, normalized on the interval R, which parametrizes points along CQ. Any point P on the surface may be expressed by the following position vector.

$$\mathbf{p} = \mathbf{c} + t(\mathbf{q} - \mathbf{c}) = \begin{bmatrix} c_x \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} C(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) - c_x \\ S(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) \\ d \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

First, t may be eliminated to produce two equations in c_x which is then eliminated to leave $f(x, y, z) = 0$.

$$c_x + (p_z/d)[C(Cc_x \pm \sqrt{r^2 - S^2 c_x^2}) - c_x] - p_x = 0$$

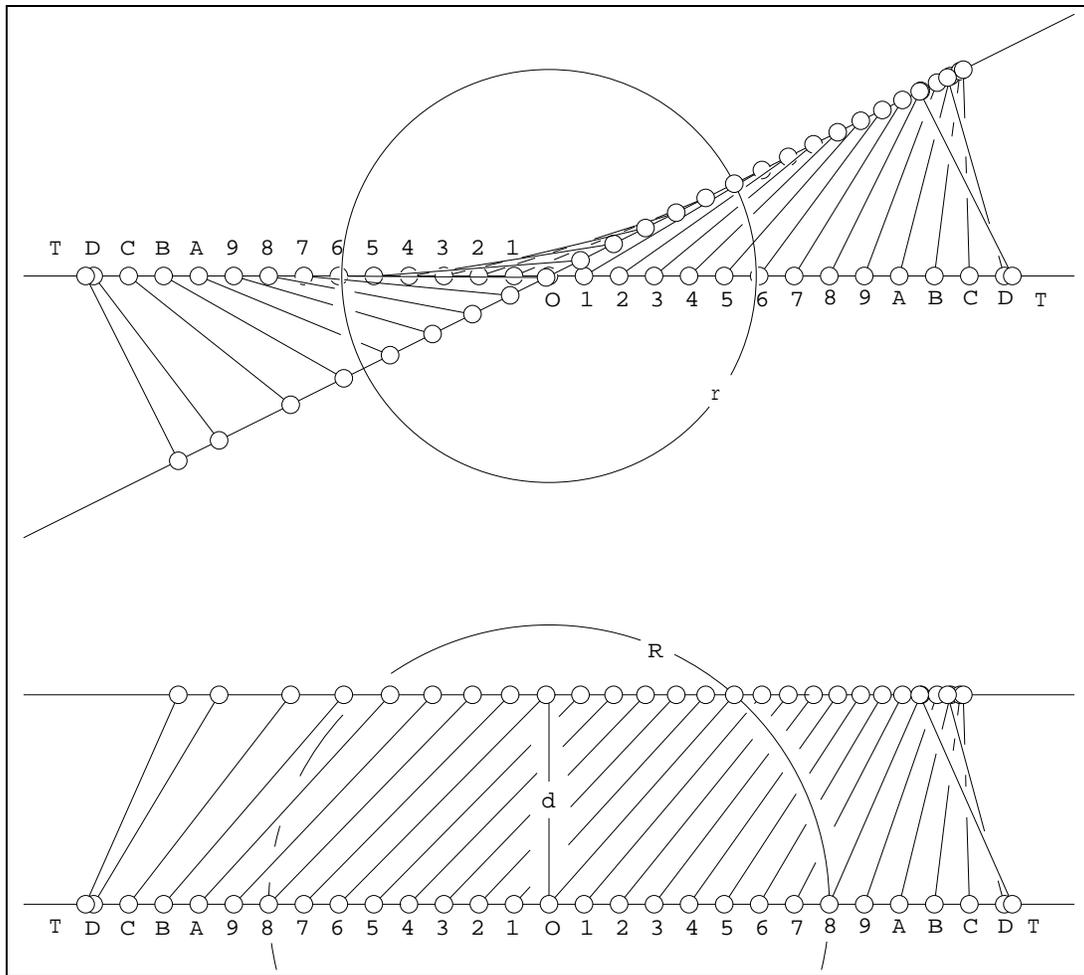$$(p_z/d)S[Cc_x \pm \sqrt{r^2 - S^2 c_x^2}] - p_y = 0 \tag{3}$$

Figure 4: Sturm class VII *Wringfläche* ruled surface

The difference between these two equations yields c_x.

$$c_x = \frac{d(Sp_x - Cp_y)}{S(d - p_z)}$$

This result can then be substituted into the second, simpler expression for p_y, Eq. 3, and simplified.

$$[C(Sx - Cy)z - (d - z)y]^2 - S^2[\rho^2(d - z)^2 - (Sx - Cy)^2]z^2 = 0 \qquad (4)$$

p_x, p_y, p_z have been replaced by x, y, z to render the implicit form more familiar. Eq. 4 can be expanded to reveal its quartic form, Eq. 5 .

$$z^2x^2 + y^2z^2 - 2dy^2z - 2\cot\alpha dxyz + \csc^2\alpha d^2y^2 + \rho^2(d - z)^2z^2 = 0 \qquad (5)$$

The species expressed in this quartic surface are the following.

$$x^2z^2, y^2z^2, y^2z, xyz, y^2, z^4, z^3, z^2$$

4   Descriptive Geometric Solution

Refering again to Fig. 4 and to Eq. 1 the following may be stated.

Lemma 1 *The subset of lines, in the congruence or two parameter family of lines intersecting two given ones, which make a constant angle β with respect to the common perpendicular, of length d, between the two given lines is identical to that subset which subtends segments of constant length $R = d\sec\beta$.*

This leads to a straightforward descriptive geometric kinematic solution. One sees in Fig. 5 various projections of the pair of leg lines and the pair of angles ψ_1, ψ_2 imposed on the respective U-joint crosses attached to these legs. The connecting line directions which satisfy this joint angle input condition are formed by a quasi-"Monge's Construction". A description of the classical "Monge's Construction" may be found in [9]. It is a method to find the connecting line direction between two given lines where it is desired that the connection makes respective angles β_1, β_2 with respect to the lines to be connected. For example, a plumber may wish to solve this problem in order to connect two skew pipelines between two straight-run "Wye"-fittings, *i.e.*, not "Tee's". The construction places two cones of revolution, half apex angle β_1, β_2, with their apices on the centre of a sphere. The line intersections of the planes of the four small circle cone-sphere intersections locate up to four real generators, common to both cones, which specify the required connecting direction. In the case at hand, one of the cones is replaced by a plane on the sphere centre. This plane is perpendicular to $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ or, conversely, parallel to the two U-joint crosses with axis parallel to $\mathbf{v}$ and $\mathbf{w}$, respectively. The compatible, real solution pair illustrated in Fig. 5 shows the EE lines CQ and CR which form such a plane on C. Notice that this plane is equally well defined by CST where CS is the U-joint cross direction on leg line 1 and CT is the U-joint cross direction on leg line 2. The construction may be summed up by the simultaneous solution of the following four equations in line coordinates. The first two define the congruence of lines with variable Plücker ray coordinates $\mathcal{P}_r\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\}$ intersecting the given leg lines with axial coordinates $\mathcal{Q}_a\{Q_{01} : Q_{02} : Q_{03} : Q_{23} : Q_{31} : Q_{12}\}$ and $\mathcal{R}_a\{R_{01} : R_{02} : R_{03} : R_{23} : R_{31} : R_{12}\}$. The third equation is the constraint imposed by the leg line mounted U-joint crosses and the fourth is the cone
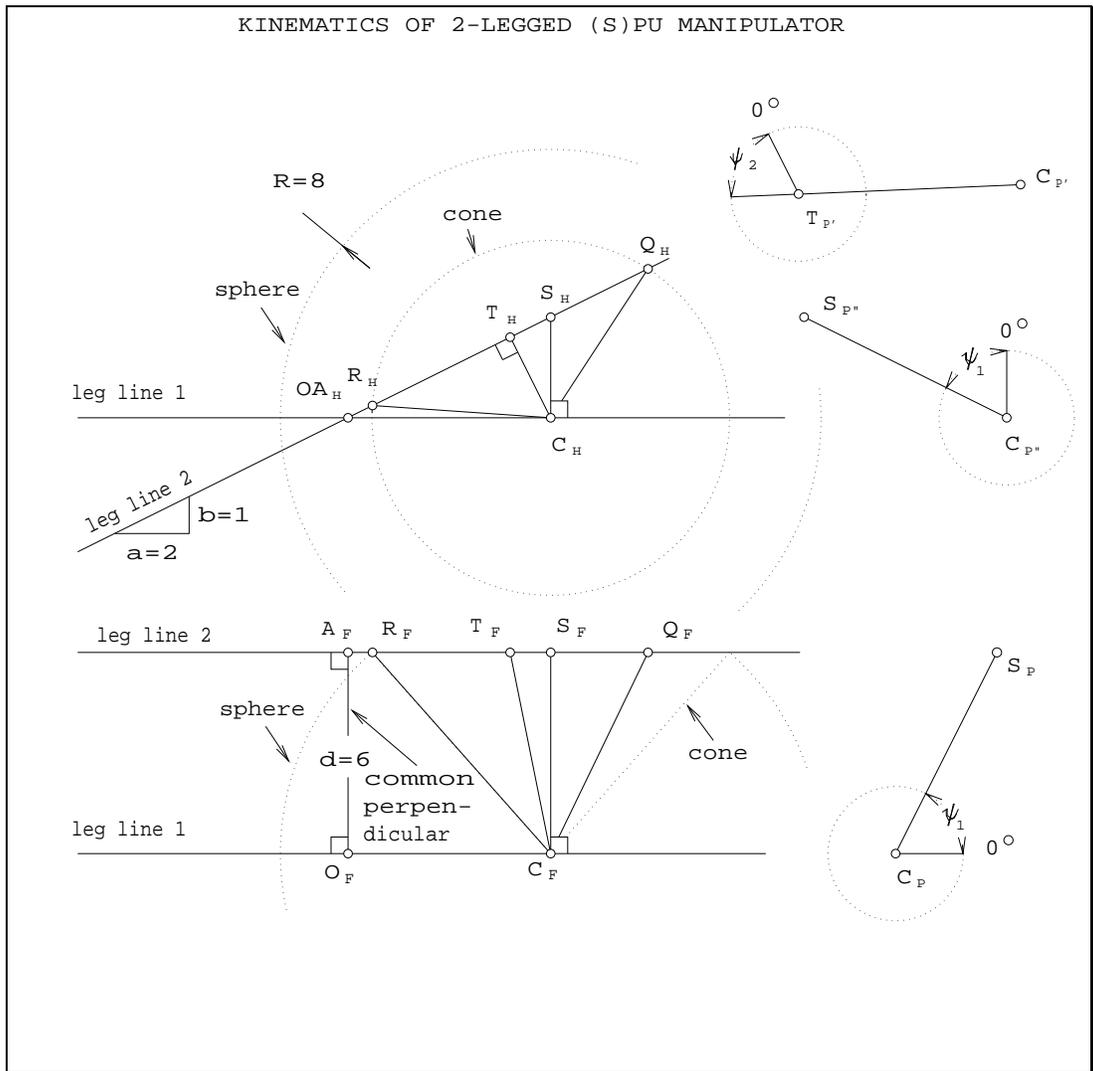
KINEMATICS OF 2-LEGGED (S)PU MANIPULATOR

0°

ψ_2

$C_{P'}$

$T_{P'}$

R=8

cone

sphere

Q_H

$S_{P''}$

0°

ψ_1

T_H S_H

leg line 1

OA_H R_H

$C_{P''}$

C_H

leg line 2

b=1

a=2

leg line 2

A_F R_F T_F S_F Q_F

S_P

sphere

d=6

common
perpen-
dicular

cone

leg line 1

ψ_1

0°

O_F

C_F

C_P

Figure 5: End effector U-joint cross direction by descriptive geometry

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apex half angle constraint.

$$Q_{01}p_{01} + Q_{02}p_{02} + Q_{03}p_{03} + Q_{23}p_{23} + Q_{31}p_{31} + Q_{12}p_{12} = 0$$
$$R_{01}p_{01} + R_{02}p_{02} + R_{03}p_{03} + R_{23}p_{23} + R_{31}p_{31} + R_{12}p_{12} = 0$$
$$u_x p_{01} + u_y p_{02} + u_z p_{03} = 0$$
$$[(P_{31}Q_{12} - P_{12}Q_{31})p_{01} + (P_{12}Q_{23} - P_{23}Q_{12})p_{02} + (P_{23}Q_{31} - P_{31}Q_{23})p_{03}]^2$$
$$- \cos^2 \beta \left[(P_{31}Q_{12} - P_{12}Q_{31})^2 + (P_{12}Q_{23} - P_{23}Q_{12})^2 \right.$$
$$\left. + (P_{23}Q_{31} - P_{31}Q_{23})^2 + p_{01}^2 + p_{02}^2 + p_{03}^2 \right] = 0$$

If the connecting line direction, *i.e.*, all three p_{01}, p_{02}, p_{03}, is known then the third and fourth equations may be replaced by the Plücker orthogonality condition $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$. Descriptive geometric solutions to various connecting line problems may be found in any elementary text, *e.g.*, [6]. It is believed however that the systematic division, of all types of these problems, into two phases, *i.e.*, first determining the required direction of connection to suit a particular requirement and then solving the so-called *kernel problem* of joining two lines with a connection in this specific direction, is mentioned only in [10].

5   Conclusion

The symmetrical workspace of this two legged manipulator is neatly defined by positional limits $\pm c_x T$ and its centre lies at the midpoint of the leg line common perpendicular. Self motion singularity arises whenever leg lines intersect, including parallel configuration. Of course there are the singularities inherent in nonredundant spherical wrists [3, 1]. The greatest challenge which remains in connection with the spherically actuated two legged (S)-P-U manipulator is to define its dextrous workspace and therefrom derive some interesting applications for this novel device which at this time is only a slightly bizarre curiosity. On the other hand the forgoing treatment shows clearly the sort of insights which may be gained by applying geometric methods.

5.1   Better Choices & Digging Deeper

In retrospect, a simpler solution results from a symmetrically placed frame and double parametrization along the lines. The z-axis remains as before but the origin O is placed midway along the common normal between the leg lines BC and AQ while the y-axis is on a plane containing the z-axis and bisecting all transversals CQ. Note that B replaces O in the original frame labelling scheme. An angle γ is defined from either leg line to the plane $y = 0$. Now scalar parameter t "pushes" Q away from A along AQ while s similarly locates C to produce

$$\mathbf{q} = \begin{bmatrix} t\cos\gamma \\ t\sin\gamma \\ d/2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -s\cos\gamma \\ s\sin\gamma \\ -d/2 \end{bmatrix}, \quad \mathbf{q} - \mathbf{c} = \begin{bmatrix} \cos\gamma(t+s) \\ \sin\gamma(t-s) \\ d \end{bmatrix}$$

Then $(\mathbf{q} - \mathbf{c}) \cdot \mathbf{u} = 0$ and the circle of radius r become

$$u_x \cos\gamma(t+s) + u_y \sin\gamma(t-s) + u_z d = 0, \quad [\cos\gamma(t+s)]^2 + [\sin\gamma(s-t)]^2 - r^2 = 0$$

Further simplification can be wrought upon these equations by setting $j = \cos\gamma(t+s)$ and $k = \sin\gamma(t-s)$, which is easily invertable to $t = t(j,k)$ and $s = s(j,k)$.

$$u_x j + u_y k - u_z d = 0, \quad j^2 + k^2 - r^2 = 0$$

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Then the ruled surface Φ on which any point P has the position vector $\mathbf{p}$ may be obtained by adding a third parameter λ to "push" P along the EE line CQ away from C towrads Q.

$$\mathbf{p} = \left[\begin{array}{c} (-\cos\gamma)s + \lambda\cos\gamma(t+s) \\ (\sin\gamma)s + \lambda\sin\gamma(t-s) \\ -\frac{d}{2} + \lambda d \end{array} \right] = \left[\begin{array}{c} x \\ y \\ z \end{array} \right]$$

This system is deparametrized by solving the first two rows for j and k then substituting for λ from the third.

$$\cos\gamma(t+s) = \frac{2d[2zx - d(\cot\gamma)y]}{(2z+d)(2z-d)}, \quad \sin\gamma(t-s) = \frac{2d[2yz - d(\tan\gamma)x]}{(2z+d)(2z-d)}$$

Finally the circle equation supplies $\Phi(x,y,z)$.

$$4d^2\{[(d\tan\gamma)x - 2yz]^2 + [(d\cot\gamma)y - 2zx]^2\} - r^2(4z^2 - d^2)^2 = 0$$

This surface has been studied extensively for well over 150 years and its properties are well known. Contributions by Burmeister [2], Magnus [4], Müller [5] and Sturm [7] are cited in this regard. Furthermore, it is pointed out in conclusion that this literature, through its treatment of such ruled surfaces, reveals even more secrets concerning this two-legged manipulator. *E.g.*, the desired connecting lines are also contained in a second order line series; a hyperbolic paraboloid. However this and the treatment of manipulator singularities and workspace is left for a sequel.

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