Sampling and Convolution on Motion Groups Using Generalized Gaussian Functions

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Abstract

Most engineers are familiar with the concepts of convolution, Gaussian functions, and the heat equation on the line. These concepts all extend when dealing with functions on the groups of interest in kinematics - the rotation and rigid-body-motion groups. In this paper we first review notation from kinematics and collect some results concerning differential operators and partial differential equations on the rotation group and three-sphere. We present these results (which have long been known in the mathematics literature) in a manner which is palatable to kinematicians. I.e., we do not introduce excessive new notations or definitions. We then build a new sampling theory for the group of rigid body motions on these results, and demonstrate its computational features in the context of manipulator workspace generation and error propagation in serial (and hybrid serial/parallel) linkages.

1 Introduction

In this paper we develop a sampling theory for functions on the group of rigid-body motions[3], and illustrate why this is a useful computational tool in kinematics.

In a number of recent papers, it has been shown how the workspace boundaries of manipulators, and workspace density function for binary manipulators[4] in particular, can be generated in a computationally efficient manner using the concept of group-theoretical convolution [5, 4]. Inverse problems associated with manipulator design can be solved using techniques of noncommutative harmonic analysis [1, 2]. Images of planar and spatial binary manipulators in action can be found at the website: *http://caesar.me.jhu.edu.*

The scope of applications for which techniques from harmonic analysis are useful has been expanded to include the generation of configuration-space obstacles for single-body mobile robots [6], statistical mechanics of chain molecules, and the radiotherapy treatment planning problem [3]. Fast Fourier transform techniques for groups reduce even further the

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[3]Often called the Euclidean Motion group, or Special Euclidean Group, and denoted as $SE(N)$
[4]Manipulators composed of a serial cascade of units containing two-state actuators

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig2-1.gif

Figure 1: Frames of Reference

time required to compute the generalized convolutions that arise in these various application areas. Such techniques have been derived for a variety of finite groups and compact Lie groups [8], but have yet to be implemented for noncompact noncommutative groups like $SE(N)$.

In all of those previous works, it is assumed that the density of points (or frames) is large enough so that direct discretization of $\mathbb{R}^N$ or $SE(N)$ leads to histograms which accurately reflect the density. While this is valid for the case when a large number of points (or frames) occupy the space under consideration, it can lead to unacceptable errors and memory overhead when points are too sparsely dispersed. The technique presented here is an alternative to discretization by counting points and forming histograms. Instead, the approach we take here is to allow each point (or frame) to diffuse. Hence, if each point is originally viewed as a Dirac delta function, after it diffuses it will be a Gaussian. Gaussians convolved with Gaussians result in functions of the same form, and the result of a convolution can be resampled and replaced by a sum of a smaller number of Gaussians. This leads to a sampling method for the motion groups.

The contribution of this paper is to formulate exactly what is meant by a Gaussian function on the Euclidean groups of the plane and three space, to present the corresponding sampling technique, and to illustrate the technique with numerical examples.

The remainder of this paper is formulated as follows. Section 2 reviews the kinematics of rigid body motion using quaternion notation. Section 3 discusses diffusion equations on the circle and quaternion sphere, and defines Gaussians as the solution of these equations. Section 4 examines properties of convolution on groups. Section 5 shows how the convolution of functions on the motion groups can be decomposed into sums of generalized Gaussians. Section 6 presents a sampling technique for sums of Gaussians, and Sections 7 and 8 generate numerical results in the context of kinematics problems.

2 A Review of Theoretical Kinematics

2.1 Frames and Motions

A *frame* is characterized by a pair $f = (\vec{p}, Q)$ where $\vec{p} \in \mathbb{R}^d$ represents a special point called the *origin* of the frame and $Q \in SO(d)$ stands for the description of the *orientation* of the frame. The set $SO(d)$ of *special orthogonal matrices* is defined as

$$SO(d) = \{Q \in \mathbb{R}^d \times \mathbb{R}^d \,|Q^T Q = 1,\ \det Q = 1\} \tag{1}$$

Usually, the spatial dimension d is 2 or 3. The case $d = 1$ is trivial since $Q \equiv 1$ (no orientation concept). However, the characteristics of the 1-D case are inherited by the higher dimensions and its low dimension is useful for illustrative purposes (easy plots). Figure 1 gives a representation of both vector $\vec{p} = \overrightarrow{OA}$ and matrix $Q = [\vec{x}\ \vec{y}]$ in the two-dimensional case.

Note that vector $\vec{p}$ has a unit of length $[\ell]$ whereas vectors $\vec{x}$ and $\vec{y}$ are unitless. A *motion* can be described by the pair $g = (\vec{d}, R)$, where $\vec{d} \in \mathbb{R}^d$ represents a *translation* and $R \in SO(d)$ is a *rotation*. The motion $g = (\vec{d}, R)$ applied to the frame $f = (\vec{p}, Q)$ results in the new frame

$$f' = gf = (\vec{p}', Q') \tag{2}$$

where the product between g and f is the product of the *special Euclidean group* $SE(d)$. That is,

$$\vec{p}' = \vec{d} + R\vec{p} \tag{3}$$
$$Q' = RQ. \tag{4}$$

Motions and frames are usually treated in a unified mathematical framework. Moreover, it makes sense to consider the composition $g_2 g_1$ of two motions since

$$g_2(g_1 f) = (g_2 g_1) f. \tag{5}$$

2.2 Unit quaternions

The most obvious drawback of describing a rotation by the matrix R is the dimension of the required space. For example, in the two-dimensional case, rotations can be well characterized by only one angle θ whereas a matrix description needs four entries. On the other hand, an angle has to be considered modulo 2π to describe rotations: θ and $\theta + 2\pi$ are two different angles, but they represent the same rotation. The three-dimensional case is much worse. Instead of the nine inputs of the rotation matrix, three parameters are sufficient *e.g.*, the Euler's angles. However, three parameters, whichever they are, cannot describe the space of rotations homogeneously (in analogy with the longitude-latitude coordinate system used in earth cartography where the blocks of the mesh are smaller near the poles). Even worse, a three parameters description contains some singularities: locally, some rotations have more than one parameterization (what is the longitude at the earth's poles?). All these drawbacks are eliminated if one uses the *unit quaternions* [9, 10], though this comes at the price of an extra parameter. The parameters of a unit quaternion $q = (q_1, q_2, q_3, q_4)$, also known as *Euler's parameters*, have the following physical meaning

$$q_1 = \cos\frac{\rho}{2} \qquad \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix} = \vec{u} \cdot \sin\frac{\rho}{2} \tag{6}$$

where $\vec{u}$ is a unit vector representing the rotation axis and ρ the rotation angle. Vectors $\vec{u}$ and $-\vec{u}$ define the same axis, the pairs $(\pm\rho, \pm\vec{u})$ describe the same unit quaternions, and the unit quaternions $\pm q$ parameterize the same rotation. Hence, the unit quaternions form a $2-1$ covering of the rotations. The product of two quaternions is defined as

$$rq = \begin{bmatrix} r_1 q_1 - r_2 q_2 - r_3 q_3 - r_4 q_4 \\ r_2 q_1 + r_1 q_2 - r_4 q_3 + r_3 q_4 \\ r_3 q_1 + r_4 q_2 + r_1 q_3 - r_2 q_4 \\ r_4 q_1 - r_3 q_2 + r_2 q_3 + r_1 q_4 \end{bmatrix}. \tag{7}$$

Rotations in two-dimensional space are actually rotations in 3D where the axis $\vec{u}$ is constant, for example $\vec{u} = (1, 0, 0)$. A subgroup of the quaternions with $q_3 = q_4 = 0$ can be considered with the product

$$rq = \begin{bmatrix} r_1 q_1 - r_2 q_2 \\ r_2 q_1 + r_1 q_2 \end{bmatrix} \tag{8}$$

but as this group directly follows from the more general three-dimensional case, only the latter will be detailed in the sequel. The *magnitude* of a quaternion is defined as

$$\|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}. \tag{9}$$

For unit quaternions, the magnitude is 1. Between two quaternions q_1 and q_2 the *Euclidean distance* is $\|q_1 - q_2\|$. This distance is not altered under rotation since

$$\|rq\| = \|r\|\|q\| = \|q\| \tag{10}$$

where r is a unit quaternion and q is any quaternion.

In the sequel, we will use $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ to denote the length of vectors in any $\mathbb{R}^N$ (not only $\mathbb{R}^4$).

2.3 Definition of the Motion Group

Both matrix R and unit quaternion r represent rotations. They are related to each other by [10]

$$R(r) = \begin{bmatrix} r_1^2 + r_2^2 - r_3^2 - r_4^2 & 2r_2 r_3 - 2r_1 r_4 & 2r_2 r_4 + 2r_1 r_3 \\ 2r_2 r_3 + 2r_1 r_4 & r_1^2 - r_2^2 + r_3^2 - r_4^2 & 2r_3 r_4 - 2r_1 r_2 \\ 2r_2 r_4 - 2r_1 r_3 & 2r_3 r_4 + 2r_1 r_2 & r_1^2 - r_2^2 - r_3^2 + r_4^2 \end{bmatrix} \tag{11}$$

$$r_1(R) = \frac{\sqrt{\mathrm{tr}R + 1}}{2}; \qquad \begin{bmatrix} r_2(R) \\ r_3(R) \\ r_4(R) \end{bmatrix} = \frac{1}{2\sqrt{\mathrm{tr}R + 1}} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \tag{12}$$

Redefining motions $g = (\vec{d}, r)$ and frames $f = (\vec{p}, q)$ with the concept of unit quaternions the product of the *motion group* is defined as

$$f' = gf = (\vec{p}', q') \tag{13}$$

where

$$\vec{p}' = \vec{d} + R(r)\vec{p} \tag{14}$$

$$q' = rq. \tag{15}$$

Translational distance is defined as

$$d(\vec{d}_1, \vec{d}_2) = \|\vec{d}_1 - \vec{d}_2\| \tag{16}$$

whereas one measure of *rotation distance* is

$$D(r_1, r_2) = \min\{\|r_1 + r_2\|, \|r_1 - r_2\|\}. \tag{17}$$

This is one of any number of metrics for $SO(3)$ that are invariant under left and right shifts [11, 12, 13, 14, 15].

3 Rotational Diffusion and Averaging

3.1 Diffusion on the Circle and the Rotation Group $SO(2)$

In this subsection we review diffusion on the circle. The purpose of this subsection is purely to serve as a connection between what is commonly known to engineers, and that which is presented in the next subsection, which is not known to most engineers.

Rotations in the plane are given by matrices of the form

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

All the information contained in the rotation matrix R is also contained in the unit vector

$$\vec{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and so we write

$$R(\vec{r}) = \begin{bmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{bmatrix}.$$

Similarly, the vector $\vec{r}$ is extracted from the rotation matrix R as $\vec{r}(R) = R\vec{e}_1$ where $\vec{e}_1 = [0,1]^T$.

The identity rotation then corresponds to the vector $\vec{r} = \vec{e}_1$, and an infinitesimal rotation close to the identity is of the form

$$R(\vec{e}_1 + \epsilon\vec{e}_2) = \mathbf{1}_{2\times 2} + \epsilon X,$$

where

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We note that $R(\vec{r}(\theta)) = \exp(\theta X)$ where $\exp(\cdot)$ is the matrix exponential.

Given a real-valued function which takes its argument from the unit circle, $f(\cdot)$ (we will use both the notation $f(\theta)$ and $f(\vec{r}(\theta))$ to mean the same thing) one defines the differential operator

$$Df = \frac{df}{da}(\exp(aX)\vec{r}(\theta))|_{a=0}.$$

It is easy to see that

$$Df = \frac{df}{da}(\vec{r}(a+\theta))|_{a=0} = \frac{df}{da}(a+\theta)|_{a=0} = \frac{df}{d\theta}.$$

If f is a function of time, t, in addition to θ, then the full derivative above is replaced with a partial one, and the Laplacian is defined as

$$\nabla^2 f = D^2 f = \frac{\partial^2 f}{\partial\theta^2}.$$

The heat equation on the circle is then

$$\frac{\partial f}{\partial t} = K\nabla^2 f.$$

It is well known that the Fourier series solution of this equation under the initial condition $f(\theta, 0) = \delta(\theta - 0)$ is of the form

$$f(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-n^2 K t} e^{in\theta} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 K t} \cos n\theta. \tag{18}$$

when the integration measure on the circle is $d\theta$. Another well-known form of the solution to the heat equation on the circle is what results from "wrapping" the solution of the heat equation on the line [5],

$$F(x, t) = \frac{1}{\sqrt{2\pi K t}} e^{-x^2/2Kt}$$

for initial conditions $F(x, 0) = \delta(x - 0)$, around the circle. I.e., shifting all intervals on the line of the form $[2\pi n, 2\pi(n + 1)]$ for $n \in \mathbf{Z}$ to the interval $[0, 2\pi]$, and superposing the values of the function. This is written as

$$f(\theta, t) = \sum_{n=-\infty}^{\infty} F(\theta - 2\pi n). \tag{19}$$

A nice feature of the expansion in Equation 19 is that when Kt is small, only one or at most a few terms in the expansion need to be retained since the Gaussian function decays so rapidly. Another nice feature is that the function $f(\theta, t)$ is always positive (as it should be) when using this expansion, whereas negative values and convergence problems are possible when using truncated Fourier expansions.

3.2 Diffusion on the Three-Sphere and the Rotation Group $SO(3)$

Spherical coordinates in $\mathbb{R}^4$ can be chosen as

$$\vec{r}(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \sin\theta_1 \sin\theta_2 \sin\theta_3 \end{bmatrix}$$

for $\theta_1 \in [0, \pi]$, $\theta_2 \in [0, \pi]$, $\theta_3 \in [0, 2\pi]$.

Using the four-dimensional matrix representation of rotation corresponding to $\vec{r}$, i.e., by defining $R_{4\times4}(r)$ such that

$$R_{4\times4}(r)q = rq$$

is consistent with Equation 7, one has:

$$R_{4\times4}(\vec{r}) = \begin{bmatrix} r_1 & -r_2 & -r_3 & -r_4 \\ r_2 & r_1 & -r_4 & r_3 \\ r_3 & r_4 & r_1 & -r_2 \\ r_4 & -r_3 & r_2 & r_1 \end{bmatrix}.$$

[5] Called a Gaussian or normal distribution

Hence, $R_{4\times4}(\vec{e}_1)$ is the identity matrix, and corresponds to no rotation. If one considers infinitesimal motions on the sphere S^3 in each of the remaining coordinate directions in the vicinity of the point $\vec{r} = \vec{e}_1$, the corresponding rotations will be of the form

$$R_{4\times4}(\vec{e}_1 + \epsilon\vec{e}_i) = \mathbf{1}_{4\times4} + \epsilon X_{i-1}$$

for $i = 2, 3, 4$ and $|\epsilon| << 1$. It is easy to see that $X_{i-1} = R_{4\times4}(\vec{e}_{i-1})$, and explicitly have the form

$$X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad X_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad X_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

In analogy with the way the cross product in $\mathbb{R}^3$ takes unit basis vectors into unit basis vectors as $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$, one finds that under the Lie bracket operation (which in this case we take to be half of matrix commutator),

$$[X_i, X_j] = \frac{1}{2}(X_i X_j - X_j X_i),$$

that

$$[X_1, X_2] = X_3; \quad [X_2, X_3] = X_1; \quad [X_3, X_1] = X_2.$$

This indicates that the matrices X_i are representations of the basis elements of the Lie algebra $so(3)$. These elements are not unique, as can be seen by the fact that

$$R_{4\times4}(\vec{e}_1 + \epsilon R(\vec{r})\vec{e}_i) = \mathbf{1}_{4\times4} + \epsilon R_{4\times4} X_{i-1} R^T_{4\times4}.$$

Hence any set of basis elements of the form $\tilde{X}_i = R_{4\times4} X_i R^T_{4\times4}$ is equally acceptable.

Differential motions on the surface of the sphere S^3 can be used to define differential operators which act on real valued-functions of the form $f(\vec{r}(\theta_1, \theta_2, \theta_3))$ as:

$$D_i f = \frac{d}{dt} f(\exp(tX_i)\vec{r}(\theta_1, \theta_2, \theta_3))|_{t=0}$$

for $i = 1, 2, 3$. Here

$$\exp(A) = \mathbf{1}_{4\times4} + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

is the matrix exponential for any $A \in \mathbb{R}^{4\times4}$.

After a little work, it can be shown that explicit form of these operators (in a basis, $\{\tilde{X}_i\}$ different than ours) is [16]

$$\tilde{D}_1 = -\cos\theta_2 \frac{\partial}{\partial\theta_1} + \sin\theta_2 \cot\theta_1 \frac{\partial}{\partial\theta_2} - \frac{\partial}{\partial\theta_3};$$

$$\tilde{D}_2 = -\sin\theta_2 \cos\theta_3 \frac{\partial}{\partial\theta_1} + (\sin\theta_3 - \cot\theta_1 \cos\theta_2 \cos\theta_3) \frac{\partial}{\partial\theta_2} + \left(\cot\theta_1 \frac{\sin\theta_3}{\sin\theta_2} + \cot\theta_2 \cos\theta_3\right) \frac{\partial}{\partial\theta_3};$$

$$\tilde{D}_3 = -\sin\theta_2 \sin\theta_3 \frac{\partial}{\partial\theta_1} - (\cos\theta_3 + \cot\theta_1 \cos\theta_2 \sin\theta_3) \frac{\partial}{\partial\theta_2} + \left(-\cot\theta_1 \frac{\cos\theta_3}{\sin\theta_2} + \cot\theta_2 \sin\theta_3 \right) \frac{\partial}{\partial\theta_3}.$$

The Laplacian, which is invariant under the choice of $so(3)$ basis, is defined as

$$\nabla^2 = (D_1)^2 + (D_2)^2 + (D_3)^2 = (\tilde{D}_1)^2 + (\tilde{D}_2)^2 + (\tilde{D}_3)^2 =$$

$$\frac{\partial^2}{\partial\theta_1^2} + 2\cot\theta_1 \frac{\partial}{\partial\theta_1} + \frac{\cot\theta_2}{\sin^2\theta_1} \frac{\partial}{\partial\theta_2} + \frac{1}{\sin^2\theta_1} \frac{\partial^2}{\partial\theta_2^2} + \frac{1}{\sin^2\theta_1 \sin^2\theta_2} \frac{\partial^2}{\partial\theta_3^2}.$$

The heat equation is

$$\frac{\partial f}{\partial t} = K\nabla^2 f. \tag{21}$$

The solution of this equation for $f(\theta_1, \theta_2, \theta_3, t)$ under the initial condition that $f(\theta_1, \theta_2, \theta_3, 0)$ is a Dirac delta function on S^3 has been known for many years. It was first derived by Perrin [17] in the late 1920s, and has been elaborated on by many mathematicians since then. Perrin's solution is of the form

$$f(\theta_1, \theta_2, \theta_3, t) = \frac{1}{\pi^2 \sin\theta_1} \sum_{n=0}^{\infty} (2n+1) e^{-[(2n+1)^2 - 1](Kt)} \sin(2n+1)\theta_1. \tag{22}$$

In a subsequent section we shall present an alternative form of the solution for small values of Kt which is analogous to the wrapped (folded) Gaussian for the case of the circle. It is this alternate form that will be central to our formulation of a sampling method for efficient generation of convolutions on $SE(3)$. The main issues regarding these convolutions are discussed in the following section.

4 Convolution on Groups

Given a group $(G, \circ)$ for which a left and right invariant integration measure ($d\mu(g) = d\mu(g \circ h) = d\mu(h \circ g)$ for all $h, g \in G$) exists[6], the convolution of functions square-integrable with respect to this measure is defined as

$$(f_1 \star f_2)(g) = \int_G f_1(h) f_2(h^{-1} \circ g) d\mu(h). \tag{23}$$

Here $\circ$ is the group operation, which is matrix multiplication in the case of rotations and rigid-body motions represented as homogeneous transforms. The physical meaning and applications of this kind of integral when $G = SE(N)$ has been studied extensively in [1, 2, 3, 4]. In all the discussions that follow, G is assumed to be this kind of a group, which is called a *unimodular* group. In general, the convolution of arbitrary square-integrable functions on such a noncommutative group is not commutative. A notable exception to this is when both functions are class functions, i.e., functions satisfying $f(g \circ h) = f(h \circ g)$ for all $g, h \in G$. This is equivalent to the statement $f(h^{-1} \circ g \circ h) = f(g)$, which clearly shows that a class function is invariant under conjugation, and hence is constant on *conjugacy classes* of

[6] This includes the rotation and motion groups of all $\mathbb{R}^N$ for finite N

the group (hence the name *class* function). With this definition, we can prove the following theorems which will be of importance in the sequel.

One example of a class function on a unimodular group is the Dirac delta. In complete analogy with the usual Dirac delta function on the real line, we have the properties

$$\int_G \delta(g)d\mu(g) = 1 \qquad \delta(h^{-1} \circ g) = \delta(g^{-1} \circ h)$$

and

$$\int_G f(h)\delta(h^{-1} \circ g)d\mu(h) = f(g).$$

Using these properties, it is easy to see that the convolution of a function with a shifted delta function, $\delta(g_i^{-1} \circ g)$, results in a shifted version of the original function.

THEOREM 1: The convolution of any function and a class function on a unimodular group is commutative, and the convolution of two class functions results in a class function.
Proof: By the definition of convolution, and the bi-invariance of the integration measure, a change of variables $k = h^{-1} \circ g$ allows one to write the convolution as

$$(f_1 \star f_2)(g) = \int_G f_1(g \circ k^{-1}) f_2(k)d\mu(k)$$

regardless of whether f_i are class functions or not. Assuming f_1 is a class function, then $f_1(g \circ k^{-1}) = f_1(k^{-1} \circ g)$, and since multiplication of scalar functions is commutative we have $f_1 \star f_2 = f_2 \star f_1$. The same follows if f_2 had been a class function instead of f_1.

To see that the convolution of two class functions is a class function, one observes that

$$(f_1 \star f_2)(k^{-1} \circ g \circ k) = \int_G f_1(h) f_2(h^{-1} \circ k^{-1} \circ g \circ k)d\mu(h).$$

If f_2 is a class function, then $f_2(h^{-1} \circ k^{-1} \circ g \circ k) = f_2(k \circ h^{-1} \circ k^{-1} \circ g)$. Making the change of variables $s^{-1} = k \circ h^{-1} \circ k^{-1}$, one finds

$$(f_1 \star f_2)(k^{-1} \circ g \circ k) = \int_G f_1(k^{-1} \circ s \circ k) f_2(s^{-1} \circ g)d\mu(s) = \int_G f_1(s) f_2(s^{-1} \circ g)d\mu(s),$$

where the last equality follows when f_1 is a class function. Hence we have

$$(f_1 \star f_2)(k^{-1} \circ g \circ k) = (f_1 \star f_2)(g),$$

which means $f_1 \star f_2$ is a class function.

THEOREM 2: The Convolution of two left(right)-shifted functions on a nonabelian unimodular group is equal to a left(right)-shifted version of the convolution of the unshifted functions if the original functions are class functions.
Proof: Let $\hat{f}_i(g) = f_i(g_i^{-1} \circ g)$. Then $\hat{f}_2(h^{-1} \circ g) = f_2(g_2^{-1} \circ (h^{-1} \circ g))$ and

$$(\hat{f}_1 \star \hat{f}_2)(g) = \int_G f_1(g_1^{-1} \circ h) f_2(g_2^{-1} \circ h^{-1} \circ g)d\mu(h).$$

Since f_1 is a class function, then

$$f_1(g_1^{-1} \circ h) = f_1(h \circ g_1^{-1}).$$

Letting $s = h \circ g_1^{-1}$, and using the fact that f_2 is a class function, then

$$(\hat{f}_1 \star \hat{f}_2)(g) = \int_G f_1(s) f_2(g_2^{-1} \circ g_1^{-1} \circ s^{-1} \circ g) d\mu(s) = (f_1 \star f_2)(g \circ g_2^{-1} \circ g_1^{-1}).$$

From Theorem 1, $f_1 \star f_2$ is a clas function, and so

$$(f_1 \star f_2)(g \circ g_2^{-1} \circ g_1^{-1}) = (f_1 \star f_2)((g_1 \circ g_2)^{-1} \circ g).$$

A similar proof follows for right shifts.

THEOREM 3: A function on $SO(3)$ is a class function if and only if it is a function solely of the angle of rotation.
Proof: By definition, matrix invariants are functions of matrix valued argument that are invariant under similarity transformation: $I_i(A) = I_i(SAS^{-1})$ for all invertible S of compatible dimension. The class functions for $SO(3)$ must be functions of the three matrix invariants of $A \in SO(3)$. That is, $f(A) = f_1(I_1(A), I_2(A), I_3(A))$, where $\det(\lambda 1 - A) = \lambda^3 - I_1(A)\lambda^2 + I_2(A)\lambda - I_3(A) = 0$. Note that $I_3(A) = \det(A) = +1$, and any element of $SO(3)$ can be written as $A(\theta, \omega) = R\, Z(\theta)\, R^T$ where θ and ω are the angle and axis of rotation, $Z(\theta)$ is rotation around z-axis, and R is any rotation matrix R with third column ω. Hence, any element of $SO(3)$ can be parametrized as

$$A(\theta, \omega) = e^{\theta[\omega]} = \mathbf{1}_{3\times 3} + \sin\theta[\omega] + (1 - \cos\theta)[\omega]^2$$

where θ is the angle of rotation, and ω is the unit vector specifying the axis of rotation. The matrix $[\omega]$ is the skew-symmetric matrix such that $[\omega]\mathbf{x} = \omega \times \mathbf{x}$ for any vector $\mathbf{x} \in I\!R^3$. From this, it is clear that $I_1(A) = tr(A) = 2\cos\theta + 1$ and $I_2(A) = \frac{1}{2}[(tr(A))^2 - tr(A^2)]$ are functions of the rotation angle only, and so class functions on $SO(3)$ must be of the form $f(A) = f_1(\theta)$.

THEOREM 4: No nontrivial square-integrable class functions exist for $SE(3)$.
Proof: See [7]

In the definitions of this section we have used standard mathematical notations. However, in the sequel, this would become somewhat awkward. Therefore, a different, and somewhat more compact, notation in the context of the problems to be addressed later is introduced in the following subsection.

4.1 Mathematician's Notation versus Physicist's

Mathematicians and physicists use generally a different notation for functions. The physicist's notation can be sometimes ambiguous and must be handled with care. In return, its connection with the concrete world is generally stressed. For example, a mathematician denotes a function as $f : I\!R^n \to I\!R$. For him, f is the name of the function. If he needs to give symbols to the independent variable and to the dependent one, he will write $f : \vec{x} \mapsto f(\vec{x})$ or $f : \vec{x} \mapsto y$. For him, $f(\vec{x})$ is *not* a function, but an element like $\vec{x}$ or y. A physicist, on the other hand, will write $\vec{v}(t)$ to denote a velocity depending on the time t. For him, $\vec{v}(t)$ can as well be a *function*. Following his same logic, he will understand $\vec{v}(\vec{F})$ as being another function, here the velocity $\vec{v}$ depending of the force $\vec{F}$. To agree with the mathematician, he

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Figure 2: Examples of Dirac Density Functions

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Figure 3: Examples of Gaussian Density Functions

should write $f : t \mapsto \vec{v}$ and $g : \vec{F} \mapsto \vec{v}$ which means that functions f and g apply onto the same variable $\vec{v}$. He does not do that simply because this is implicit in his mind and because the writing emphasizes $\vec{v}$ instead of some rather abstract symbols such as f or g. Of course, one could choose $\vec{v}_t$ or $\vec{v}_{\vec{F}}$ instead of f and g, but that requires an index and complicates things when other indices are needed. Therefore, one can as well write $\vec{v}(t)$ or $\vec{v}(\vec{F})$. This presents other advantages, like considering a shifted function simply written $\vec{v}(\vec{F} - \vec{F}_0)$ instead of $h : \vec{F} \mapsto h(\vec{F}) = g(\vec{F} - \vec{F}_0)$, where $g : \vec{F} \mapsto \vec{v}$. The drawbacks are that the physicist's notation requires some "interpretation" or understanding of what it represents and may lead to contradictions or errors if used in a simply mechanical way. Another example of this different notation is convolutions. Mathematicians are used to writing $h = g \star f$. With the physicist's notation where the inside of the parenthesis is relevant for the identification of the functions at hand, one may see $u(t - \tau) = \delta(t - \tau) * u(t)$ or, in the more abstract case $f_3(z) = f_1(y) * f_2(x)$ which in a mathematician's notation means simply $f_3 = f_1 * f_2$ or $f_3(x) = (f_1 * f_2)(x)$.

This physicist's notation will be extensively used in the sequel because it is very convenient when dealing with the convolution of shifted functions.

In addition, we will henceforth drop the explicit use of the symbol $\circ$ for group operation. This product will henceforth be implied whenever two group elements are juxtaposed.

Another notational change will be that if a pure translation (or rotation) is applied to an arbitrary group element, we will use the following shorthand. If $g(\vec{r}_i, q_i) \in SE(3)$ is a full motion (with q_i being the rotation described as a unit quaternion) then, instead of writing the product $g^{-1}(\vec{r}_i, 0)h$ or $g^{-1}(\vec{0}, q_i)h$ for any $h \in SE(3)$, we will write $\vec{r}_i^{-1}h$ and $q_i^{-1}h$. This shorthand may seem awkward to a mathematician, but there is no ambiguity and it reduces the use of a substantial number of symbols.

4.2 Density Functions

Several frames f_i can be collected in a *density functions* $\varphi(f)$ in the form of a sum of *Dirac delta functions* $\delta(f_i^{-1}f)$. A density function $\psi(g)$ is likewise defined for the motions g. Figure 2 illustrates this concept in the one-dimensional case. The density function $\varphi(p)$ includes 2 frames, or merely their origin, localized in p_1, p_2 with a "weight" of 1 since no coefficient accompanies the δ functions. The density function $\psi(d)$ similarly represents 2 motions *i.e.*, translations, of value d_1, d_2.

The natural extension of the density function defined so far is to consider the possibility of "diffused" frames (or motions) as well. Figure 3 gives an example.

Sometimes, when the position $\vec{p}$ and the orientation q are independent, the density func-

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Figure 4: Example of Density Functions with Independent Translational and Rotational Parts

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Figure 5: The Result of Convolving a Sum of Diracs

tion $\varphi(f)$ can be broken up into two subfunctions[7] $\varphi(\vec{p})$ and $\varphi(q)$

$$\varphi(f) = \varphi(\vec{p})\varphi(q) \tag{24}$$

For such functions, the two-dimensional case can be displayed, as one can see in Figure 4.

4.3 Multiple Convolutions

Often, several convolutions have to be performed

$$\varphi_N = \psi_N \star \ldots \star \psi_1 \star \varphi_0 \tag{25}$$

Suppose first that $N = 1$ and, as an example, consider the convolution of the two density functions of Figure 2. The result is illustrated in Figure 5-a. Now, if $N = 2$, Figure 5-b is obtained, assuming that $\psi_2 = \psi_1$. Two Dirac functions have double height. As a matter of fact, if one goes on with $\psi_i = \psi_1$, as in this example the group product $(+)$ is commutative, the number of Dirac functions will grow *linearly* because some of them pile up. However, as soon as either ψ_i is different from ψ_1 or the group product is not commutative, the number of Dirac functions will increase *exponentially*, like 2^{N+1}. This means that for a relatively small N, the computation of the density function φ_N is generally impossible.

As we will see in Section 7 the workspace description of a 15-module binary robot is given by the density function obtained after 14 convolutions. The number of Dirac functions would be 8^{15} which is an astronomical number. Also in Section 7 we will discuss how one can compute an approximation of such a discrete density function to any desired degree of accuracy. This will involve the concept of diffusion discussed in Section 3.

5 Decomposition into Basic Convolutions

5.1 Decomposition

5.1.1 Distributivity

Suppose that in the convolution $\varphi'(f') = \psi(g) \star \varphi(f)$ the density functions consist of sums of more *basic functions*

$$\psi(g) \;=\; \sum_{j=1}^{n} \psi_j(g_j^{-1} g) \tag{26}$$

[7] Remember that the physicist's notation is used (subsection 4.1) and therefore that $\varphi(f)$, $\varphi(\vec{p})$, $\varphi(q)$ are three different functions, which is clear from their arguments. In a mathematician's notation it would be $\varphi : f \mapsto \varphi_{\vec{p}}(f)\varphi_q(f)$.

$$\varphi(f) \quad = \quad \sum_{i=1}^{m} \varphi_i(f_i^{-1} f). \tag{27}$$

Then, the sums can be taken out of the convolution since integrals and sums commute

$$\left[\sum_{j=1}^{n} \psi_j(g_j^{-1} g)\right] \star \left[\sum_{i=1}^{m} \varphi_i(f_i^{-1} f)\right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \psi_j(g_j^{-1} g) \star \varphi_i(f_i^{-1} f) \tag{28}$$

5.1.2 Normalization

The next possible simplification is to "center" the density function $\psi_j(g_j^{-1} g)$

$$\psi_j(g_j^{-1} g) = \delta(g_j^{-1} g) \star \psi_j(g) \tag{29}$$

The same can be applied to the density function $\varphi_j(f_j^{-1} f)$.

5.1.3 Swapping of Rotations

The expression $\delta(f_i^{-1} f)$ for $f_i = (\vec{p}_i, q_i)$ means that a rotation $\delta(q_i^{-1} f)$ is followed by a translation $\delta(\vec{p}_i^{\leftarrow 1} f)$. In other words

$$\varphi_i(f_i^{-1} f) = \delta(\vec{p}_i^{\leftarrow 1} f) \star \delta(q_i^{-1} f) \star \varphi_i(f). \tag{30}$$

Now, assume that $\varphi_i(f)$ for $f = (\vec{p}, q)$ may be written as

$$\varphi_i(f) = \varphi_i(p)\varphi_i(\sigma) \tag{31}$$

where $p = \sqrt{\vec{p}^T \vec{p}}$ and $\sigma = \arccos(q_1)$. In this case, the rotation part commutes and one obtains

$$\varphi_i(f_i^{-1} f) = \delta(\vec{p}_i^{\leftarrow 1} f) \star \varphi_i(f) \star \delta(q_i^{-1} f). \tag{32}$$

5.1.4 Reduction of Translations

The translation component $\delta(\vec{p}_i^{\leftarrow 1} f)$ is determined by the vector $\vec{p}_i$ which has a direction $r_{p_i} \vec{e}_1$ and a magnitude p_i. Thus, a translation may be understood as[8]

$$\delta(\vec{p}_i^{\leftarrow 1} f) = \delta(r_{p_i}^{-1} f) \star \delta(p_i^{-1} f) \star \delta(r_{p_i} f) \tag{33}$$

where r_{p_i} describes the rotation that moves vector $(p_i, 0, 0)$ to $\vec{p}_i$. Taking into account this property, one obtains

$$\varphi_i(f_i^{-1} f) = \delta(r_{p_i}^{-1} f) \star \delta(p_i^{-1} f) \star \delta(r_{p_i} f) \star \varphi_i(f) \star \delta(q_i^{-1} f) \tag{34}$$

which is also

$$\varphi_i(f_i^{-1} f) = \delta(r_{p_i}^{-1} f) \star \delta(p_i^{-1} f) \star \varphi_i(f) \star \delta(r_{p_i} f) \star \delta(q_i^{-1} f) \tag{35}$$

or, finally

$$\varphi_i(f_i^{-1} f) = \delta(r_{p_i}^{-1} f) \star \delta(p_i^{-1} f) \star \varphi_i(f) \star \delta(q_i^{-1} r_{p_i} f). \tag{36}$$

[8] $p_i^{-1} f$ still means that a frame f is translated, here along the x-axis, by a distance p_i. In other words, the product is still that of a translation of arbitrary element of $SE(3)$. Do not get confused and think of p_i as a scalar, since f/p_i would have no meaning.

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Figure 6: First Basic Convolution

5.1.5 First Basic Convolution

Putting things together, the decomposition process has led to

$$\varphi^{'}(f^{'}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \delta(g_j^{-1}g) \star \psi_j(g) \star \delta(r_{p_i}^{-1}f) \star \delta(p_i^{-1}f) \star \varphi_i(f) \star \delta(q_i^{-1}r_{p_i}f). \tag{37}$$

Supposing that the same assumption as for $\varphi_i(f)$ may be made for $\psi_j(g)$. That is,

$$\psi_j(g) = \psi_j(d)\psi_j(r) \tag{38}$$

where $d = \sqrt{\vec{d}^T\vec{d}}$, the rotation $\delta(r_{p_i}^{-1}f)$ can be moved in front and combined with the rotation part r_j of g_j

$$\varphi^{'}(f^{'}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \delta(\vec{d_j}^{-1}g) \star \delta(r_{p_i}^{-1}r_j^{-1}g) \star \psi_j(g) \star \delta(p_i^{-1}f) \star \varphi_i(f) \star \delta(q_i^{-1}r_{p_i}f). \tag{39}$$

Suppose that the convolution $\psi_j(g) \star \delta(p_i^{-1}f)$ is now simple enough to allow a direct computation. This operation is called the *first basic convolution*. An illustration can be seen in Figure 6 for the two-dimensional case. Note that the resulting density function cannot be split, because the two variables $\vec{p}$ and q are not independent. Thus, a complete plot cannot be drawn. In Figure 6, the plots for two specific values of θ have been superposed, though one should keep in mind that θ is a continuous variable and, since the space is four-dimensional, there is in reality no such overlay. We define

$$\psi_{ij}^{'}(g^{'}) = \psi_j(g) \star \delta(p_i^{-1}f). \tag{40}$$

5.1.6 Second Basic Convolution

Section 6 will describe how one can transform the density function $\psi_{ij}^{'}(g^{'})$ into a sum of basic functions like

$$\psi_{ij}^{'}(g^{'}) = \sum_{k=1}^{l} \psi_{ijk}^{'}((g_k^{'})^{-1}g^{'}) \tag{41}$$

Then the sum can be moved again out the convolution

$$\varphi^{'}(f^{'}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \delta(\vec{d_j}^{-1}g) \star \delta(r_{p_i}^{-1}r_j^{-1}g) \star \psi_{ijk}^{'}((g_k^{'})^{-1}g^{'}) \star \varphi_i(f) \star \delta(q_i^{-1}r_{p_i}f) \tag{42}$$

and the function $\psi_{ijk}^{'}((g_k^{'})^{-1}g^{'})$ can be "centered"

$$\varphi^{'}(f^{'}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \delta(\vec{d_j}^{-1}g) \star \delta(r_{p_i}^{-1}r_j^{-1}g) \star \delta((g_k^{'})^{-1}g^{'}) \star \psi_{ijk}^{'}(g^{'}) \star \varphi_i(f) \star \delta(q_i^{-1}r_{p_i}f). \tag{43}$$

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Figure 7: Second Basic Convolution

The convolution $\psi'_{ijk}(g') \star \varphi_i(f)$ is now assumed to be simple enough to be carried out by direct computation. This is the *second basic convolution* (see illustration in Figure 7)

$$\varphi'_{ijk}(f') = \psi'_{ijk}(g') \star \varphi_i(f). \tag{44}$$

5.1.7 Decentering

The last step is to "move" this function according to the original uncentered functions:

$$\varphi'(f') = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \varphi'_{ijk}(g_k^{-1} r_{p_i}^{-1} r_j^{-1} \vec{d}_j^{-1} f' q_i^{-1} r_{p_i}). \tag{45}$$

5.2 Basic Functions

5.2.1 The One-Dimensional Case

To thoroughly exploit the power of decomposition the basic functions should belong to a single family generated by one or two parameters only. Moreover, the convolution of two basic functions should still result in a basic function. For the one-dimensional case the chosen basic function is a Gaussian function

$$\psi(d) = b\mathrm{e}^{-\pi \mathrm{b}^2 \mathrm{d}^2} \tag{46}$$

$$\varphi(p) = a\mathrm{e}^{-\pi \mathrm{a}^2 \mathrm{p}^2} \tag{47}$$

because it fulfills the above requirements and also because the Gaussian function is rapidly decreasing in both the spatial and Fourier domains. It is therefore a good tradeoff between finitely supported functions and spectrums, which is a necessary property for numerical computation, as will be discussed through Section 6. Note that the spectrum of these functions is simply a Gaussian function. An example of such basic functions can be found in Figure 8. The convolution for the one-dimensional case degenerates into a usual convolution, and the usual convolution of two Gaussian functions still results in a Gaussian function

$$\varphi'(p') = \int_{-\infty}^{\infty} \psi(x)\varphi(-x+p')\mathrm{dx} \tag{48}$$

$$= \int_{-\infty}^{\infty} \left[b\mathrm{e}^{-\pi \mathrm{b}^2 \mathrm{x}^2}\right] \left[a\mathrm{e}^{-\pi \mathrm{a}^2(-\mathrm{x}+\mathrm{p}')^2}\right] \mathrm{dx} \tag{49}$$

$$= a'\mathrm{e}^{-\pi(\mathrm{a}')^2(\mathrm{p}')^2} \tag{50}$$

where

$$\frac{1}{a'^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{51}$$

We remark that the convolution is simply a product in the spectrum (frequency) domain and the product of two Gaussian functions is still a Gaussian function.

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Figure 8: One-Dimensional Basic Functions

5.2.2 The Two-Dimensional Case

The two-dimensional case involves also rotations and orientations. One can first factorize the density function as

$$\psi(\vec{d},\vec{r}) = \psi(\vec{d})\psi(\vec{r}) \quad (52)$$

$$\varphi(\vec{p},\vec{q}) = \varphi(\vec{p})\varphi(\vec{q}). \quad (53)$$

The translation and position part are still defined as Gaussian functions, that is

$$\psi(\vec{d}) = b^2 \mathrm{e}^{-\pi \mathrm{b}^2 \vec{\mathrm{d}}^{\mathrm{T}} \vec{\mathrm{d}}} \quad (54)$$

$$\varphi(\vec{p}) = a^2 \mathrm{e}^{-\pi \mathrm{a}^2 \vec{\mathrm{p}}^{\mathrm{T}} \vec{\mathrm{p}}}. \quad (55)$$

For the rotation and orientation part, $\vec{r}$ and $\vec{q}$ are used and parameterized in the following way

$$\vec{r} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad (56)$$

$$\vec{q} = \begin{bmatrix} \cos\sigma \\ \sin\sigma \end{bmatrix}. \quad (57)$$

The basic function is defined as a Gaussian function "spread on the circle"

$$\psi(\theta) = \beta \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi\beta^2(\theta - 2\mathrm{k}\pi)^2} \quad (58)$$

$$\varphi(\sigma) = \alpha \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi\alpha^2(\sigma - 2\mathrm{k}\pi)^2}. \quad (59)$$

Note that the spectrum of these functions is simply a sampled Gaussian function (as a result of the periodicity). See Figure 9 for a plot of such basic functions. For these basic functions, rotations and translations are decoupled, since the rotation/orientation part is independent of the translation/position part and since $\varphi(\vec{p})$ is invariant under rotation. Thus, the result of the convolution is

$$\varphi'(\vec{p}',\vec{q}') = \varphi'(\vec{p}')\varphi'(\vec{q}') \quad (60)$$

where

$$\varphi'(\vec{p}') = (a')^2 \mathrm{e}^{-\pi(\mathrm{a}')^2(\vec{\mathrm{p}}')^{\mathrm{T}}\vec{\mathrm{p}}'} \quad (61)$$

$$\varphi'(\sigma') = \alpha' \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi(\alpha')^2(\sigma' - 2\mathrm{k}\pi)^2} \quad (62)$$

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Figure 9: Two-Dimensional Basic Functions

with

$$\frac{1}{(a^{'})^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{63}$$

$$\frac{1}{\alpha^{'2}} = \frac{1}{\alpha^2} + \frac{1}{\beta^2}. \tag{64}$$

5.2.3 The Three-Dimensional Case

The extension of the translation/position part to the three-dimensional case is straight forward

$$\psi(\vec{d}) = b^3 \mathrm{e}^{-\pi \mathrm{b}^2 \vec{\mathrm{d}}^{\mathrm{T}} \vec{\mathrm{d}}} \tag{65}$$

$$\varphi(\vec{p}) = a^3 \mathrm{e}^{-\pi \mathrm{a}^2 \vec{\mathrm{p}}^{\mathrm{T}} \vec{\mathrm{p}}}. \tag{66}$$

For the rotation/orientation part the following parameterization is proposed

$$\vec{r} = \begin{bmatrix} \cos\theta \\ \vec{v}\sin\theta \end{bmatrix} \tag{67}$$

$$\vec{q} = \begin{bmatrix} \cos\sigma \\ \vec{u}\sin\sigma \end{bmatrix} \tag{68}$$

where $\vec{v}$ and $\vec{u}$ are unit vectors, that is

$$\vec{v} = \begin{bmatrix} \cos\nu_1 \\ \cos\nu_2 \sin\nu_1 \\ \sin\nu_2 \sin\nu_1 \end{bmatrix} \tag{69}$$

$$\vec{u} = \begin{bmatrix} \cos\mu_1 \\ \cos\mu_2 \sin\mu_1 \\ \sin\mu_2 \sin\mu_1 \end{bmatrix}. \tag{70}$$

Then, the basic functions are defined as

$$\psi(\theta) = \frac{\beta^3}{\sin\theta} \sum_{k=-\infty}^{\infty} (\theta - 2k\pi) \mathrm{e}^{-\pi\beta^2(\theta - 2\mathrm{k}\pi)^2} \tag{71}$$

$$\varphi(\sigma) = \frac{\alpha^3}{\sin\sigma} \sum_{k=-\infty}^{\infty} (\sigma - 2k\pi) \mathrm{e}^{-\pi\alpha^2(\sigma - 2\mathrm{k}\pi)^2}. \tag{72}$$

See Figure 10 for a plot of such basic functions.

The result of the convolution of the translation/position part is again a basic function

$$\varphi^{'}(\vec{p}^{'}) = (a^{'})^3 \mathrm{e}^{-\pi (\mathrm{a}^{'})^2 (\vec{\mathrm{p}}^{'})^{\mathrm{T}} \vec{\mathrm{p}}^{'}} \tag{73}$$

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Figure 10: Three-Dimensional Basic Functions (rotation/orientation part)

where

$$\frac{1}{a^{'2}} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{74}$$

The convolution of the rotation/orientation part is more difficult to perform. First, note that the basic functions do not depend on $\vec{u}$ or $\vec{v}$ and neither does the density function $\varphi^{'}(\sigma^{'})$ resulting from the convolution. Thus, without any loss of generality, one can assume that $\mu_1^{'} = 0$ and $\mu_2^{'} = 0$. The convolution to perform is then

$$\begin{aligned} \varphi^{'}(q^{'}) &= \psi(r) \star \varphi(q) & (75)\\ &= \int_0^{\pi}\int_0^{\pi}\int_{-\pi}^{\pi} \psi(\theta)\varphi(\sigma)\sin^2\theta \sin\nu_1 \mathrm{d}\nu_2 \mathrm{d}\nu_1 \mathrm{d}\theta & (76) \end{aligned}$$

where

$$\cos\sigma = \cos\theta\cos\sigma^{'} - \cos\nu_1\sin\theta\sin\sigma^{'}. \tag{77}$$

Since neither $\psi(\theta)$ nor $\varphi(\sigma)$ depend on ν_2, one obtains

$$\varphi^{'}(\sigma^{'}) = 2\pi\int_0^{\pi}\psi(\theta)\sin^2\theta\int_0^{\pi}\varphi(\sigma)\sin\nu_1\mathrm{d}\nu_1\mathrm{d}\theta. \tag{78}$$

As σ is a function of ν_1, the variable of the integral can be changed:

$$-\sin\sigma\mathrm{d}\sigma = \sin\nu_1\sin\theta\sin\sigma^{'}\mathrm{d}\nu_1 \tag{79}$$

$$\varphi^{'}(\sigma^{'}) = -\frac{2\pi}{\sin\sigma^{'}}\int_0^{\pi}\tilde{\psi}(\theta)\int_{\sigma^{'}+\theta}^{\sigma^{'}-\theta}\tilde{\varphi}(\sigma)\mathrm{d}\sigma\mathrm{d}\theta \tag{80}$$

where

$$\begin{aligned} \tilde{\psi}(\theta) &= \psi(\theta)\sin\theta & (81)\\ \tilde{\varphi}(\sigma) &= \varphi(\sigma)\sin\sigma. & (82) \end{aligned}$$

Assuming that $\tilde{\Phi}$ is the indefinite integral of $\tilde{\varphi}$, the convolution can be rewritten as

$$\begin{aligned} \varphi^{'}(\sigma^{'}) &= \frac{2\pi}{\sin\sigma^{'}}\int_0^{\pi}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta - \frac{2\pi}{\sin\sigma^{'}}\int_0^{\pi}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}+\theta)\mathrm{d}\theta & (83)\\ &= \frac{2\pi}{\sin\sigma^{'}}\int_0^{\pi}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta - \frac{2\pi}{\sin\sigma^{'}}\int_{-\pi}^{0}\tilde{\psi}(-\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta & (84)\\ &= \frac{2\pi}{\sin\sigma^{'}}\int_0^{\pi}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta + \frac{2\pi}{\sin\sigma^{'}}\int_{-\pi}^{0}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta & (85)\\ &= \frac{2\pi}{\sin\sigma^{'}}\int_{-\pi}^{\pi}\tilde{\psi}(\theta)\tilde{\Phi}(\sigma^{'}-\theta)\mathrm{d}\theta & (86)\\ &= \frac{(\alpha^{'})^3}{\sin\sigma^{'}}\sum_{k=-\infty}^{\infty}(\sigma^{'}-2k\pi)\mathrm{e}^{-\pi(\alpha^{'})^2(\sigma^{'}-2k\pi)^2} & (87) \end{aligned}$$

where

$$\frac{1}{\alpha'^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2}. \tag{88}$$

Observe that the convolution of two basic functions is commutative, which is not usual in $SO(3)$.

We note also, that in addition to being closed under the operation of convolution, these functions are solutions to the equation

$$\frac{\partial f}{\partial t} = K(\nabla^2 f - f)$$

(where ∇^2 is the Laplacian on the quaternion sphere) with initial conditions $f(g; t = 0) = \delta(g)$ when

$$\beta = \frac{1}{2\sqrt{\pi K t}}.$$

Hence, these "folded normal" functions are not exact solutions to the heat equation on $SO(3)$ (or, equivalently, the quaternion sphere),(21, though for small enough values of Kt (large values of β) the solutions become indistinguishable from those of the heat equation. The benefit here is, of course, analogous to the planar case where the folded normal is very well approximated by the first term in the series, whereas the Fourier solution requires a large number of terms for the same accuracy when β is large. Some nice properties of these functions result as well, since they are functions of the angle of rotation (recall $\theta = \rho/2$), and so the theorems of Section 4 apply.

6 Sampling and Reconstruction

Methods of harmonic analysis of functions on the circle, line, and $\mathbb{R}^N$ are well known to many engineers. The Shannon sampling theorem is of particular importance in regard to the reconstruction of band-limited functions on the line. We present an alternative sampling technique for functions on the line based on Gaussians, and show how this formulation is applicable to convolution of functions on $SE(N)$.

6.1 Band-Limited Functions and Sampling

6.1.1 Spectrum

The *spectrum* (Fourier transform) $H(\vec{f}) \in \mathbf{C}$ of a function $h : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$H(\vec{f}) = \int_{\mathbb{R}^n} h(\vec{x}) \mathrm{e}^{-i2\pi \vec{x}^T \vec{f}} \mathrm{d}\vec{\mathrm{x}} \tag{89}$$

where $\vec{f} \in \mathbb{R}^n$ is the so-called *frequency*. Note that a "translation" $\vec{d}$ of the initial function does not alter the magnitude of a spectrum and a "rotation" R (with $R^T R = \mathbf{1}$) simply rotates the spectrum

$$h(\vec{x} - \vec{d}) \leftrightarrow H(\vec{f}) \mathrm{e}^{-i2\pi \vec{d}^T \vec{f}} \tag{90}$$

$$h(R\vec{x}) \leftrightarrow H(R\vec{f}). \tag{91}$$

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Figure 11: Example of a Function and its Spectrum (Gaussian)

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig4-2.gif

Figure 12: Example of a Band-Limited Function and its Spectrum (sinc)

The Gaussian function is special in that its spectrum is also a Gaussian function:

$$h(\vec{x}) = \prod_{i=1}^{n} a_i \mathrm{e}^{-\pi a_i^2 x_i^2} \tag{92}$$

$$H(\vec{f}) = \prod_{i=1}^{n} \mathrm{e}^{-\pi f_i^2 / a_i^2}. \tag{93}$$

Figure 11 gives an illustration for $x \in I\!R$.

6.1.2 Band-Limited Functions

A function $h(\vec{x})$ is called a *band-limited function* with band-limit ν if its spectrum $H(\vec{f})$ fulfills the following condition[9]

$$H(\vec{f}) = 0 \quad \text{when} \quad \|\vec{f}\| \geq \nu \tag{94}$$

where $\|\vec{f}\| = \sqrt{\vec{f}^T \vec{f}}$. Note that a rigidly "moved" band-limited function $h(R\vec{x} + \vec{d})$ is a band-limited function with the same band limit.

As an example of a function $h(x)$ band-limited to $\nu = \frac{1}{2\Delta}$, consider the *sine cardinal* and its spectrum $H(f)$ (see Figure 12)

$$h(x) = \frac{\sin \pi \frac{x}{\Delta}}{\pi \frac{x}{\Delta}} \tag{95}$$

$$\tag{96}$$

$$H(f) = \begin{cases} \Delta & \text{if } |f| < \frac{1}{2\Delta} \\ 0 & \text{elsewhere} \end{cases}. \tag{97}$$

The Gaussian function (92) with $a_i = a$ is not, strictly speaking, band-limited (see Figure 11), but for $w = \frac{\nu}{a}$ sufficiently big, one can make the approximation $H(\vec{f}) \approx 0$. Moreover, Gaussian functions tend so rapidly to zero that, for $w > 1.32$ typically, every small increase of w leads to a bigger and bigger amelioration of the approximation.

6.1.3 Sampling and Shannon's Reconstruction

The *sampling* of a function $h(x)$ consists of a multiplication with an array of Dirac functions $\delta_\Delta(\vec{x})$

$$\hat{h}(\vec{x}) = \delta_\Delta(\vec{x}) h(\vec{x}) \tag{98}$$

[9] In other contexts, sometimes a less conservative definition is used: $H(\vec{f}) = 0$ when $\|f_i\| \geq \nu$.

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Figure 13: Arrays of Dirac Functions (One and Two Dimensions)

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig4-4.gif

Figure 14: Sampled Function and its Spectrum (Gaussian)

where $\delta_\Delta(\vec{x})$ is defined as (see Figure 13)

$$\delta_\Delta(\vec{x}) = \prod_{i=1}^{n} \sum_{j=-\infty}^{\infty} \delta(x_i - j\Delta_i). \tag{99}$$

The spectrum of an array of Dirac functions $\delta_\Delta(\vec{x})$ is still an array of Dirac functions, though with a step of $\frac{1}{\Delta}$ and an altered weight

$$\delta_\Delta(\vec{x}) \leftrightarrow \prod_{i=1}^{n} \frac{1}{\Delta_i} \sum_{j=-\infty}^{\infty} \delta(f_i - \frac{j}{\Delta_i}) \tag{100}$$

When two functions are multiplied, their spectrums are convolved. Consequently, sampling a function $h(\vec{x})$ duplicates periodically its spectrum $H(\vec{f})$ (and alters the magnitude) according to the sampling rates Δ_i (compare Figures 11 and 14).

According to *Shannon's theorem*, a band-limited function $h(x)$ can be sampled without any loss of information provided that the sampling rates Δ_i fulfill the condition

$$\frac{1}{\Delta_i} \geq 2\nu. \tag{101}$$

No loss of information means that the original function $h(\vec{x})$ can be exactly reconstructed from the sampled function $\hat{h}(\vec{x})$. This is done by using a low-pass filter which "selects" the original spectrum among their aliases. When the above Shannon condition is not fulfilled, the original function cannot be reconstructed, since its spectrum is altered by too-narrow aliases. Below is the reconstruction formula

$$h(\vec{x}) = \hat{h}(\vec{x}) * \prod_{i=1}^{n} \frac{\sin \pi \frac{x_i}{\Delta_i}}{\pi \frac{x_i}{\Delta_i}} \tag{102}$$

where $*$ is the usual convolution on $\mathbb{R}^n$. Figure 15 depicts schematically the application of this reconstruction on the sampled function of the Gaussian function of Figure 14. The result of the reconstruction is identical (or almost, depending on w) to the initial Gaussian function of Figure 11.

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Figure 15: Schematic Plot of Reconstruction (Sine Cardinal)

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig4-6.gif

Figure 16: Schematic Plot of Gaussian Reconstruction

6.2 Gaussian Reconstruction

6.2.1 Functions of Finite Support

A numerical computation can be carried out only providing that i) the involved functions $h(\vec{x})$ can be sampled without any loss of information and ii) the *support of the functions* $h(\vec{x})$ is finite, for example

$$h(x) = \begin{cases} \text{whatever} & \text{if } |x| < \xi \\ 0 & \text{elsewhere} \end{cases} . \tag{103}$$

Unfortunately these two conditions cannot be fulfilled simultaneously. A reasonable tradeoff is again the Gaussian function (see Figure 11) since for $w = \xi a$ sufficiently big (typically $w > 1.32$), one can assume in practice that the support of the function is finite. At any rate, if the result of a computation based on this assumption is $r(w)$, owing to the rapid decreasing of Gaussian functions, $r(w)$ will converge rapidly to its asymptotic value.

6.2.2 Reconstruction

A function $h(\vec{x})$ can be reconstructed from its sampled version $\hat{h}(\vec{x})$ by a sum of sine cardinal functions as stated in Equation 102. However, such an operation can be costly to perform since the sine cardinal function does not have a finite support which means that to interpolate between two samples, all the other samples have to be considered (see Figure 15). It would be more convenient if instead of the sine cardinal, one had a function with finite support, since only the nearest neighbors would have to be considered.

What is the price to pay if a Gaussian function like the one in Figure 15-a) is used instead of the sine cardinal of Figure 16-a)? The answer can be "read" in the frequency domain of Figure 16-b). As a side effect, the rejection of the aliases by a Gaussian alters the original spectrum. Consequently, the reconstructed function is somewhat different from the original one. The following equation holds, which means that the reconstructed function is low-pass filtered in comparison with the original function

$$h(\vec{x}) * (\hat{a}\Delta)^n \mathrm{e}^{-\pi \hat{a}^2 \vec{x}^T \vec{x}} \approx \hat{h}(\vec{x}) * (\hat{a}\Delta)^n \mathrm{e}^{-\pi \hat{a}^2 \vec{x}^T \vec{x}} \tag{104}$$

where $\hat{a} = \frac{1}{2w\Delta}$.

6.2.3 High-pass Filtering

In the previous subsection it was shown that the reconstructed function by Gaussian filtering is low-pass filtered in comparison with the original function $h(\vec{x})$. If one samples a high-pass filtered version $h'(\vec{x})$ instead of $h(\vec{x})$, one can compensate the effect of the subsequent low-pass process. For example, assume that

$$h(\vec{x}) = \sum_i c_i \mathrm{e}^{-\pi a^2 (\vec{x} - \vec{x}_i)^T (\vec{x} - \vec{x}_i)} . \tag{105}$$

Then, the high-pass filtered version is

$$h'(\vec{x}) = \left[\frac{a'}{a\Delta}\right]^n \sum_i c_i \mathrm{e}^{-\pi a'^2(\vec{x}-\vec{x}_i)^T(\vec{x}-\vec{x}_i)} \tag{106}$$

where

$$\frac{1}{a'^2} = \frac{1}{\hat{a}^2} - \frac{1}{a^2}. \tag{107}$$

The reconstructed version of the sampled function $\hat{h}'(\vec{x})$ by a Gaussian filtering is then (almost, depending on w) equal to the original function $h(\vec{x})$

$$h(\vec{x}) \approx \hat{h}'(\vec{x}) * (\hat{a}\Delta)^n \mathrm{e}^{-\pi \hat{a}^2 \vec{x}^T \vec{x}}. \tag{108}$$

However, the maximum sampling step Δ must be smaller for $h'(\vec{x})$ than for $h(\vec{x})$.

6.2.4 Application to Motion-Group Convolutions

The Gaussian reconstruction technique outlined previously in this section is directly applicable to the numerical computation of motion-group convolutions calculated in the order

$$\psi_n \star ... \star \psi_1 \star \varphi_0 = \psi_n \star (\psi_{n-1} \star ... \star (\psi_1 \star \varphi_0)...)$$

when ψ_i are general functions on the motion group and φ_0 is a function only of position (i.e., it is constant over all orientations).

Geometrically, this can be viewed as φ_0 being swept by ψ_i, then the result, $\psi_1 \star \varphi_0$ being swept by ψ_2, etc. At each stage the quantity being swept is a function of position only, and hence can be expanded as a sum of basic (Gaussian) functions. The formulation of Section 5 is then used to perform the convolution of basic functions.

7 Discrete Convolutions

7.1 Approximation of Discrete Convolutions

As was shown in subsection 4.3, the computation of multiple convolutions is intractable for *discrete density functions* $\psi(g)$ and $\varphi(f)$ which are a weighted sum of Dirac functions. This is because the number of Dirac functions increases exponentially with the number of convolutions. On the other hand, sections 5 and 6 describe how one can perform multiple convolutions to any desired accuracy and with a polynomial computational complexity in the case of band-limited density functions of finite support. Thus, for discrete density functions, an approximation by lowpass filtering can be computed (see Figure 18 for examples of a discrete density function and some approximations by lowpass filtering with a Gaussian function of different widths a). It is important to notice that such approximation is independent of the number of convolutions, contrary to other methods like [5] or [4]. The effect of any approximation is probably not so important in the middle of the support of the resulting density function, but it could be significant on the boundaries. That is good news since a brute force approach by a Monte Carlo method can as well approximate the middle of the density function but fails to give accurate results on the boundaries which is critical in workspace generation.

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Figure 17: A 15-Module Binary Variable-Geometry-Truss Manipulator

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig5-1.gif

Figure 18: Discrete and Filtered Density Functions ($a = 100, 20, 10$)

7.2 Workspace of Binary Robots

The binary robot of Figure 17 is taken as an example.

Figure 19 summarized the results one can obtain for different numbers of modules. The last column counts the number of Dirac functions of the density function $\varphi(\vec{p})$. For 16 modules, this number is so huge that only the filtered density function $h(\vec{p})$ can be computed. The next to last column stands for the number of samples for the filtered density function $h(\vec{p})$. This number is limited in our program to a chosen value of about 250000 and in order to not exceed this bound the program computes the optimal value for a. For 16 modules almost all the combinations of the binary actuators lead to a banana shape cluster. However, some configurations exist outside. For example, the drawn position of the 16 modules binary robot in Figure 19 illustrates that the workspace can be actually larger than what could be obtained by a Monte Carlo approach. Better representations include Figure 20 which uses a "continuous rescaling" which is simply a logarithmic plot.

8 Error Accumulation in Serial Linkages

Error accumulation in serial linkages and cascades of platform manipulators is another problem that is easily quantified using generalized Gaussian functions. Intuitively, the errors due to manufacturing inaccuracies and backlash of the constituent components "add up" as one traverses the length of an open kinematic chain. This intuitive notion is quantified below using the concepts of convolution of functions on the group of rigid body motions. The discussion presented here extends the formulation presented in [18]

8.1 Model Formulation

Suppose we are given a manipulator consisting of two six degree-of-freedom sub-units. These units could be Stewart-Gough platforms or six serial links connected with revolute joints. One unit is stacked on top of the other one. The proximal unit will be able to reach each frame $h_1 \in SE(3)$ with some error. This error may be different for each different frame h_1. This is expressed mathematically as a real-valued function on $SE(3)$ which has a peak in the neighborhood of h_1 and decays rapidly away from h_1. If the unit could reach h_1 exactly, this function would be a delta function. Explicitly this function may have many

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Figure 19: Binary Robots and Filtered Density Functions

http://caesar.me.jhu.edu/directory/gregpaper/ejck/fig5-3.gif

Figure 20: Filtered Density Function for 16 Modules

http://caesar.me.jhu.edu/directory/gregpaper/ejck/error.gif

Figure 21: Error Propagation in Linkages

forms depending on what error model is used. However, it will always be the case that it is of the form $\rho_1(h_1, g_1)$ for $h_1, g_1 \in SE(3)$. That is, the error will be a function on $g_1 \in SE(3)$ for each frame h_1 that the top of the module tries to attain relative to its base. Likewise, the second module will have an error function $\rho_2(h_2, g_2)$ for $h_2, g_2 \in SE(3)$ that describes the distribution of frames around h_2 that might be reached when h_2 is the actual goal.

The error that results from the concatenation of two modules with errors $\rho_1(\cdot)$ and $\rho_2(\cdot)$ results from sweeping the error of the second module by that of the first. This is written mathematically as:

$$(\rho_1 \otimes \rho_2)(h_1 \circ h_2, g) = \int_{SE(3)} \rho_1(h_1, g_1)\rho_2(h_2, g_1^{-1} \circ g)d\mu(g_1). \tag{109}$$

In the case of no error, the multiplication of homogeneous transforms h_1 and h_2 as $h_1 \circ h_2$ represents the composite change in position and orientation from the base of the lower unit to the interface between units, and from the interface to the top of the upper unit. In the case of inexact kinematics, the error density function for the upper unit is shifted by the lower unit $(\rho_2(h_2, g_1^{-1} \circ g))$, weighted by the error of the lower unit $(\rho_1(h_1, g_1))$ and integrated over the support of the error density function of the lower unit (which is the same as integrating over all of $SE(3)$ since outside of the support of the error function the integral is zero). The result of this integration is by definition the error function around the frame $h_1 \circ h_2$, and this is denoted as $(\rho_1 \otimes \rho_2)(h_1 \circ h_2, g)$. We illustrated Equation (109) in Fig. 8.1.

8.2 Related Mathematical Issues

To test this formulation, consider the case of exact kinematics. In this case, the error functions have a very special form: they are Dirac delta functions on $SE(3)$. In exact kinematics, the error functions for both units may be written as

$$\rho_i(h_i, g) = \delta(h_i^{-1} \circ g).$$

Then in this special case Eq. 109 reduces to

$$(\delta \otimes \delta)(h_1 \circ h_2, g) = \int_{SE(3)} \delta(h_1^{-1} \circ g_1)\delta(h_2^{-1} \circ g_1^{-1} \circ g)d\mu(g_1) = \delta(h_2^{-1} \circ h_1^{-1} \circ g) = \delta((h_1 \circ h_2)^{-1} \circ g). \tag{110}$$

In other words, in the case of exact kinematics, we have exactly the result which is expected. Moreover, the delta function is a tool that also allows us to reexamine the general error propagation equation in (109) as a convolution of functions on the direct product group $SE(3) \times SE(3)$. Recall that the group law for the direct product of group with itself, $G \times G$, is simply $(h_1, g_1)\hat{\circ}(h_2, g_2) = (h_1 \circ h_2, g_1 \circ g_2)$ for $h_i, g_i \in G$. Likewise, the convolution of functions on the direct product is defined as

$$(\alpha \star \beta)(h, g) = \int_G \int_G \alpha(h_3, g_1)\beta(h_3^{-1} \circ h, g_1^{-1} \circ g) d\mu(h_3) d\mu(g_1) \tag{111}$$

By letting $\alpha(h, g) = \rho_1(h, g)\, \delta(h_1^{-1} \circ h)$ and $\beta = \rho_2(h, g)$ in the above equation and using the properties of the Dirac delta function, it is clear that error accumulation (109) may be written as a convolution on the direct product of $SE(3)$ with itself.

8.3 The Role of Generalized Gaussians in Error Propagation

The decomposition and sampling techniques developed in Sections 5 and 6 are required for workspace generation because in that context the density functions are not class functions, and hence the equality

$$(\hat{f}_1 \star \hat{f}_2)(g) = (f_1 \star f_2)((g_1 \circ g_2)^{-1} \circ g) \tag{112}$$

does not hold where $\hat{f}_i(g) = f_i(g_i^{-1} \circ g)$.

However, in the case of error propagation, the density functions are *close to* being Dirac delta functions, and so (112) becomes a good approximation. Hence, for small errors we have the closed-form analytical result that the error of two concatenated segments, each with error density functions of the form

$$\rho_i(h_i, g) = f(h_i^{-1} \circ g; \beta(h_i), b(h_i))$$

is

$$(\rho_1 \otimes \rho_2)(h_1 \circ h_2, g) = f((h_1 \circ h_2)^{-1} \circ g; \alpha(h_1, h_2), a(h_1, h_2)).$$

Here f is the product of rotational and translational Gaussians and β and b describe the orientational and positional dispersion of the density function for $i = 1, 2$.

The parameters α and a follow from discussions in earlier sections, and are given as

$$\alpha(h_1, h_2) = \frac{\beta(h_1)\beta(h_2)}{\sqrt{\beta^2(h_1) + \beta^2(h_2)}}$$

and

$$a(h_1, h_2) = \frac{b(h_1)b(h_2)}{\sqrt{b^2(h_1) + b^2(h_2)}}.$$

9 Conclusion

We have formulated and implemented a general numerical proceedure for decomposing functions on motion groups into sums of shifted generalized Gaussians. These generalized Gaussians are the product of Gaussians in $\mathbb{R}^N$ and "folded normal" distributions on the circle and quaternion sphere. The relationship between these Gaussians and solutions of the heat equation on the circle and quaternion sphere were explored. It was concluded that in the case of the circle the solutions are identical, while for the quaternion sphere they are not identical, but become indistinguishable in the range of parameter values of greatest interest in applications. It was shown that the convolution of these Gaussians is closed under the operation of convolution on $SE(N)$. By extending Shannon's sampling theorem, a numerical tool for generating manipulator workspaces and analyzing error propogation in serial and hybrid manipulators resulted.

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