Linking Rigid Bodies Symmetrically

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An important application of rigidity theory is the rigidity and flexibility analysis of biomolecules and proteins, where an ideal molecule is modeled as a *body-hinge framework*, that is, a structural model consisting of rigid bodies connected, in pairs, by revolute hinges along assigned lines. A result by Tay [6, 7] and Whiteley [9] asserts that a generic body-hinge framework is infinitesimally rigid in $\mathbb{R}^3$ if and only if $5G$ contains six edge-disjoint spanning trees, where G denotes the underlying graph obtained by identifying each body with a vertex and each hinge with an edge, and $5G$ denotes the graph obtained from G by replacing each edge by five parallel copies. Based on this result, efficient combinatorial algorithms have been used for analyzing the rigidity properties of proteins (see, e.g., [10, 3, 1, 5]), even though body-hinge frameworks arising from molecules do not fit the genericity assumption in Tay-Whiteley's theorem. A recent result by Katoh and Tanigawa [2] successfully eliminated this assumption, and hence this approach for analyzing the flexibility of proteins is now proven to be mathematically rigorous.

However, many molecules and proteins (as well as many man-made structures such as buildings or mechanical linkages) exhibit non-trivial point group symmetries, and recent work has shown that symmetry can sometimes lead to additional flexibility in a structure [4]. Thus our goal is to develop a symmetric extension of generic rigidity theory which permits a rigidity analysis of structures that possess non-trivial symmetries.

In this talk we will give an extension of Tay-Whiteley's theorem which characterizes infinitesimally rigid symmetric body-hinge structures in terms of a graph packing condition when the underlying point group symmetry is of a form $\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$. This result leads to an efficient combinatorial algorithm for checking the infinitesimal (or static) rigidity properties of body-hinge frameworks in the presence of symmetry.

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