Linking Rigid Bodies Symmetrically

Bernd Schulze[1] and Shin-ichi Tanigawa[2]

[1]Lancaster Unviersity,

[2]Kyoto University

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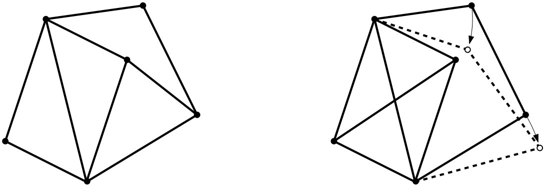
Rigidity of Frameworks

- A bar-joint framework is a pair (G, p) of a simple graph $G = (V, E)$ and $p : V \to \mathbb{R}^d$
- (G, p) is flexible if $\exists$ a continuos "deformation" keeping the edge lengths; otherwise (G, p) is rigid



Infinitesimal Rigidity

- $\dot{p} : V \to \mathbb{R}^d$ is an infinitesimal motion of (G, p) if
$$\langle p(i) - p(j), \dot{p}(i) - \dot{p}(j) \rangle = 0 \qquad (\forall ij \in E).$$
- (G, p) is infinitesimally rigid if every infinitesimal motion $\dot{p}$ of (G, p) is trivial, i.e., $\exists$ a skew symmetric matrix S and $t \in \mathbb{R}^d$ such that $\dot{p}(i) = Sp(i) + t$ for $i \in V$.

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- Proposition (Asimov and Roth 79) Suppose p is generic. Then (G, p) is rigid iff (G, p) is infinitesimally rigid.
- Theorem (Laman 1970) Suppose p is generic. Then (G, p) is minimally rigid in $\mathbb{R}^2$ if and only if
  - $|E| = 2|V| - 3$ and
  - $|E(G')| \leq 2|V(G')| - 3$ for any $G' \subseteq G$ with $|E(G')| \geq 2$.

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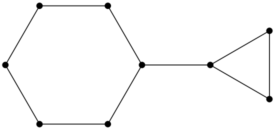
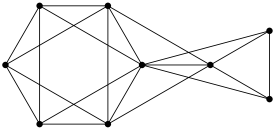
- It is still open to give a 3-dimensional counterpart of Laman's theorem

Molecular Frameworks

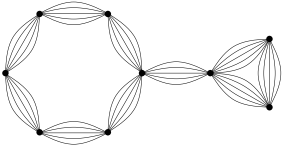
- A molecular framework is a bar-joint framework whose underlying graph is G^2 of some G.
- Theorem (Katoh&T11) Suppose p is generic. Then (G^2, p) is rigid in $\mathbb{R}^3$ if and only if $5G$ contains six edge-disjoint spanning trees.



G



G^2



$5G$

Symmetric Frameworks

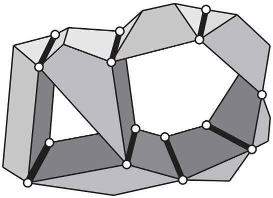
Symmetry in proteins...

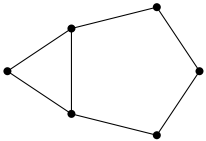
Rigidity of symmetric frameworks

- Symmetry-forced rigidity (asking symmetry-preserving motions)
  - well understood
- Infinitesimal rigidity
- Rigidity

Body-hinge Frameworks

- A body-hinge framework is a structure consisting of rigid bodies connected by hinges...
- A body-hinge framework is a pair $(G = (V, E), h)$;
  - vertex $\Leftrightarrow$ body
  - edge $\Leftrightarrow$ hinge
  - $h(e) := \{h(e)_1, \ldots, h(e)_{d-1}\}$, affinely independent $d - 1$ points in $\mathbb{R}^d$, for each $e \in E$.





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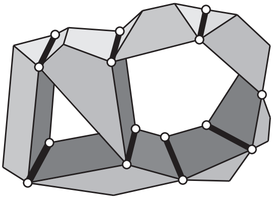
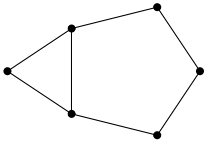
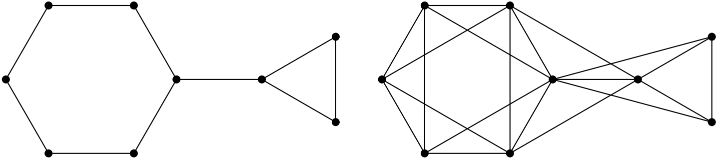
- Theorem (Tay 89, Whiteley 88). Suppose h is generic. Then (G, h) is infinitesimally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)G$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees.





Molecular Frameworks as Body-hinge Frameworks

- In (G^2, p), $N_G(v) \cup \{v\}$ forms a clique, which is rigid
- (G^2, p) can be regarded as a hinge-concurrent body-hinge framework (G, h)
  - h is not generic; for each v, $\mathrm{span}(h(e))$ intersects at $p(v)$ for every e incident to v in G



G G^2

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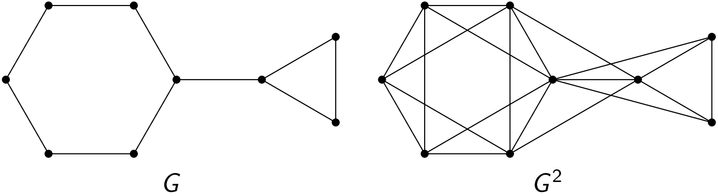
- Theorem (Katoh&T11). Suppose (G, h) is hinge-concurrent generic. Then (G, h) is infinitesimally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)G$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees.



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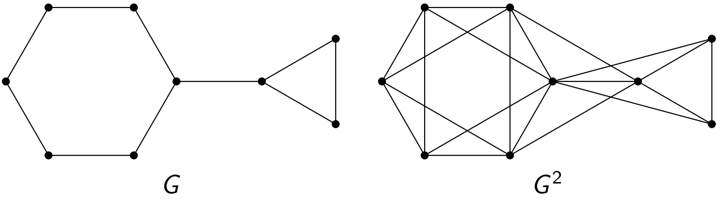
- Here we give a symmetric version of Tay-Whiteley's theorem for body-hinge frameworks.



Symmetric Body-hinge Frameworks

- A graph $G = (V, E)$ is (Γ, θ)-symmetric (or, simply, Γ-symmetric) if Γ is isomorphic to a subgroup of $\mathrm{Aut}(G)$ through $\theta : \Gamma \to \mathrm{Aut}(G)$.

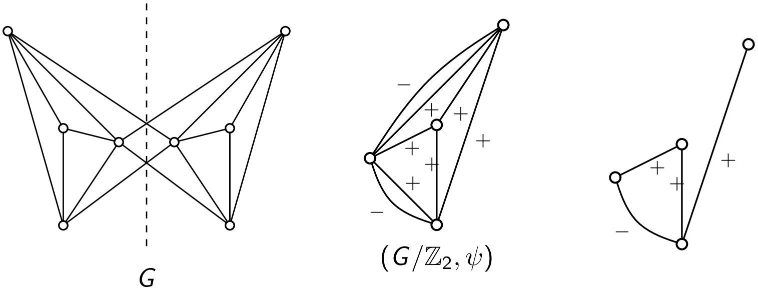
- A body-hinge framework (G, h) is (Γ, θ, τ)-symmetric (or, simply, Γ-symmetric) if

$$\tau(\gamma)h(e)_i = h(\theta(\gamma)e)_i \qquad (\forall e \in E, \forall i \in \{1, \ldots, d-1\})$$

where $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$

Quotient Signed Graphs

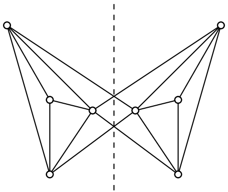
- Definition For a $\mathbb{Z}_2$-symmetric graph G, the quotient signed graph is a pair $(G/\mathbb{Z}_2, \psi)$ of the quotient graph G/Γ and $\psi : E(G) \to \{-, +\}$.
- Definition A cycle is negative if it contains an odd number of negative edges
- Definition A signed graph is called an unbalanced 1-forest if each connected component contains exactly one cycle, which is negative.

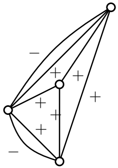


Combinatorial Characterization for $\mathcal{C}_s$ in $\mathbb{R}^3$

- $\mathcal{C}_s$: a group generated by a reflection in $\mathbb{R}^3$
- (G, h): a $(\mathbb{Z}_2, \theta, \tau)$-symmetric body-hinge framework, where
  - $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$, freely acting on $E(G)$
  - $\tau : \mathbb{Z}_2 \to \mathcal{C}_s$, faithful
  - h: a $\mathcal{C}_s$-generic hinge-configuration

- Theorem(Schulze&T14) (G, h) is infinitesimally rigid if and only if the quotient signed graph $(5G/\mathbb{Z}_2, \psi)$ contains edge-disjoint
  - three spanning trees and
  - three spanning unbalanced 1-forests

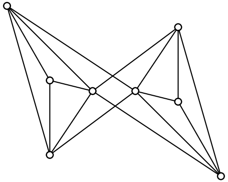


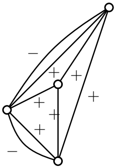


Combinatorial Characterization for $\mathcal{C}_2$ in $\mathbb{R}^3$

- $\mathcal{C}_2$: a group generated by a rotation in $\mathbb{R}^3$
- (G, h): a $(\mathbb{Z}_2, \theta, \rho)$-symmetric body-hinge framework, where
  - $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$, freely acting on $E(G)$
  - $\tau : \mathbb{Z}_2 \to \mathcal{C}_2$, faithful
  - h: a $\mathcal{C}_2$-generic hinge-configuration

- Theorem(Schulze&T14) (G, h) is infinitesimally rigid if and only if the quotient signed graph $(5G/\mathbb{Z}_2, \psi)$ contains edge-disjoint
  - two spanning trees and
  - four spanning unbalanced 1-forests





More generally

- Γ: a finite group
- $\mathcal{P}$: a point group of $\mathbb{R}^d$ isomorphic to Γ,
- $\tau : \Gamma \to \mathcal{P}$, an isomorphism
- $\hat{\tau} : \gamma \in \Gamma \mapsto \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O}(\mathbb{R}^{d+1})$
- $C_2(\hat{\tau}) : \gamma \in \Gamma \mapsto C_2(\hat{\tau}(\gamma)) \in \mathcal{O}(\bigwedge^2 \mathbb{R}^{d+1})$,
  - For a $(d+1) \times (d+1)$-matrix A, $C_2(A)$ denotes the second compound matrix of A; that is, a matrix of size $\binom{d+1}{2} \times \binom{d+1}{2}$ formed from all the 2×2 minors $\det A[\{i,j\},\{k,l\}]$ arranged with the index sets $\{i,j\}$ and $\{k,l\}$ in lexicographic order.

- Suppose $\Gamma = (\mathbb{Z}_2)^k$
- $\mathcal{P}$: a point group of $\mathbb{R}^d$ with an isomorphism $\tau : \Gamma \to \mathcal{P}$
- $C_2(\hat{\tau}) = \bigoplus_{1 \leq i \leq \binom{d+1}{2}} \tau_i$, where $\tau_i : \Gamma \to \{-,+\}$
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- Theorem (Schulze&T14). Let (G, h) be a (Γ, θ, τ)-symmetric body-hinge framework with a $\mathcal{P}$-generic hinge-configuration h. Then (G, h) is infinitesimally rigid iff, for each $1 \leq j \leq 2^k$, $(\binom{d+1}{2} - 1)G/\Gamma$ contains a spanning subgraph H_j such that
  - $|E(H_j)| = \binom{d+1}{2}|V(H_j)| - \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Trace}(\rho_j(\gamma)\hat{\tau}^{(2)}(\gamma))$
  - H_j contains $\binom{d+1}{2}$ edge-disjoint subgraphs H_j^i ($1 \leq i \leq \binom{d+1}{2}$) such that each H_j^i is a spanning unbalanced 1-forest with respect to $\psi_{i,j}$.

Concluding Remarks

- Extension to molecular frameworks (hinge-identified body-hinge frameworks)?
  - maybe possible for $\mathcal{C}_s$ or $\mathcal{C}_2$
- Extension to a wider class of point groups?
  - Our proof uses Whiteley's idea, which requires a tree-decomposition property of graphs satisfying a necessary "count" condition
- Applications to protein-function analysis?