Ultrarigid periodic frameworks

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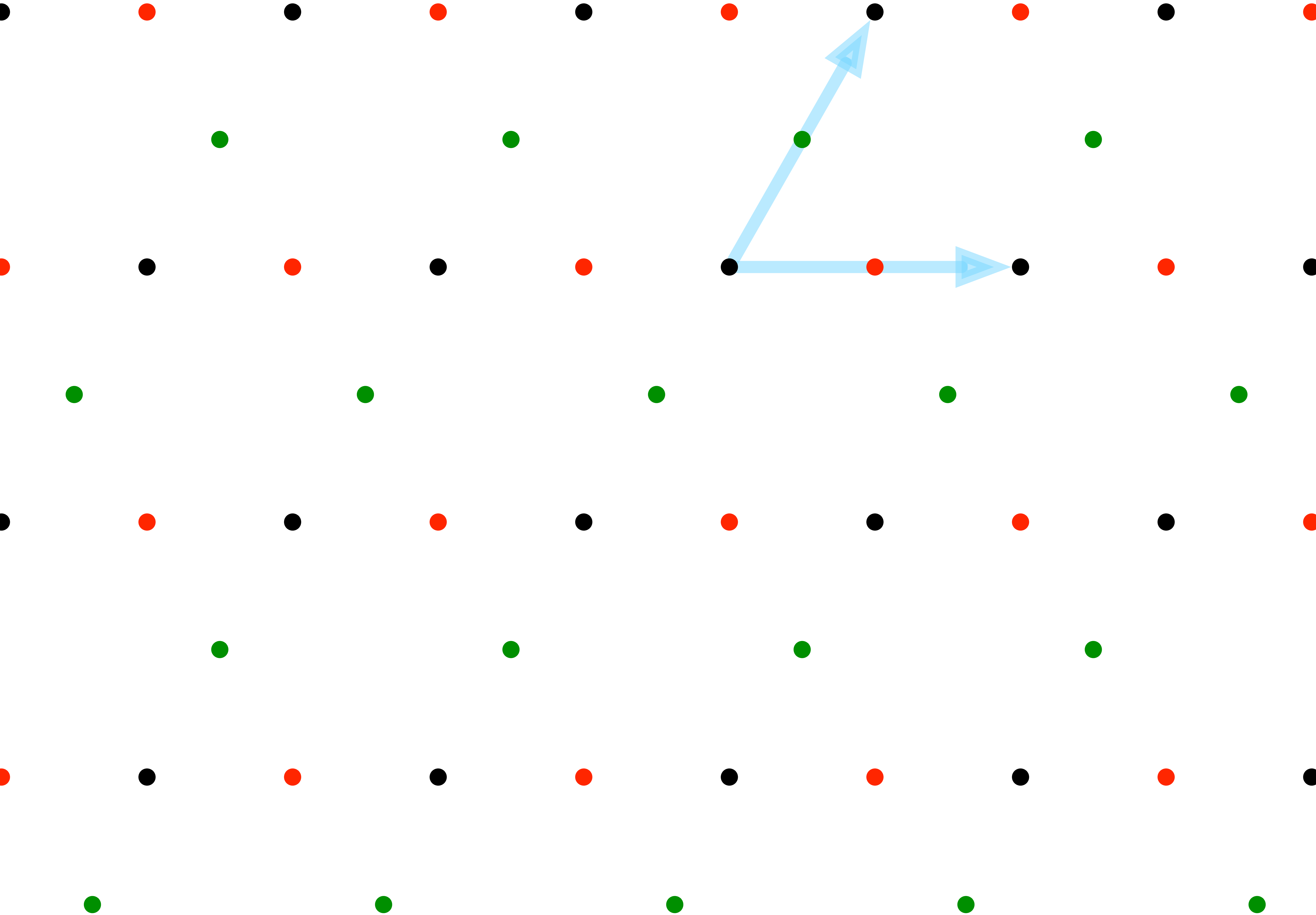
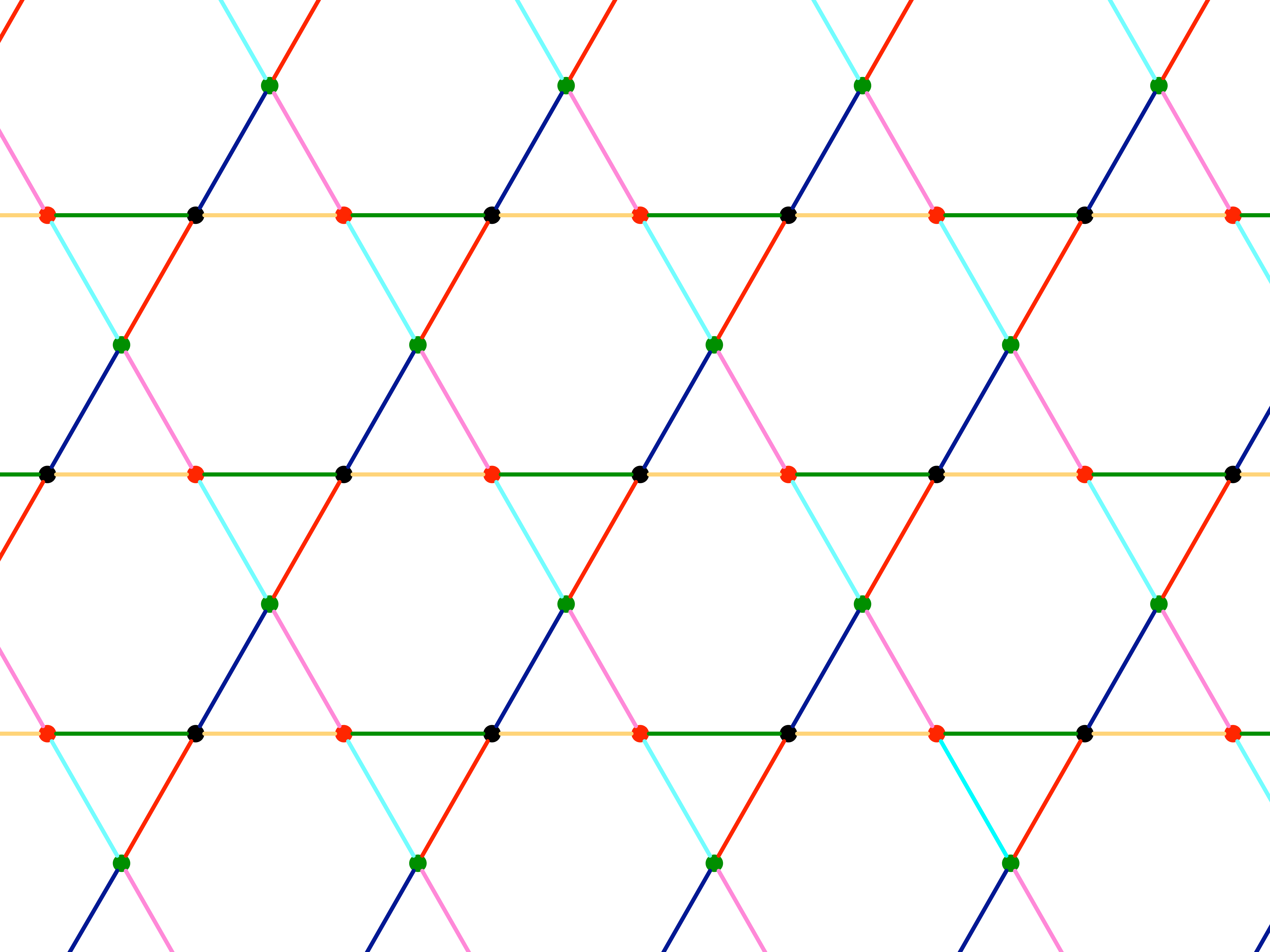
Periodic frameworks

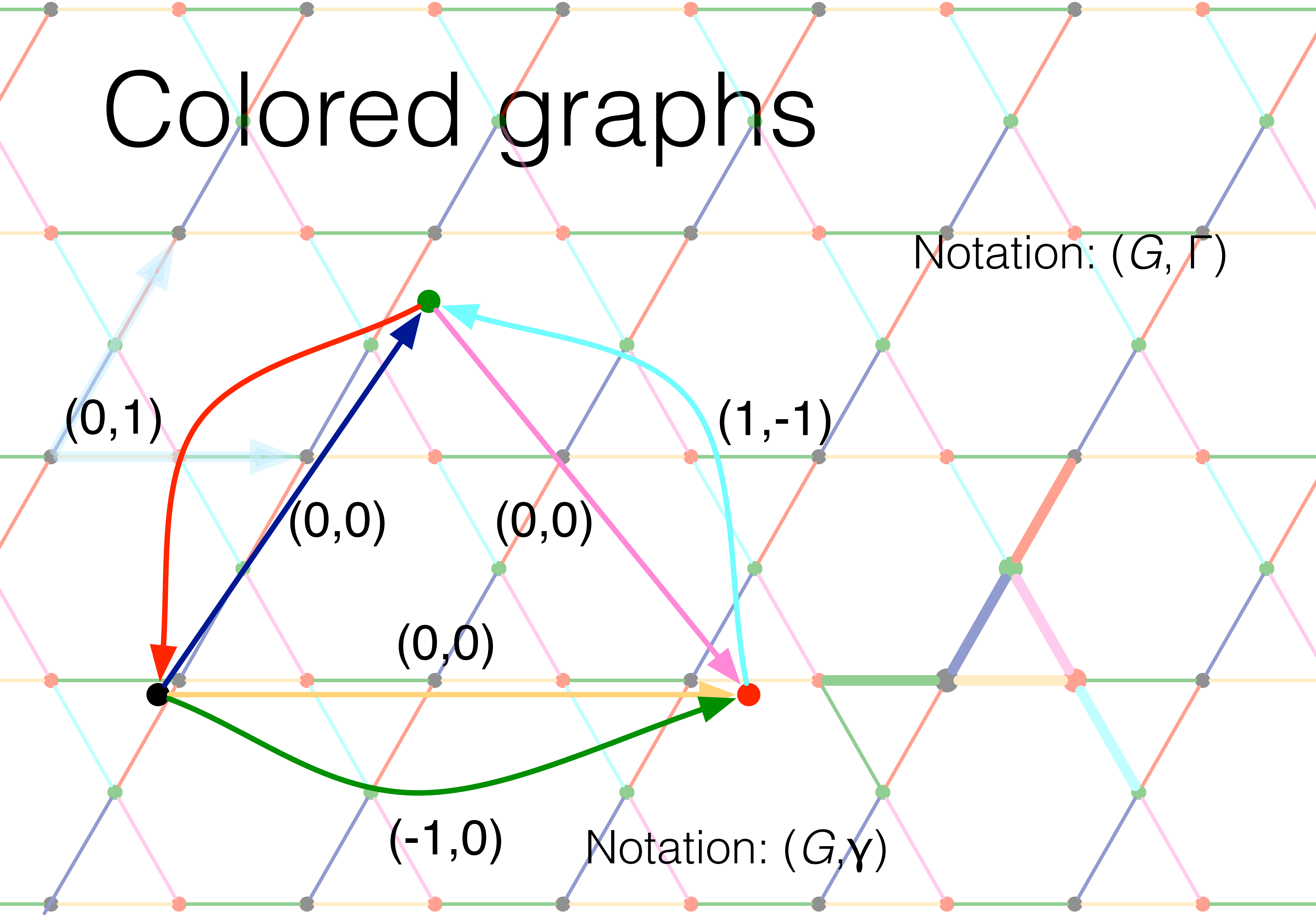
[Borcea-Streinu '10]

- A periodic framework (G, ℓ, Γ) is an *infinite* framework with
  - $\Gamma < \mathrm{Aut}(G)$ — Γ free abelian, rank d
  - $\ell(\gamma(ij)) = \ell(ij)$
- A realization G(p,L) is a realization *periodic* with respect to a *lattice of translations* L, which realizes Γ
- Motions *preserve the Γ-symmetry* (not L); *rigid* iff no non-trivial motion



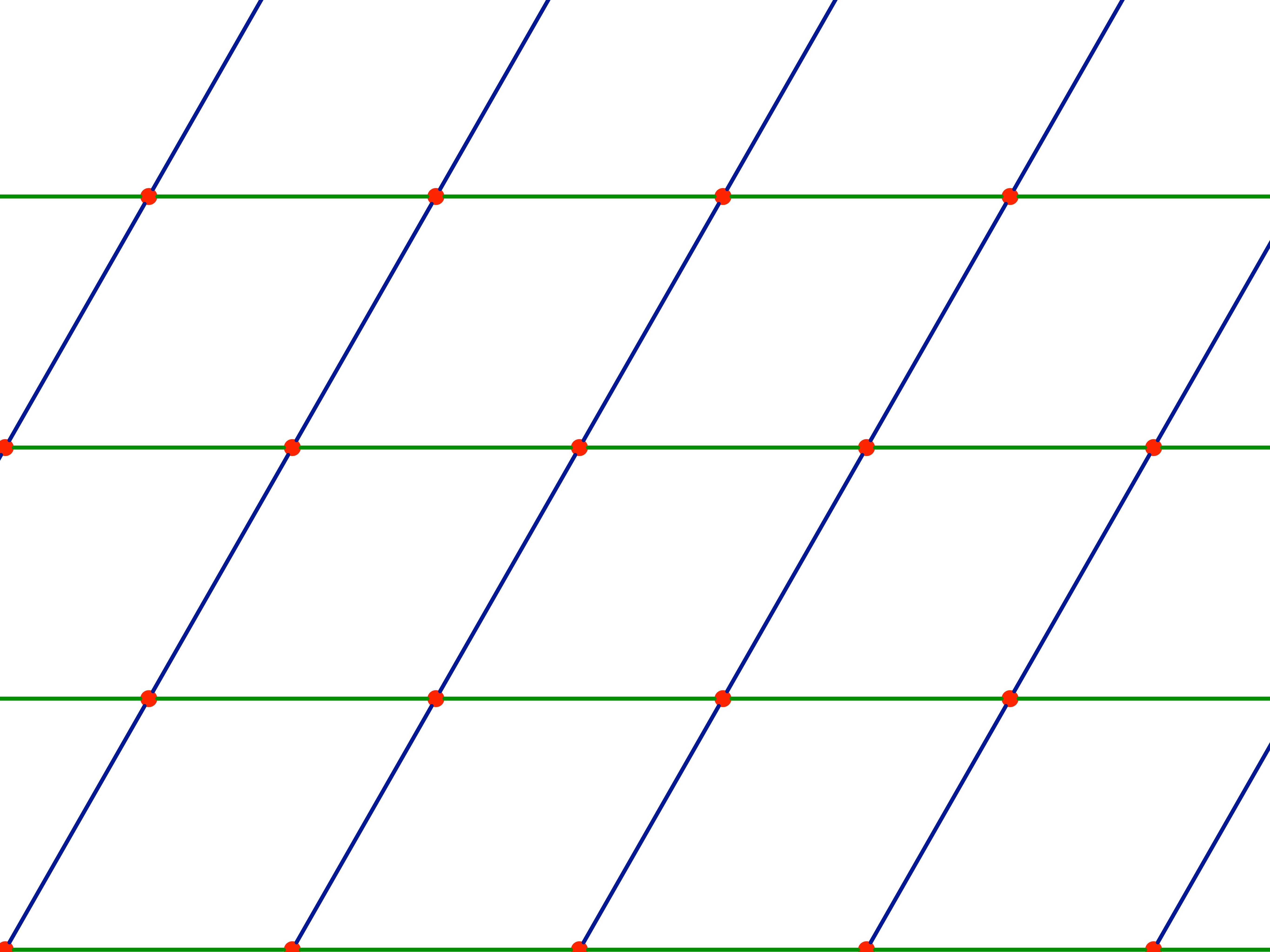


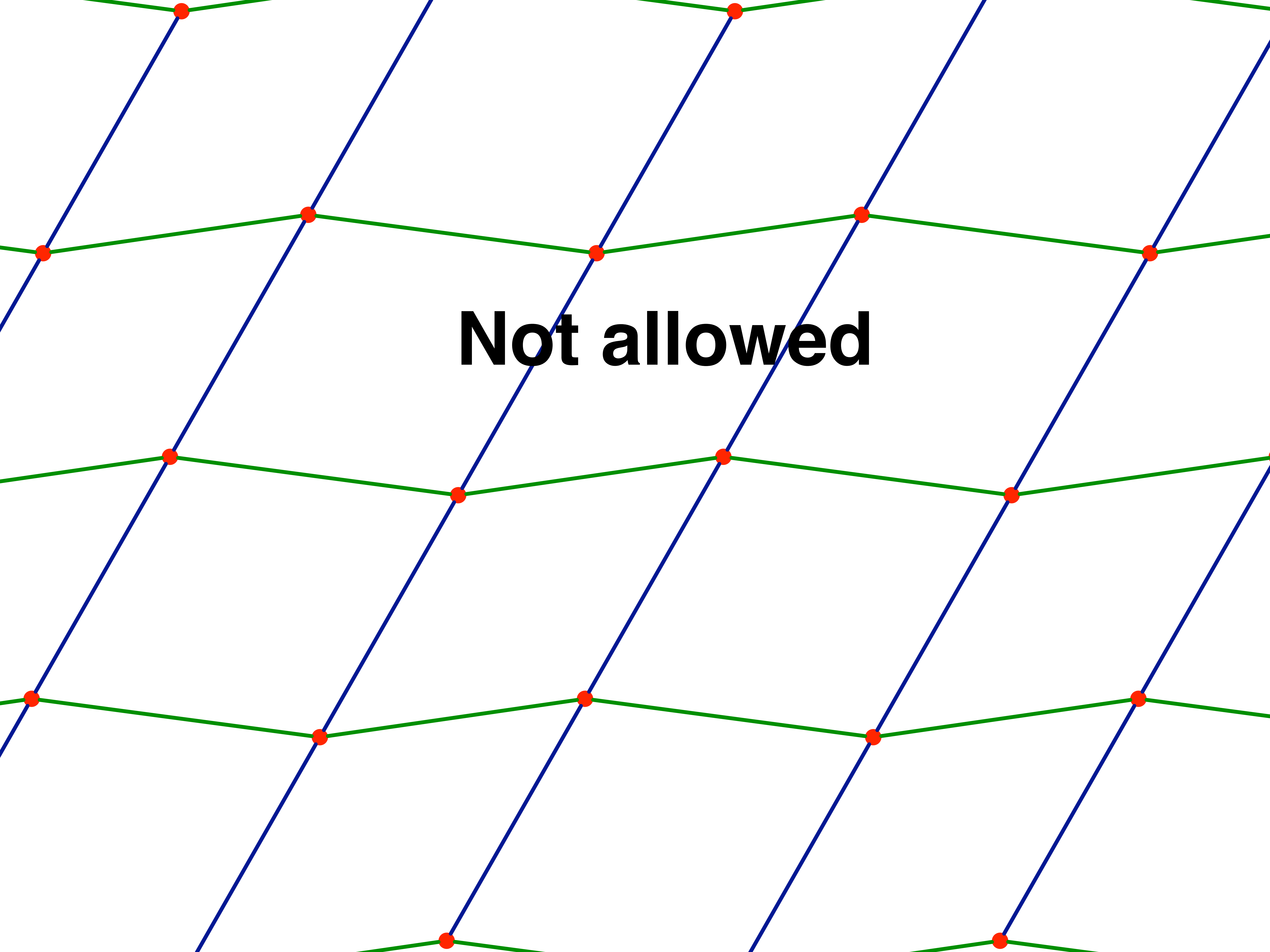
Colored graphs



Background

- Symmetry-forcing allows for *finite* algebraic treatment of configuration spaces [B-S '10]
  - Naive setup gives very wild conf. spaces [Owen-Power]
- Lots of "Laman-type" combinatorial characterizations [B-S '11], [Malestein-T …], [Tanigawa], [Jordán-Kazinitzki-Tanigawa],…
- Many variants: *fixed-lattice* [Whiteley '88], [Ross '12] …; *fixed-area/volume* [Malestein-T '13], [Treacy, et al.] …; body-bar [Borcea-Streinu-Tanigawa '12], fixed-lattice body-bar…




Not allowed

Symmetry-forcing

- Symmetry-forcing excludes some “obvious” motions
- Rigidity *does not* imply connectivity
- Old “conjecture”: periodic rigidity implies periodic rigidity with respect to sub-lattices
    - (See above…, earliest examples [B-S ’10])
- Not *exactly* what’s wanted/used in applications

"Sublattice problem"

[T '11,Malestein-T '13]

- Suppose that (G,γ) is a generically rigid colored graph, and

  - for *any* sublattice $\Lambda < \Gamma$ the associated colored graph (G,γ') is also generically rigid

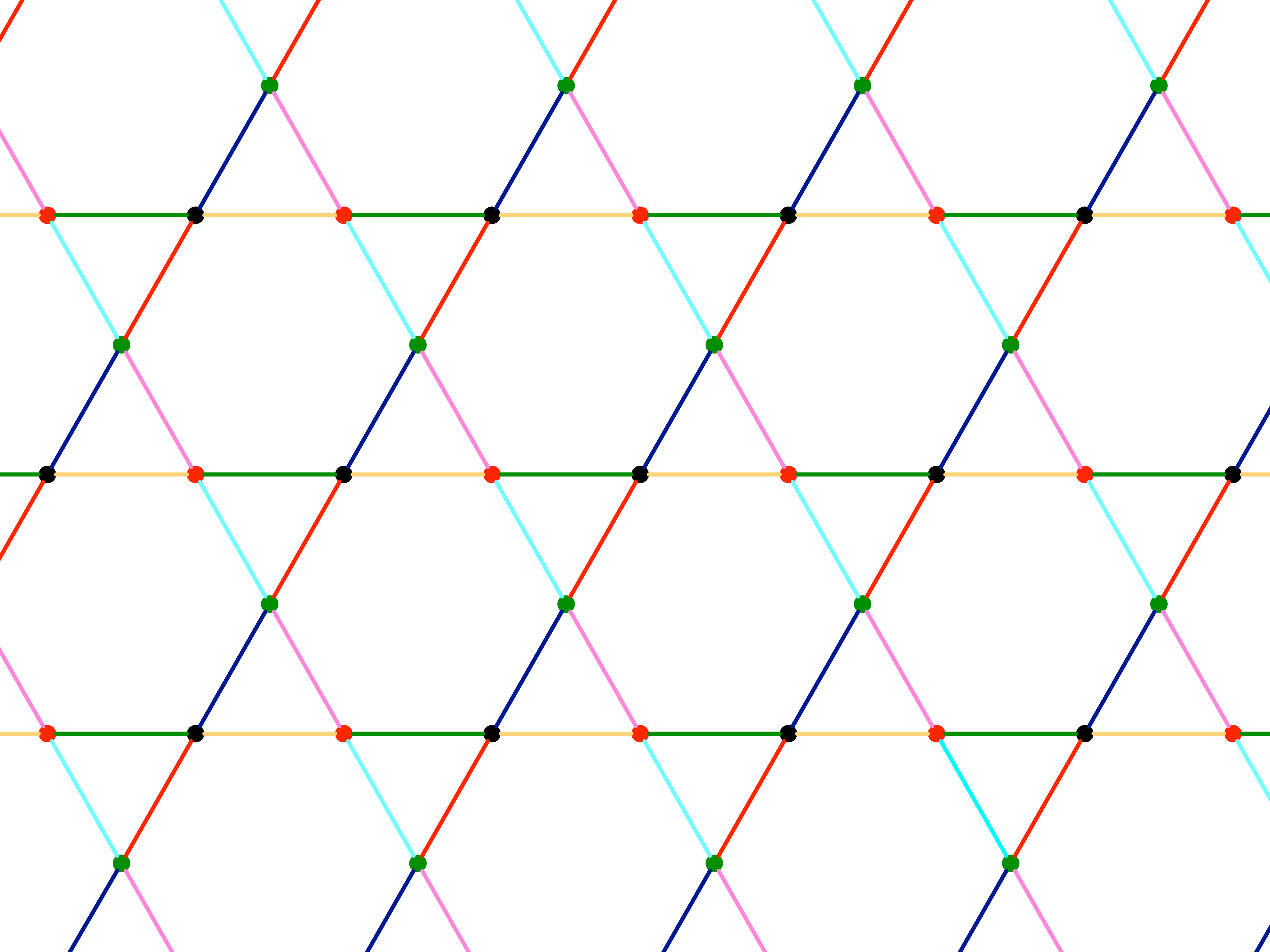
- Is it true that for a generic rigid realization (G, p, L) of the original colored graph all the rest of the relaxed frameworks are infinitesimally rigid?

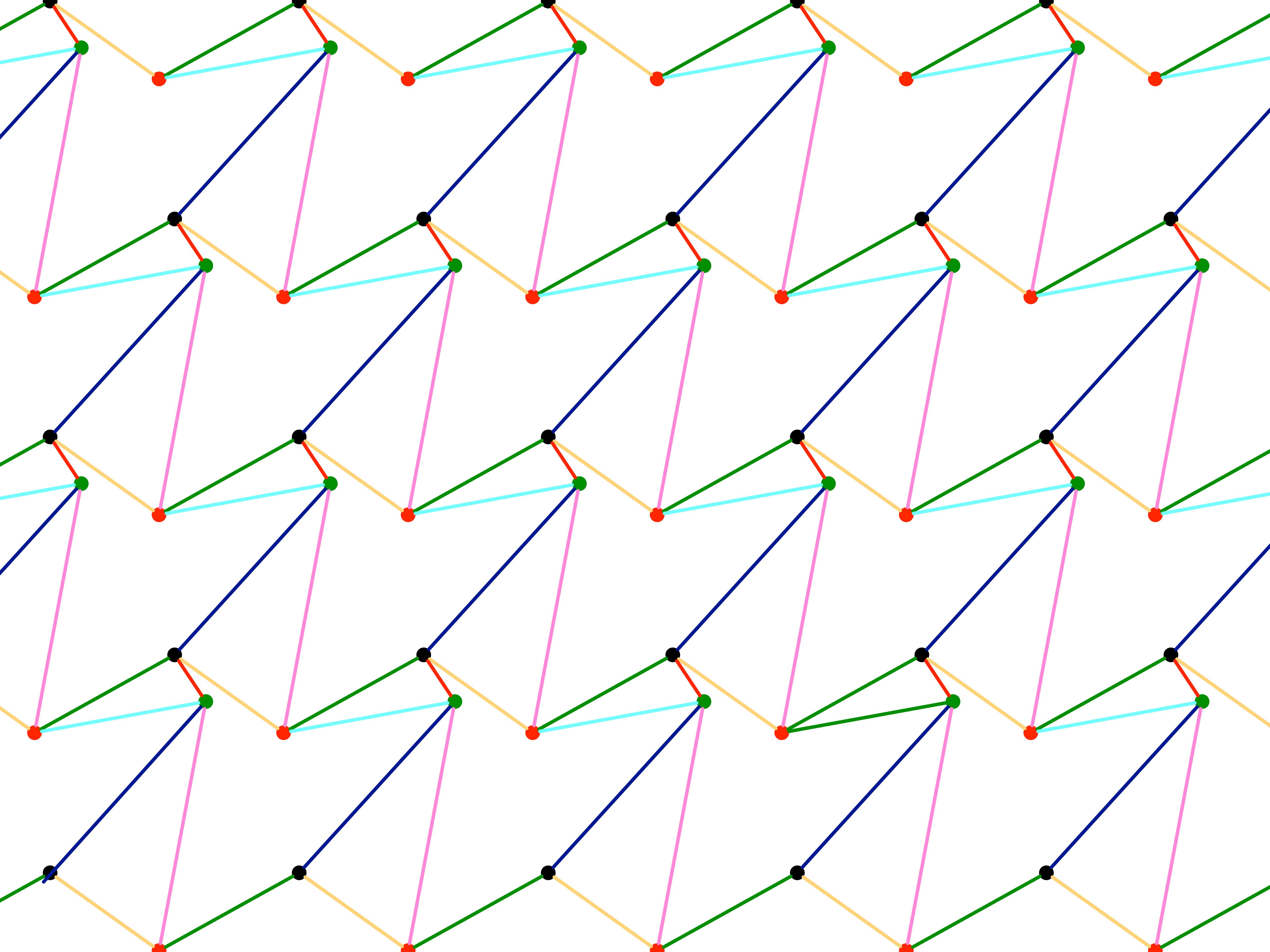
- Very likely (already?) not true.

Ultrarigidity

[Borcea '12]

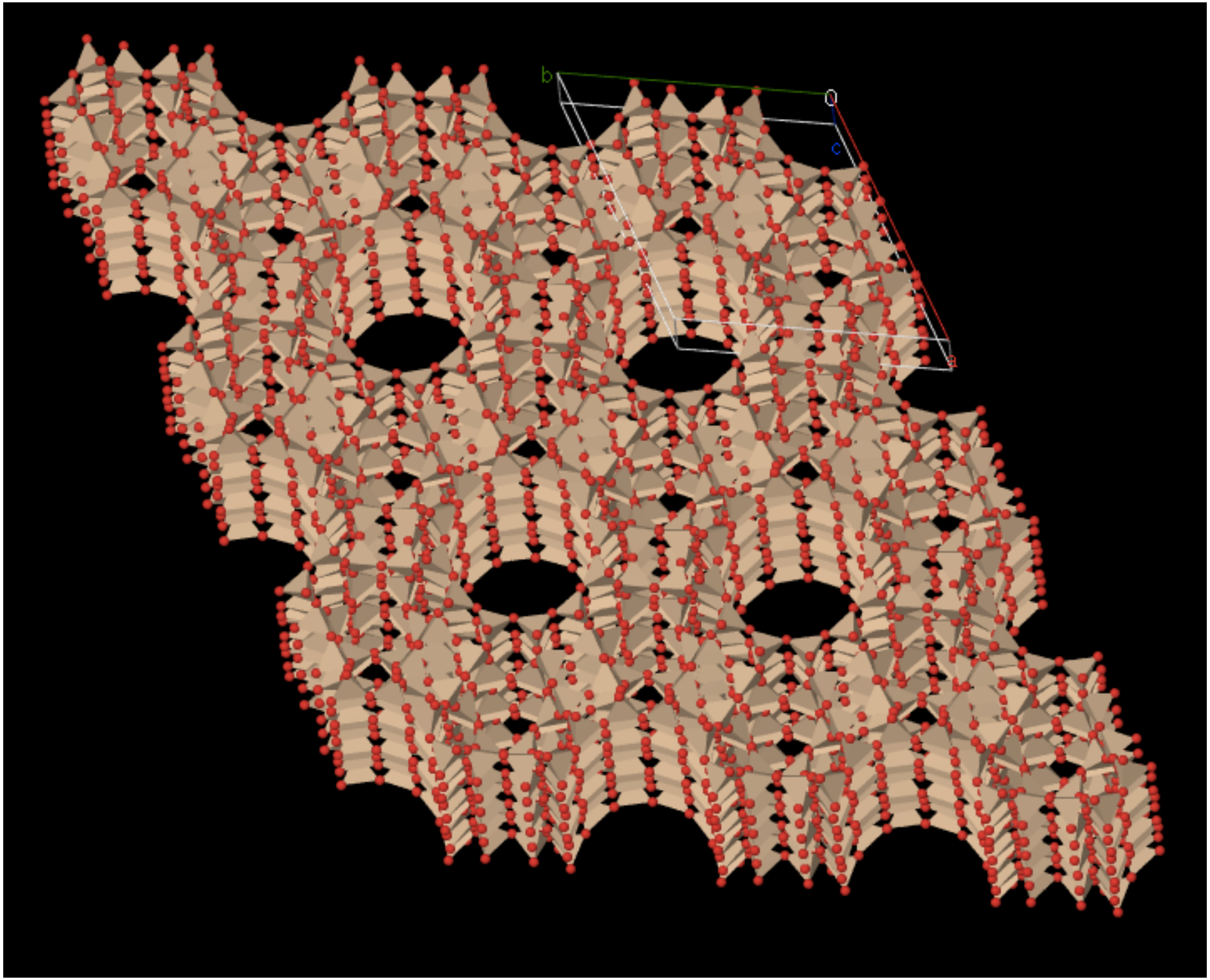
- Let (G, p, L) be a realization of (G, ℓ, Γ)
- (G, p, L) is (periodically) *ultrarigid* if
  - it is rigid
  - for any (finite-index) sub-lattice $\Lambda < \Gamma$, (G, p, L) is a rigid realization of (G, ℓ, Λ)
- Related concept: "ultra 1-d.o.f." (in 2d)





Ultrarigidity

- In between “infinite frameworks” and periodic frameworks
- Hope for effective, combinatorial characterizations
- Interesting applications
- Techniques could be applicable to “incidental symmetry” in finite frameworks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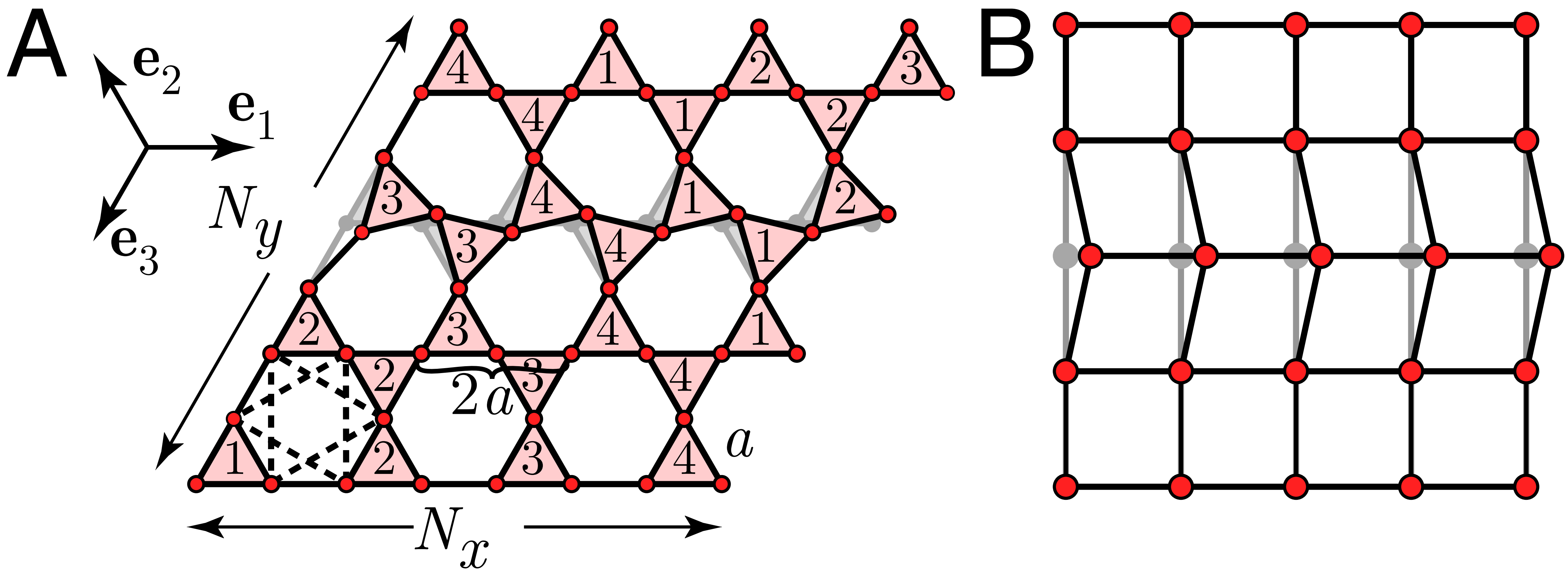

[Sun et al. PNAS '12]

Questions

- Characterize *generic* ultrarigidity by a framework's colored graph
- Find algorithms for recognizing ultrarigid graphs
- When ultrarigidity fails, describe the motions

Challenges

- Periodic rigidity, like all *finite* rigidity theories is characterized by the *rank* of *one matrix* (“rigidity matrix”)
    - Can’t just build the rigidity matrix of (G, p, L) for every Λ
- The algebraic tools underlying finite/periodic rigidity imply rigidity is a “generic property”
    - Can’t apply them the same way here

Algebraic characterization

- A realization (G, $\mathbf{p}$, $\mathbf{L}$) is infinitesimally ultrarigid if and only if:
  - It is infinitesimally periodically rigid
  - The matrix with ijth row, $ij \in E(G,\varphi)$

comp. wise mult

edge direction vector

$$\begin{array}{ccc} & i & j \\ (\ldots\ldots & -\mathbf{d}_{ij} \;\ldots\ldots\ldots & \mathbf{d}_{ij} \otimes \{\gamma_{ij}^{-1}, \omega\} \;\ldots\ldots) \end{array}$$

$\{\delta,\omega\} := (\zeta_1^{\delta_1}, \ldots, \zeta_d^{\delta_d})$, ζ_i root of unity

has rank dn for all $\omega \neq \mathbf{1}$

Ideas

- The matrices in the second part are projections of *group ring rigidity matrices*

- Re-parameterize the length map for (all) placements of (G,Γ) by

$$(\mathrm{Func}(\Gamma,\mathbf{R}^d))^n \times \mathrm{Hom}(\Gamma,\mathbf{R}^d) \xrightarrow{\ell} (\mathrm{Func}(\Gamma,\mathbf{R}^d))^n$$

Connections/corollaries

- The ω where the rank is deficient are equivalent to the rational points of the *Rigid Unit Mode Spectrum* (RUM) [Power '13]
  - derivation different, gets back/generalizes the periodic rigidity matrix
- Infinitesimal motions certifying failures of ultrarigidity are either Γ-periodic or *lattice-fixing*
  - finite motion totally different

Algorithmic characterization

- We can check the algebraic characterization with a very simple algorithm based on:

- **Theorem:** For any set of polynomials $p_1, \ldots, p_k \in$ $\mathbf{C}[\mathbf{x},\mathbf{x}^{-1}]$, there is an effectively computable constant C such that if $V(p_1, \ldots, p_k)$ contains torsion points $\neq \mathbf{1}$, there is one of order at most C

  - C depends on the degrees of the p_i and the coefficient field

- Ultrarigidity is *decidable*

Combinatorial char.

- For $d = 2$ and $|E\,(G,\gamma)| = 2n + 1$, the colored graph (G,γ) is generically infinitesimally ultrarigid if and only if
  - (G,γ) is generically periodically rigid
  - For all finite cycle groups Δ and epimorphisms $\Psi : \Gamma \to \Delta$, $(G, \Psi(\gamma))$ is Δ-(2,2) spanning
- Can be checked in time polynomial in n and $|\gamma|$

Variants

- Let d = 2, |E (G,γ)| = 2n. Then the following are equivalent:
  - (G,γ) is fixed-lattice ultrarigid
  - (G,γ) is fixed-area ultrarigid
  - (G,γ) is "unit-area-Laman" and for all finite cyclic groups Δ and epimorphisms Ψ : Γ → Δ, (G, Ψ(γ)) is "Δ-(2,2) spanning"
- Have *polynomial time algorithms*

Questions

- What is the generic set for ultrarigidity like?
  - In particular, does it contain open sets?
- Combinatorial characterization of ultrarigid frameworks in dimension 2
- Relationship between ultrarigidity and "incidental rigidity" for infinite frameworks?
- Theory of stresses for ultrarigidity?

Thanks!

[arXiv: 1404.2319]