Concentration phenomena in high dimensional geometry.

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Conjecture and thematics.

Let X be a random vector uniformly distributed on an isotropic (choice of the Euclidean structure) convex body in $\mathbb{R}^n$.

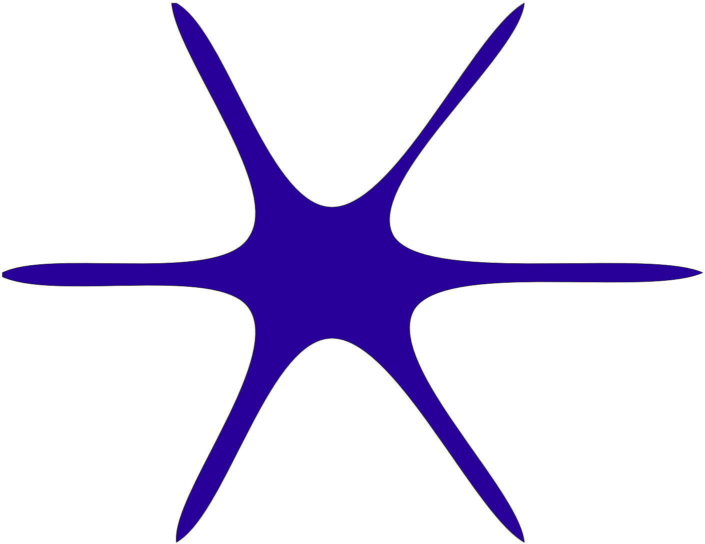
Conjecture

All the volume is concentrated in a thin Euclidean shell.

$$\mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C\,\exp(-c\,t\,\sqrt{n})$$

Hölder or reverse Hölder inequalities.

Pictures - Intuition in high dimension.



Convex body in ”isotropic position”.

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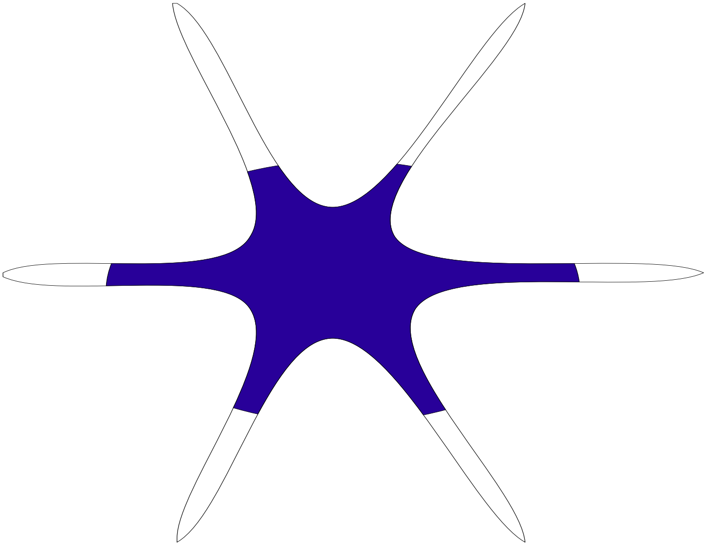

Intersection with a Euclidean ball of radius $\sqrt{n}$.

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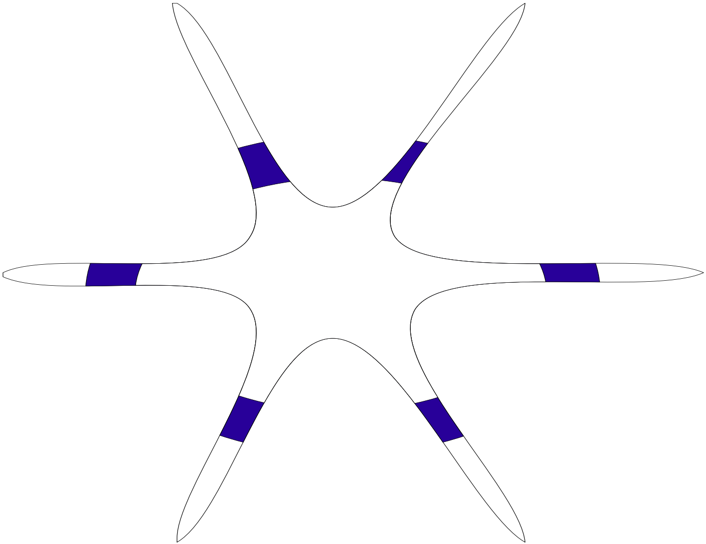

volume in a shell of radius $\sqrt{n}$ and width 1

Brunn-Minkowski inequality.

Let A and B be two compacts in $\mathbb{R}^n$ such that $|A| \cdot |B| > 0$ then

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

Geometry of convex bodies :

Let K be a convex body with non empty interior, $\lambda \in [0,1]$

$$|(1-\lambda)(K \cap A) + \lambda(K \cap B)|^{1/n} \geq (1-\lambda)|K \cap A|^{1/n} + \lambda|K \cap B|^{1/n}$$

whenever $|K \cap A| \cdot |K \cap B| > 0$

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Consequence. Let μ be the uniform measure on K then

$$\mu\left((1-\lambda)A + \lambda B\right)^{1/n} \geq (1-\lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}$$

when $\mu(A)\mu(B) > 0$.

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μ uniforme measure on K, for every compact A, B

$$\mu\left((1-\lambda)A + \lambda B\right) \geq \mu(A)^{1-\lambda}\, \mu(B)^{\lambda}$$

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We say that μ is log-concave.

Log-concave measures.

Let $f : \mathbb{R}^n \to \mathbb{R}+$ such that $\forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1]$,

$$f((1-\theta)x + \theta y) \geq f(x)^{1-\theta} f(y)^{\theta}$$

A measure with density $f \in L_1^{\text{loc}}$ is said to be log-concave and satisfies $\forall A, B \subset \mathbb{R}^n, \forall \theta \in [0, 1]$,

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Classical examples :
1) Probabilistic : $f(x) = \exp(-|x|_2^2), f(x) = \exp(-|x|_1)$
2) Geometric : $f(x) = 1_K(x)$ where K is a convex body.

Properties of log-concave measures.

Marginals

Let $w : \mathbb{R}^2 \to \mathbb{R}_+$ be a log-concave function. Then

$$x \mapsto \int w(x,y)dy$$

is log-concave on $\mathbb{R}$.
In other words, when μ is log-concave, for every subspace F, the marginal $\pi_F \mu$ is log-concave.

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Convolution

If f and g are two log-concave functions on $\mathbb{R}$ then

$$x \mapsto \int f(x-y)g(y)dy$$

is log-concave on $\mathbb{R}$.
In other words, if X et Y are random vectors with log-concave law then $X+Y$ is log-concave.

Convex geometry - Log-concave measures.

K. Ball
Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$. Studia Math. 88 (1988), no. 1, 69–84
and more recent ones of Klartag, Paouris . . .

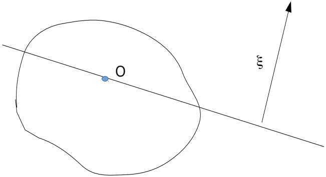
L. Lovász, M. Simonovits
Random walks in a convex body and an improved volume algorithm. Random Structures Algorithms 4 (1993), no. 4, 359–412.

R. Kannan, L. Lovász, M. Simonovits
Isoperimetric problems for convex bodies and a localization lemma. Discrete Comput. Geom. 13 (1995), no. 3-4, 541–559.
Random walks and an $O^*(n^5)$ volume algorithm for convex bodies. Random Structures Algorithms 11 (1997), no. 1, 1–50.

Convex geometry - Log-concave measures.

The hyperplane conjecture :
does there exist a constant $C > 0$ such that :

for every n and every convex body $K \subset \mathbb{R}^n$ of volume 1 and barycenter at the origin, there is a direction ξ such that $\mathrm{Vol}\,(K \cap \xi^{\perp}) \geq C$?



let K_1 and K_2 be two convex bodies with barycenter at the origin such that for every $\xi \in S^{n-1}$

$$\mathrm{Vol}\,(K_1 \cap \xi^{\perp}) \leq \mathrm{Vol}\,(K_2 \cap \xi^{\perp})$$

then $\mathrm{Vol}\,(K_1) \leq \ C\,\mathrm{Vol}\,(K_2)$?

Convex geometry - Log-concave measures.

The hyperplane conjecture : equivalent formulation

$$n\, L_K^2 = \min_{\mathcal{E},\, \mathrm{Vol}\, \mathcal{E} = \mathrm{Vol}\, B_2^n} \frac{1}{(\mathrm{Vol}\, K)^{1+\frac{2}{n}}} \int_K \|x\|_{\mathcal{E}}^2 \, dx, \qquad \sup_{n,\, K} L_K \le C\ ?$$

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Attained when K is in isotropic position :

K has barycenter at the origin and the inertia matrix is the identity

$$\frac{1}{\mathrm{Vol}\,K}\int_K x_i x_j\,dx = \delta_{i,j}. \qquad L_K = \frac{1}{(\mathrm{Vol}\,K)^{\frac{1}{n}}}$$

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Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be a log-concave isotropic function,

$$\int f(x) dx = 1, \ \int x f(x) dx = 0, \ \int x_i x_j f(x) dx = \delta_{i,j}.$$

$$\sup_{f \text{ isotropic}} f(0)^{1/n} \leq C?$$

Convex geometry - Log-concave measures.

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Theorem (Ball). These two questions are equivalent.

Convex geometry - Log-concave measures.

Theorem (Ball, '85). Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function. Then for every $p > 0$, the function $F : \mathbb{R}^n \to \mathbb{R}_+$

$$x \mapsto \left(\int_0^{+\infty} f(rx)\, r^{p-1} dr \right)^{-1/p}$$

is convex.

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When $f(0) > 0$, we define a family of convex sets

$$K_p(f) = \left\{ x \in \mathbb{R}^n,\ p \int_0^{+\infty} f(rx)\, r^{p-1} dr \geq f(0) \right\}$$

Computing the volume of a convex body

$K \subset \mathbb{R}^n$ is given by a separation oracle

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Randomization - Given ε and η, Dyer-Frieze-Kannan('89) established randomized algorithms returning a non-negative number ζ such that

$$(1-\varepsilon)\zeta < \operatorname{Vol} K < (1+\varepsilon)\zeta$$

with probability at least $1-\eta$. The running time of the algorithm is polynomial in n, $1/\varepsilon$ and $\log(1/\eta)$.

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The number of oracle calls is a random variable and the bound is for example on its expected value.

Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

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Rounding - Put the convex body in a position where

$$B_2^n \subset K \subset d\, B_2^n$$

where $d \leq n^{const}$.

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- John ('48) : $d \leq n$ (or $d \leq \sqrt{n}$ in the symmetric case).
How to find an algorithm to do so ?

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- Idea : find an algorithm which produces in polynomial time a matrix A such that AK is in an approximate isotropic position.

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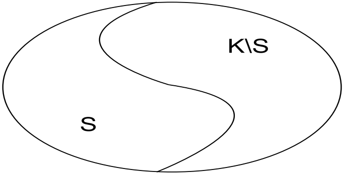
Computing the volume - Monte Carlo algorithm, estimates of local conductance.

Conjecture 1 of KLS ('95) : isoperimetric inequality - open !

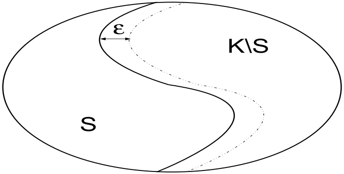
Isoperimetric problem.



Isoperimetric problem.

Define

$$\mu^+(S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B_2^n) - \mu(S)}{\varepsilon}$$



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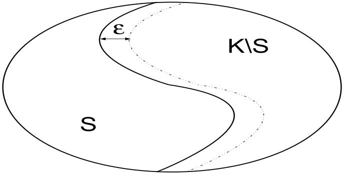

Question. Find the largest h such that

$$\forall\, S \subset K,\ \mu^+(S) \geq\ h\, \mu(S)(1-\mu(S)) \quad ?$$

μ is log-concave with log concave density f.

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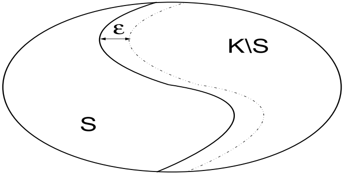

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The probability $d\mu(x) = f(x)dx$ is log-concave isotropic.

Poincaré type inequality. For every regular function F,

$$h^2 \operatorname{Var}_{\mu} F \leq \int |\nabla F(x)|_2^2 f(x)dx.$$

The conjecture is that h is a universal constant.

Kannan, Lovász, Simonovits ['95], Bobkov ['07] :

$$h \geq \frac{c}{\int_K |x - g_K|_2 dx} \qquad h \geq \frac{c}{(\mathrm{Var}\,|X|_2^2)^{1/4}} .$$

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KLS conjecture is that h is a universal constant.

Take $F(x) = |x|_2$ or $F(x) = |x|_2^p$
Strong concentration of the Euclidean norm

$$\mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C \exp(-c\, t\, \sqrt{n})$$

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Large and medium scales !

Concentration - Khintchine

Proposition. (Borell '73) Let μ be a log-concave probability, C a symmetric convex set in $\mathbb{R}^n$ such that $\mu(C) \geq 2/3$. Then for every $t \geq 1$,

$$\mu\left(\mathbb{R}^n \setminus (tC)\right) \leq \left(\frac{1}{2}\right)^{\frac{t+1}{2}}$$

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Indeed for $\alpha = \frac{t-1}{t+1}$ we have : $1 - \alpha = \frac{2}{t+1}$ and

$$(1-\alpha)\big(\mathbb{R}^n \setminus (tC)\big) + \alpha C \subset \big(\mathbb{R}^n \setminus C\big)$$

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Consequences : reverse Hölder inequality. If X is a log-concave random vector then for every $\theta \in \mathbb{R}^n$, for every $p \geq 2$,

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norm, Khintchine-Kahane

Results.

Evidence : in isotropic position, $\mathbb{E}|X|_2^2 = n$. Take the proposition with

$$C = \left\{ x \in \mathbb{R}^n,\ |x|_2 \leq \sqrt{3\,n} \right\}$$

then $\mu(C) \geq 2/3$ and for every $t \geq 1$,

$$\mu\left(\mathbb{R}^n \setminus (tC)\right) \leq \left(\frac{1}{2}\right)^{\frac{t+1}{2}}$$

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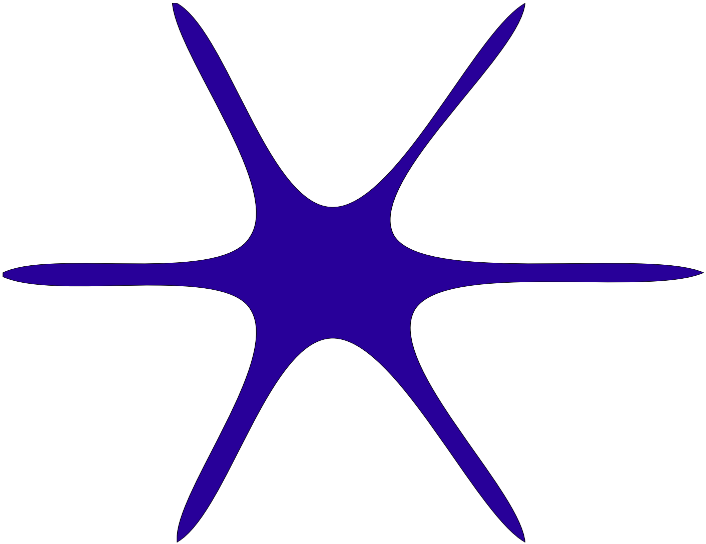
After works of Klartag, Fleury-G-Paouris, Fleury

Theorem (G-Milman 2011). For every $t \in (0,1)$

$$\mathbb{P}\left\{\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right\} \leq Ce^{-c\,t^3\,\sqrt{n}}$$

Pictures - Intuition in high dimension.



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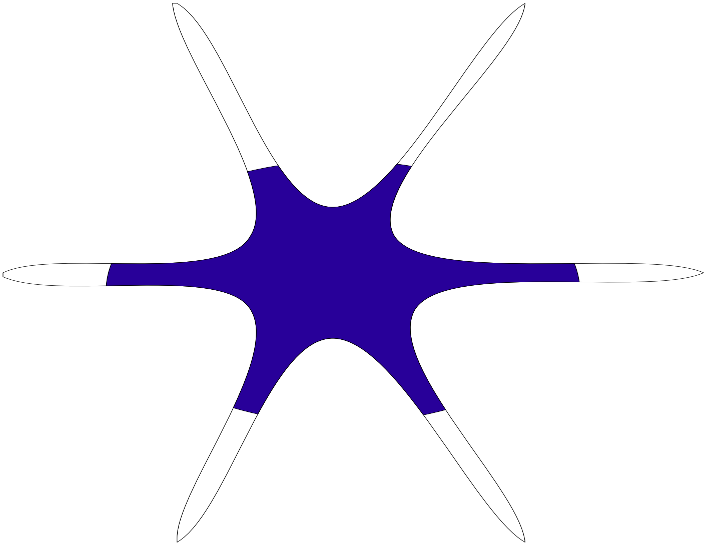
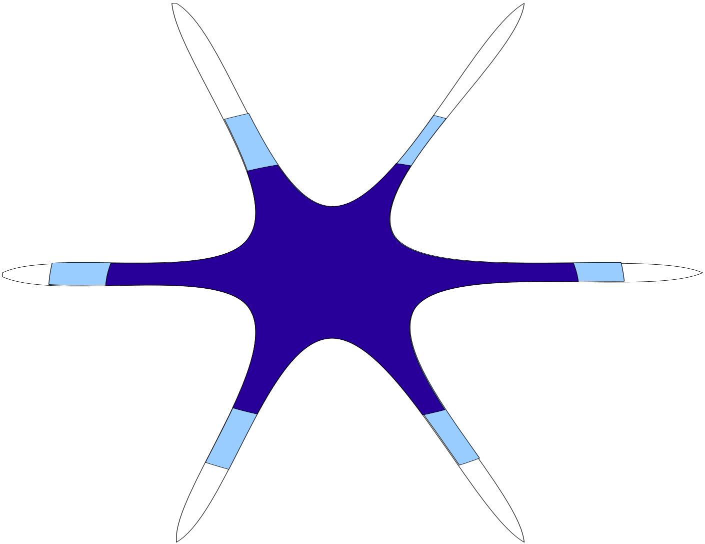


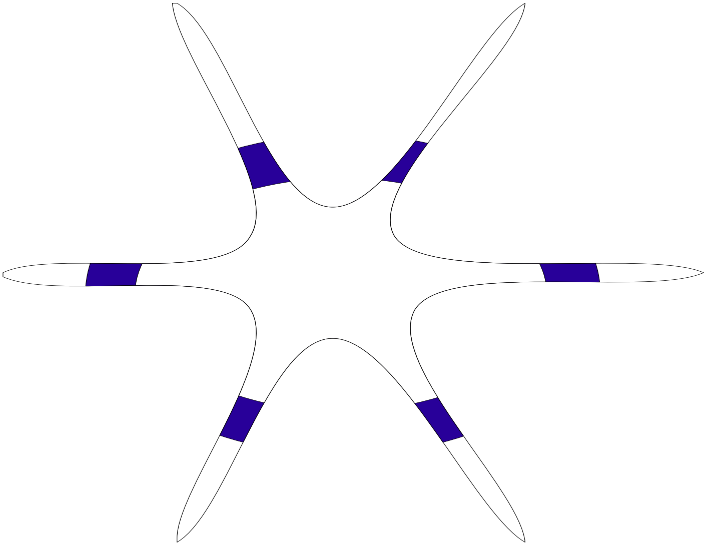
intersection with a ball of radius $\sqrt{n}$.

Pictures - Intuition in high dimension.



volume inside a ball of radius $100\sqrt{n}$

Pictures - Intuition in high dimension.



volume inside a shell of width $\sqrt{n}/n^{1/6}$

Thin shell and central limit theorem

CLT : classical case. $x_1, \ldots, x_n$, n i.i.d random variables,

$$\mathbb{E}x_i^2 = 1, \mathbb{E}x_i = 0, \mathbb{E}x_i^3 = \tau$$

then $\forall \theta \in S^{n-1}$

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^n \theta_i x_i \le t\right) - \int_{-\infty}^t e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \le \tau |\theta|_4^2 = \frac{\tau}{\sqrt{n}}.$$

Thin shell and central limit theorem

Question. [Ball '97], [Brehm-Voigt '98] Let K be an isotropic convex body, find a direction $\theta \in S^{n-1}$ such that

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Conjecture. [Anttila-Ball-Perissinaki '03]

Thin shell conjecture : $\forall n, \exists \varepsilon_n$ such that for every random vector uniformly distributed in an isotropic convex body

$$\mathbb{P}\left(\left| \frac{|X|_2}{\sqrt{n}} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n$$

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Theorem[ABP]. Thin shell $\Rightarrow$ CLT

Concentration of the mass in a Euclidean ball or shell

$\Leftrightarrow$

Behavior of $(\mathbb{E}|X|_2^p)^{1/p}$ for some values of p.

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• X log-concave random vector. Paouris Theorem (large deviation) may be written as (ALLOPT '12)

$$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\, \mathbb{E}|X|_2 + c\, \sigma_p(X)$$

where $\sigma_p(X) = \sup_{|z|_2 \leq 1} \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p}$.

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→ In isotropic position, $\mathbb{E}|X|_2 \leq (\mathbb{E}|X|_2^2)^{1/2} = \sqrt{n}$.
By Borell's inequality (Khintchine type inequality)

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Hence $$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\sqrt{n} + cp$$

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Hence $\quad \forall p \geq 1, \quad (\mathbb{E}|X|_2^p)^{1/p} \leq C\sqrt{n} + cp$

Take $p = t\sqrt{n}$, Markov gives

$$\forall t \geq 10, \quad \mathbb{P}\left(|X|_2 \geq t\,\sqrt{n}\right) \leq e^{-c\,t\,\sqrt{n}}.$$

KLS conjecture and consequences.

- Strong concentration of the Euclidean norm

$$\mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C\,\exp(-c\,t\,\sqrt{n})$$

$$\forall p \in [-c\sqrt{n}, c\sqrt{n}], \quad (\mathbb{E}|X|_2^p)^{1/p} \leq (\mathbb{E}|X|_2^2)^{1/2}\Big(1 + \frac{c\,|p|}{n}\Big).$$

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- Eldan-Klartag ['11], Eldan ['12].

Idea to attack the problem

We replace a simple quantity : $|X|_2$ by a more complicated

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But for a k-dimensional F, $P_F G_n \sim G_k$ hence

$$\mathbb{E}|X|_2^p = \frac{\mathbb{E}|G_n|_2^p}{\mathbb{E}|G_k|_2^p}\ \mathbb{E}\mathbb{E}_F|P_F X|_2^p.$$

An integration in polar coordinates proves that

$$\mathbb{E}_F\mathbb{E}|P_F X|_2^p = \mathbb{E}_U h_{k,p}(U)$$

where for every $u \in SO(n)$,

$$h_{k,p}(u) = |S^{k-1}| \int_0^{+\infty} t^{k+p-1} \pi_{u(F_0)} w(tu(\theta_0))dt$$

Other reverse Hölder inequalities ?

Log-Sobolev inequality

For every function $h \geq 0$, define

$\mathrm{Ent}\ h = \int h \log h - \int h \log \int h$.

We have a log-Sobolev inequality when there exists C such that for all smooth functions h,

$$\mathrm{Ent}\ h \leq C \int h\ |\nabla \log h|_2^2$$

Other reverse Hölder inequalities ?

Consequences : reverse Hölder inequality. Let

$$M : p \mapsto \left(\int h^p\right)^{1/p} = \exp\left(\frac{1}{p}\log\int h^p\right)$$

Then

$$\begin{aligned} M'(p) &= M(p)\left(\frac{-1}{p^2}\log\int h^p + \frac{1}{p}\frac{\int h^p \log h}{\int h^p}\right) \\ &= \frac{1}{p^2}\left(\int h^p\right)^{\frac{1}{p}-1}\left(-\int h^p \log\int h^p + \int h^p \log h^p\right) \\ &= \frac{1}{p^2}\left(\int h^p\right)^{\frac{1}{p}-1} \mathrm{Ent}\, h^p \\ &\leq C_{LS}\left(\int h^p\right)^{\frac{1}{p}-1}\int h^p\, |\nabla \log h|_2^2 \end{aligned}$$

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If the function h has a log-Lispchitz constant bounded by L, then we have

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hence for every $p > r$

$$\frac{M(p)}{M(r)} \leq \exp\left(C_{LS}\, L^2\, (p - r) \right)$$

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And $SO(n)$ satisfies the criteria curvature-dimension of Bakry-Émery and in this case, $C_{LS} \leq \frac{c}{n}$

Where disappears the geometry of convex bodies ?

Everything is hidden in the study of the log-Lipshitz constant of the function $h_{k,p}$ defined on $SO(n)$ by

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- Marginals of log-concave measure are log-concave
- The Ball's bodies
- Some reverse Hölder inequality of Borell in a log-concave setting

THANK YOU