De la géométrie algorithmique

au calcul géométrique

l'exemple

de la triangulation de Delaunay

Les grands classiques

Géométrie algorithmique

Problème géométrique

Analyse des performances

- Borne inférieure
- Borne supérieure
- Complexité asymptotique
- Analyse dans le cas le pire
- Analyse en moyenne

Borne inférieure pour la triangulation de Delaunay

$$\Omega\left(f(n)\right)$$

Il n'existe pas d'algorithme

résolvant toutes
les instances du problème

avec moins de $f(n)$ opérations

Borne inférieure pour le tri

Borne inférieure pour le tri

Trier $\Longleftrightarrow$ trouver une permutation

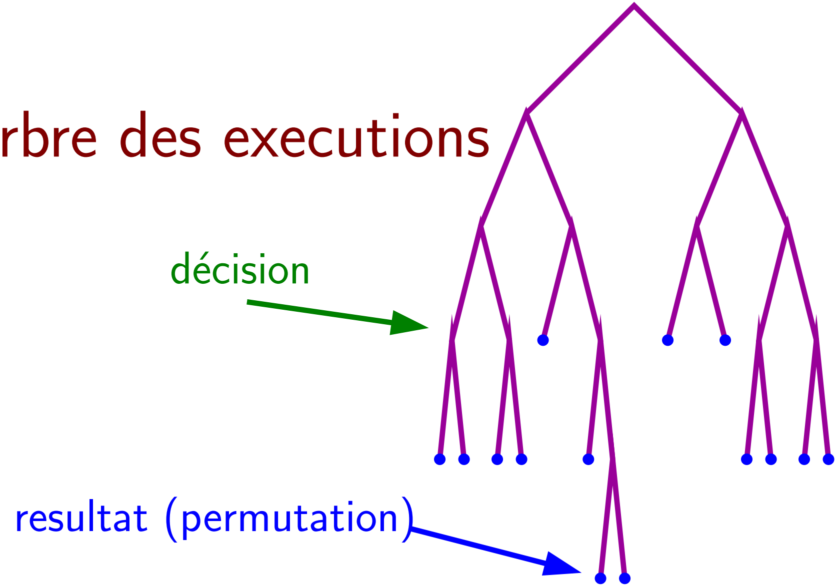
Borne inférieure pour le tri

Trier $\Longleftrightarrow$ trouver une permutation

Ordre $\neq \Longrightarrow$ exécution $\neq$

Borne inférieure pour le tri

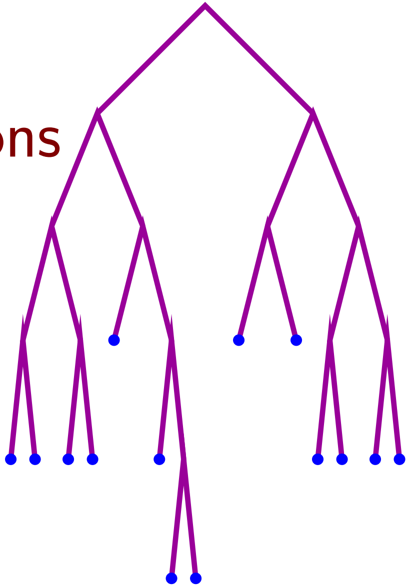
Arbre des executions



Borne inférieure pour le tri

Arbre des executions

Arbre binaire
$n!$ feuilles



hauteur $\geq \log_2 n! \simeq n \log n$

Borne inférieure pour le tri

$$\Omega(n \log n)$$

Borne inférieure pour Delaunay

Delaunay sert à trier

Borne inférieure pour Delaunay

Delaunay sert à trier

Prendre un problème de tri

Supposer que l'on sait faire Delaunay

Utiliser Delaunay pour trier

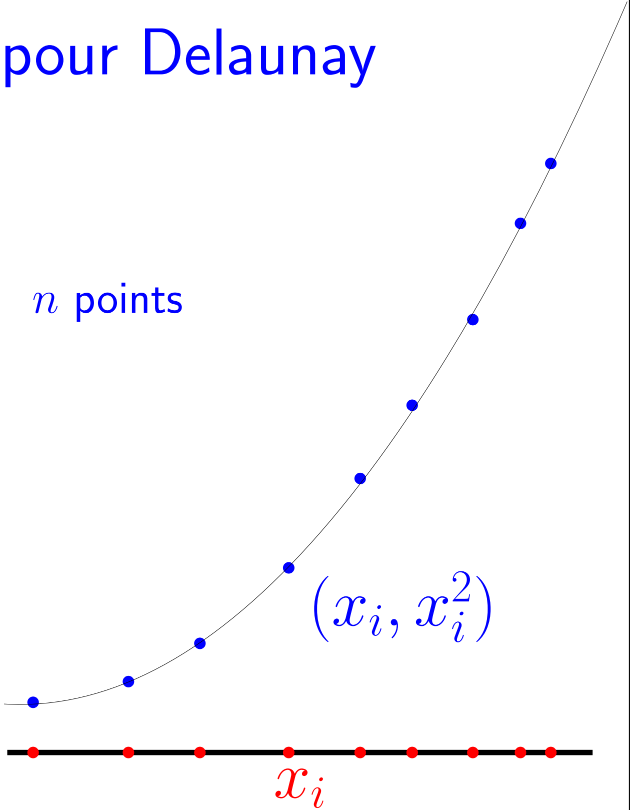
Borne inférieure pour Delaunay

$x_1, x_2, \ldots, x_n$ à trier

Borne inférieure pour Delaunay

$x_1, x_2, \ldots, x_n$ à trier

$(x_1, x_1^2), \ldots, (x_n, x_n^2)$ $\quad$ n points



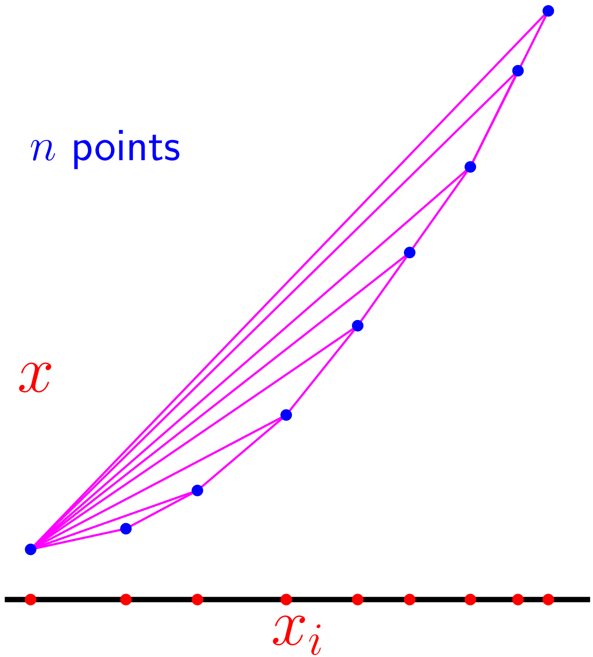
Borne inférieure pour Delaunay

$x_1, x_2, \ldots, x_n$ à trier

$(x_1, x_1^2), \ldots, (x_n, x_n^2)$ $\quad n$ points

Delaunay

$\longrightarrow$ ordre en x



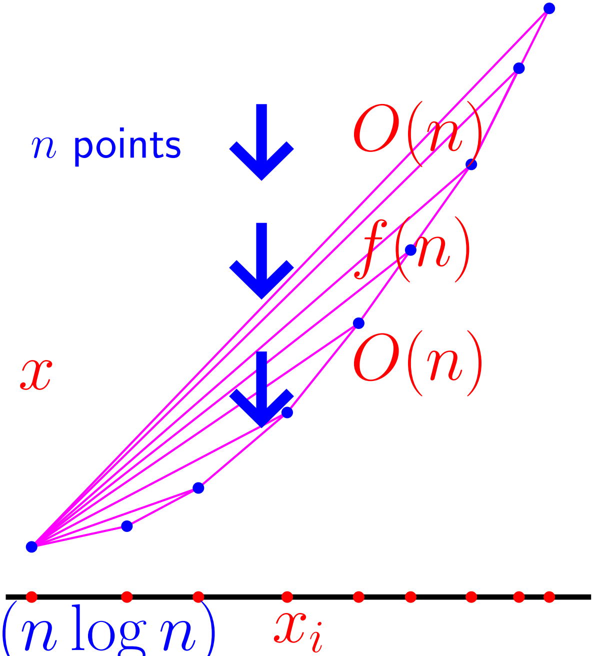
Borne inférieure pour Delaunay

$x_1, x_2, \ldots, x_n$ à trier

$(x_1, x_1^2), \ldots, (x_n, x_n^2)$ $\quad n$ points

Delaunay

$\longrightarrow$ ordre en x



$O(n) + f(n) \in O(n \log n)$

Borne inférieure pour Delaunay

$x_1, x_2, \ldots, x_n$ à trier

$$\Omega(n \log n)$$

Algorithmes optimaux pour la triangulation de Delaunay

Produire des algorithmes optimaux

Complexité asymptotique
Dans le cas le pire

Division-fusion
Algorithme de balayage

Algorithmes optimaux pour la triangulation de Delaunay

Balayage

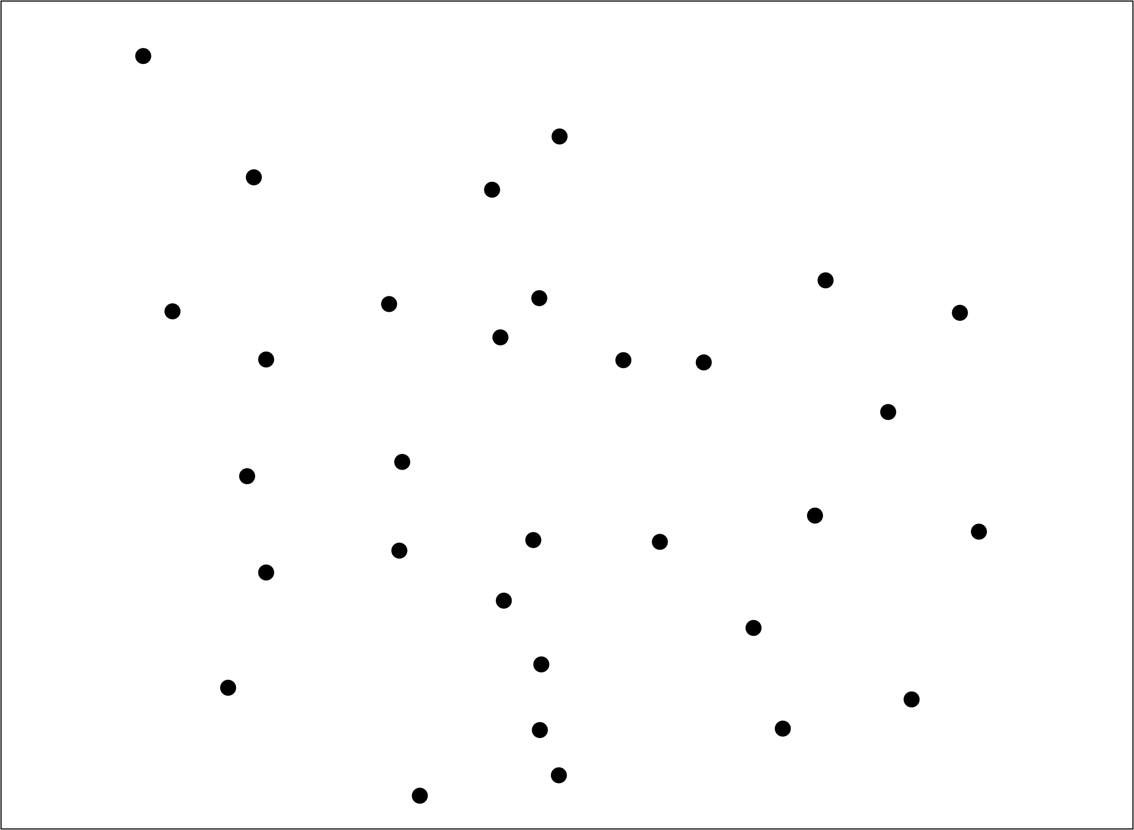
Balayage

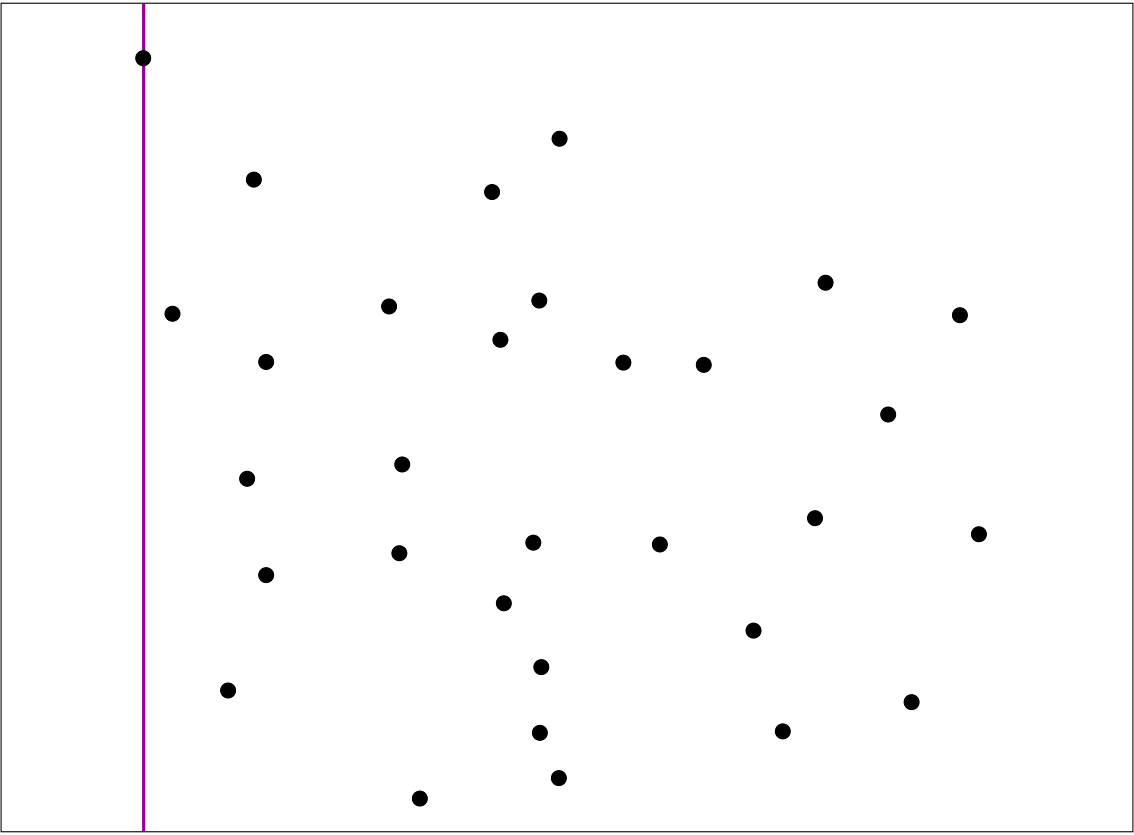
Couper le problème 2D par un objet 1D

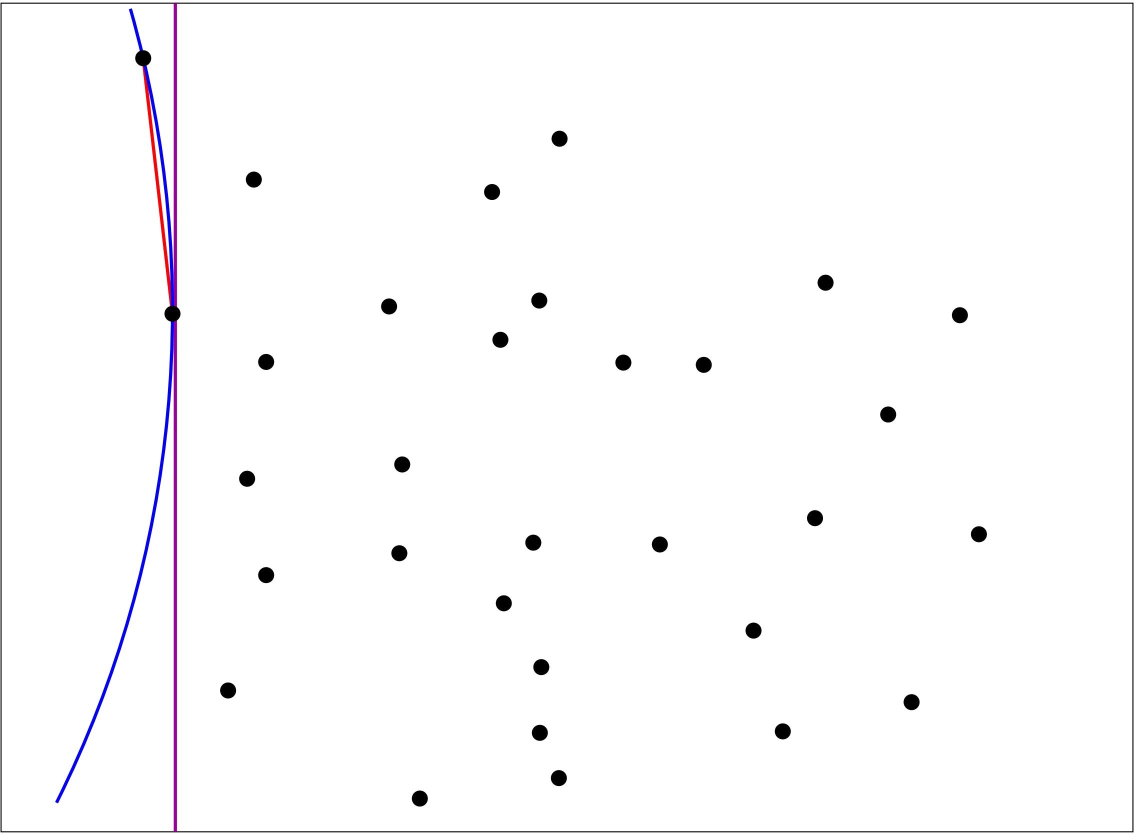
Déplacer l'objet

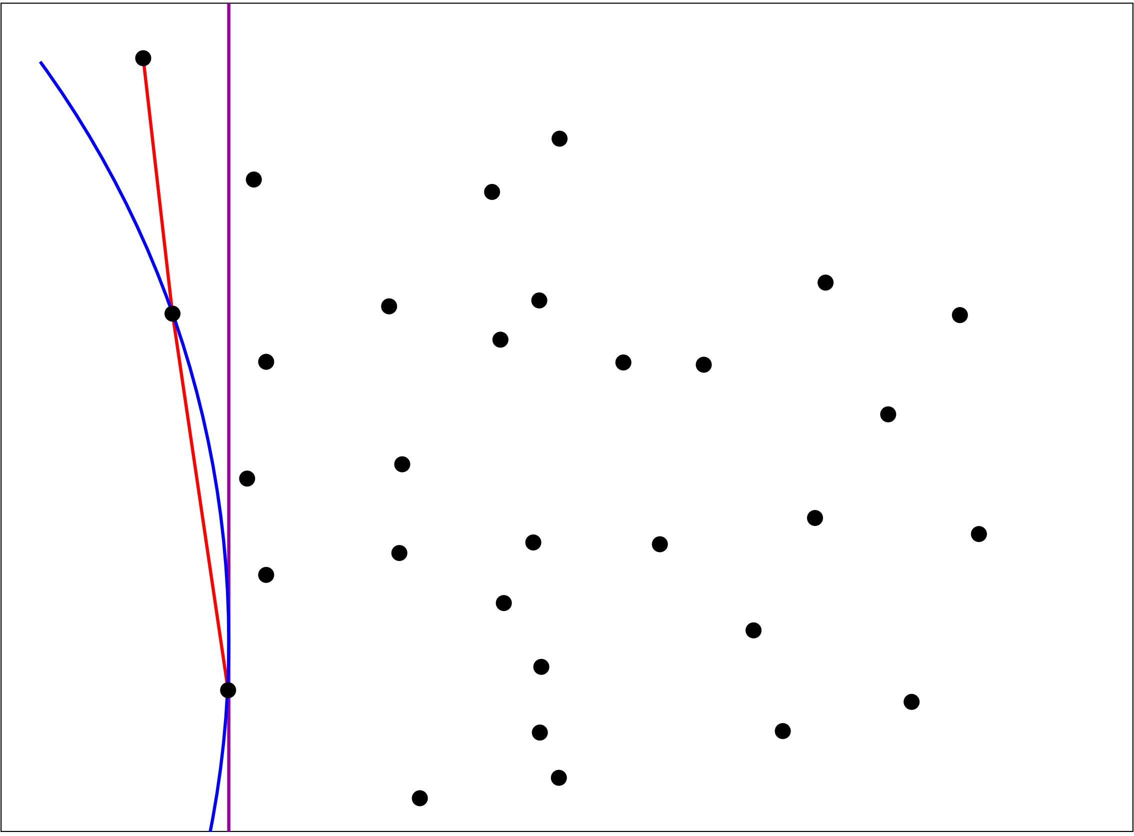
Voronoi [Fortune]

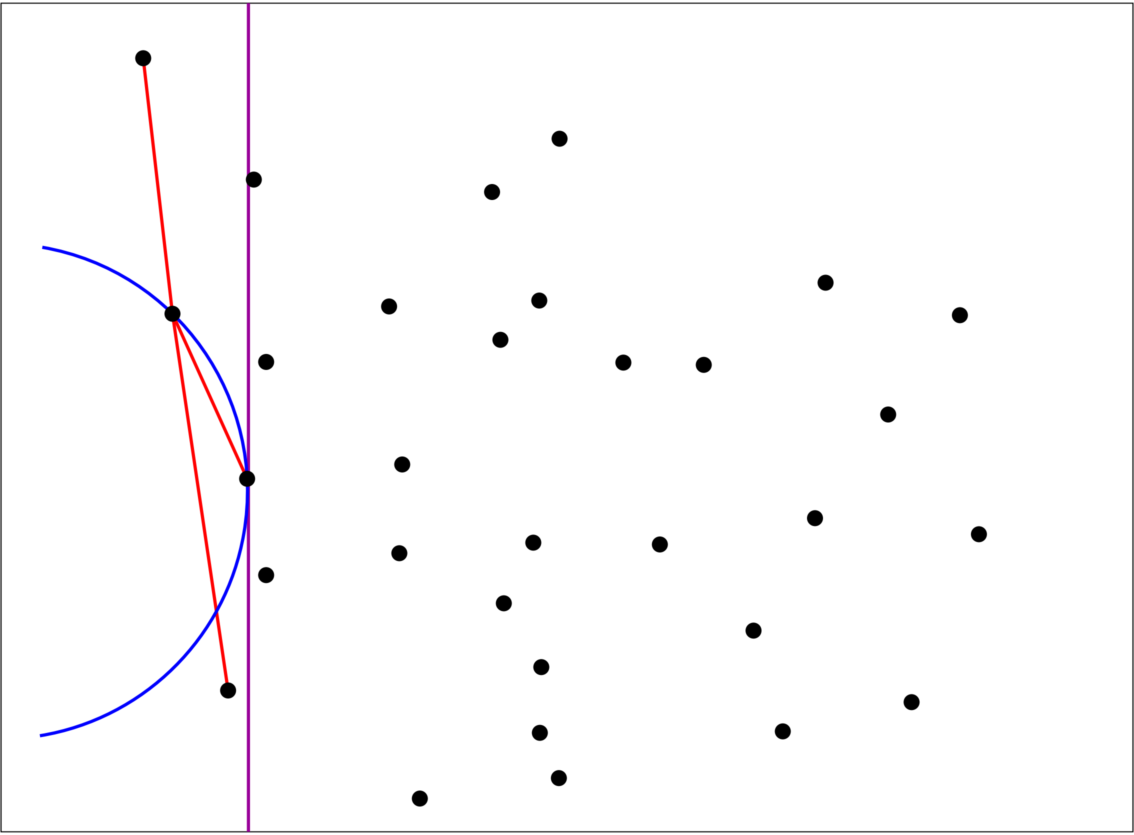
Présentation duale sur Delaunay

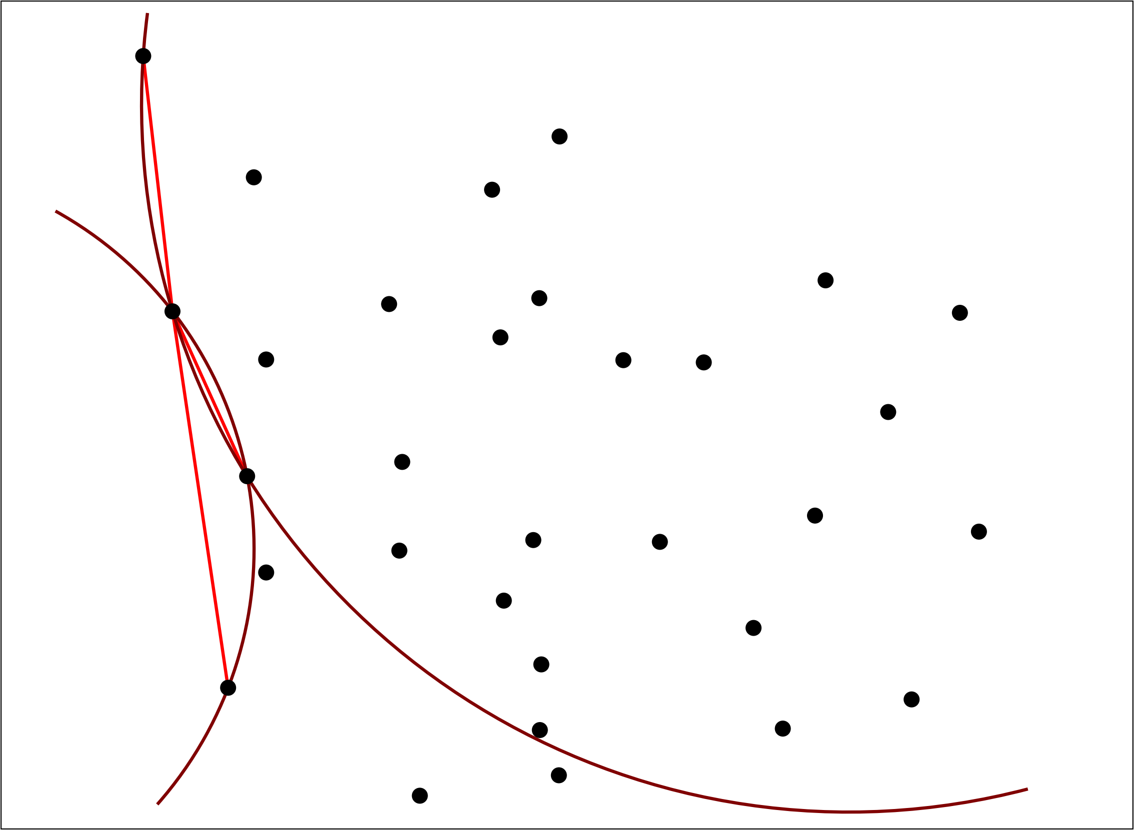




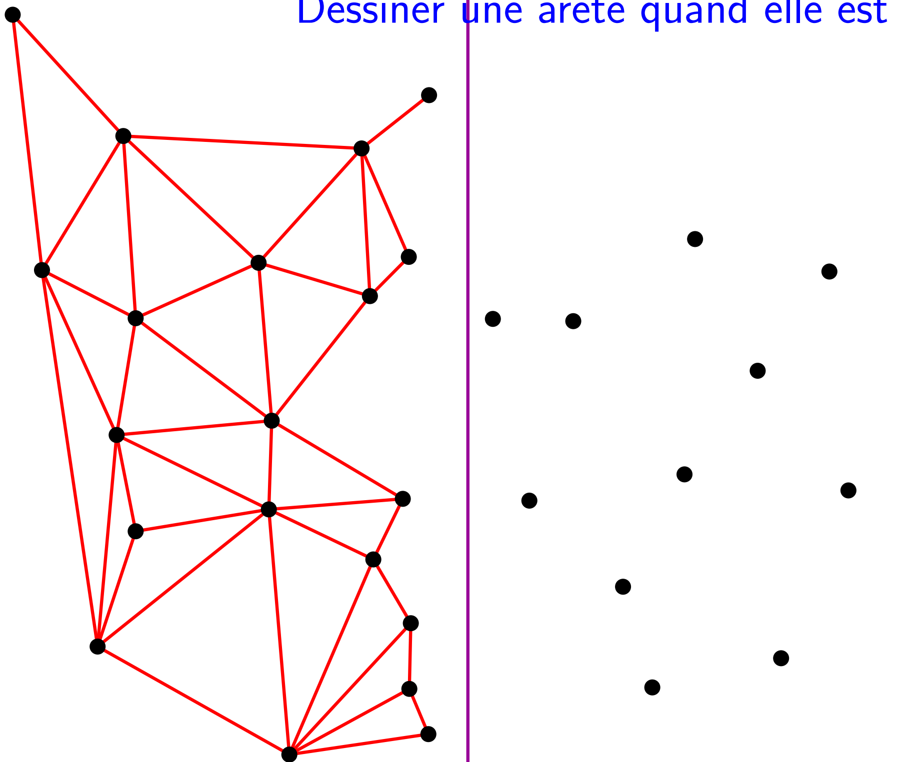


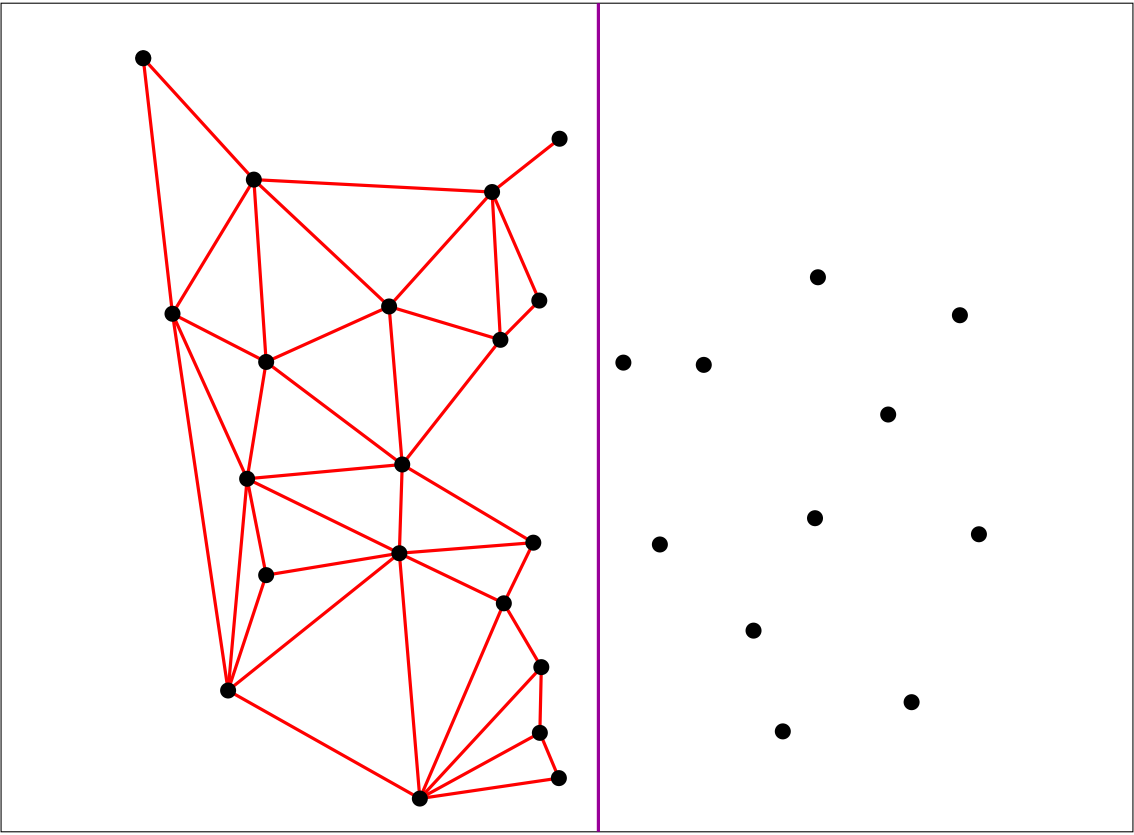




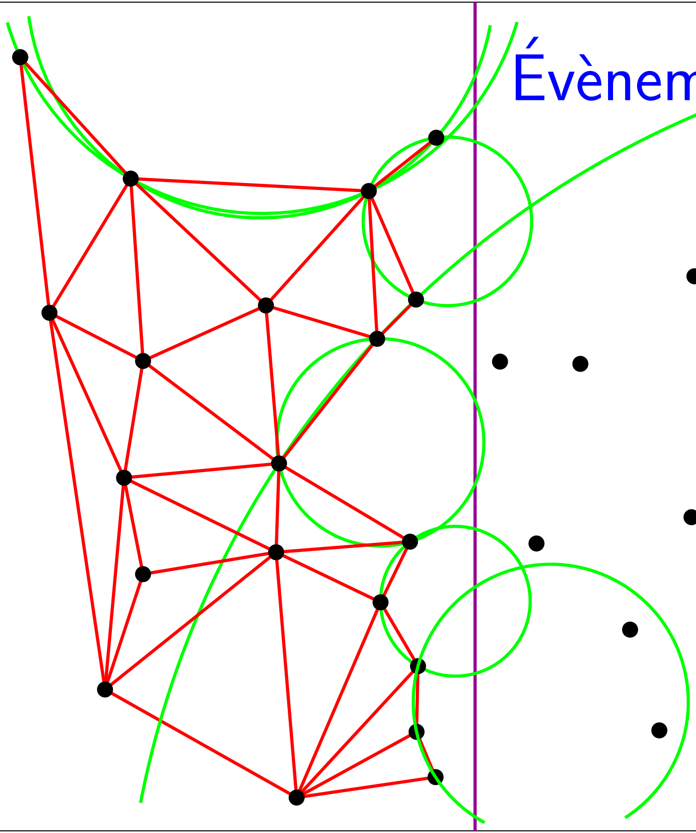


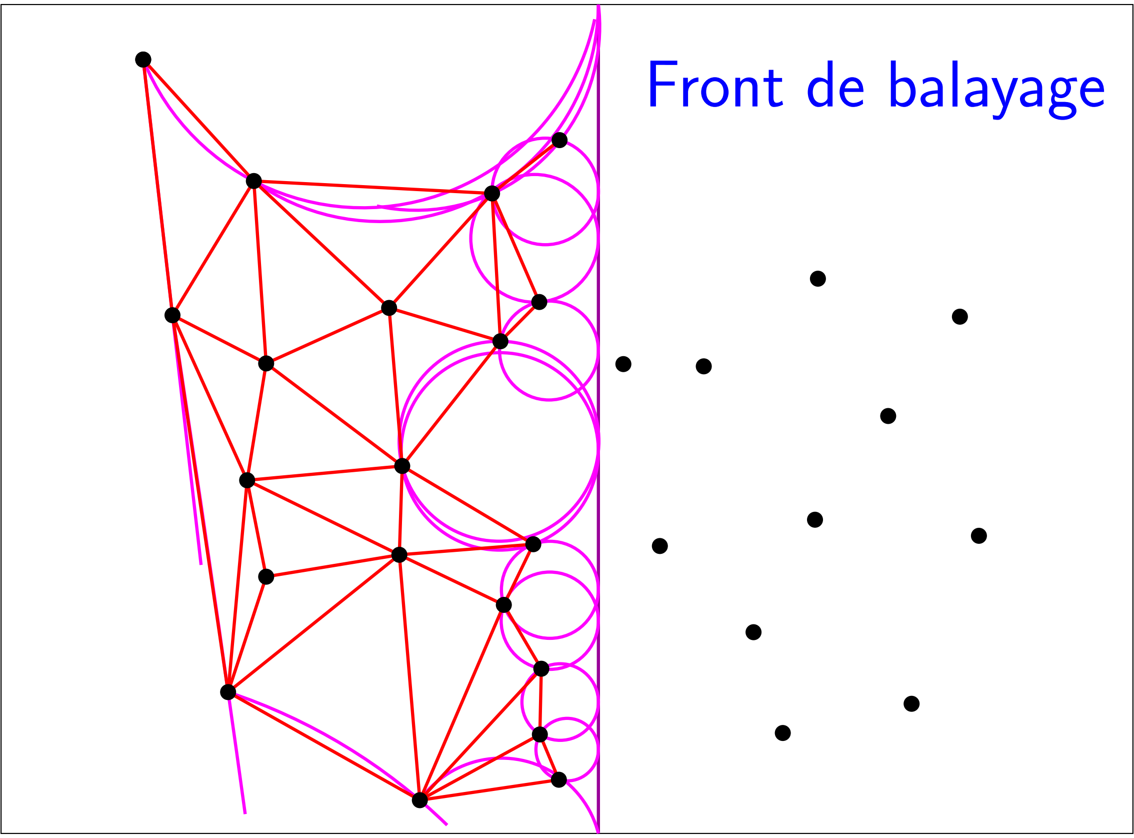
Dessiner une arête quand elle est sure



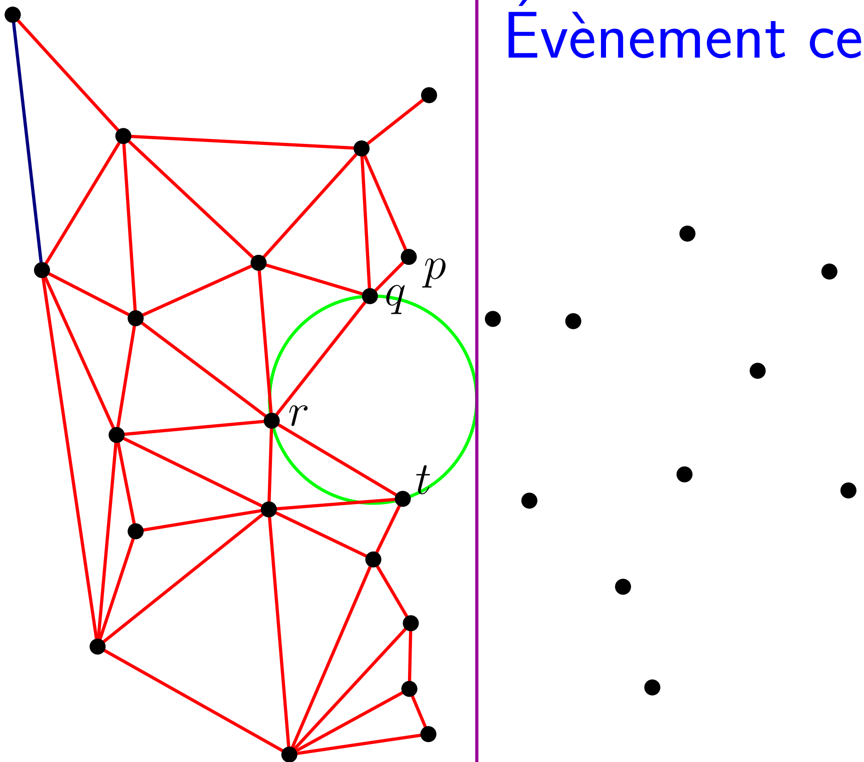


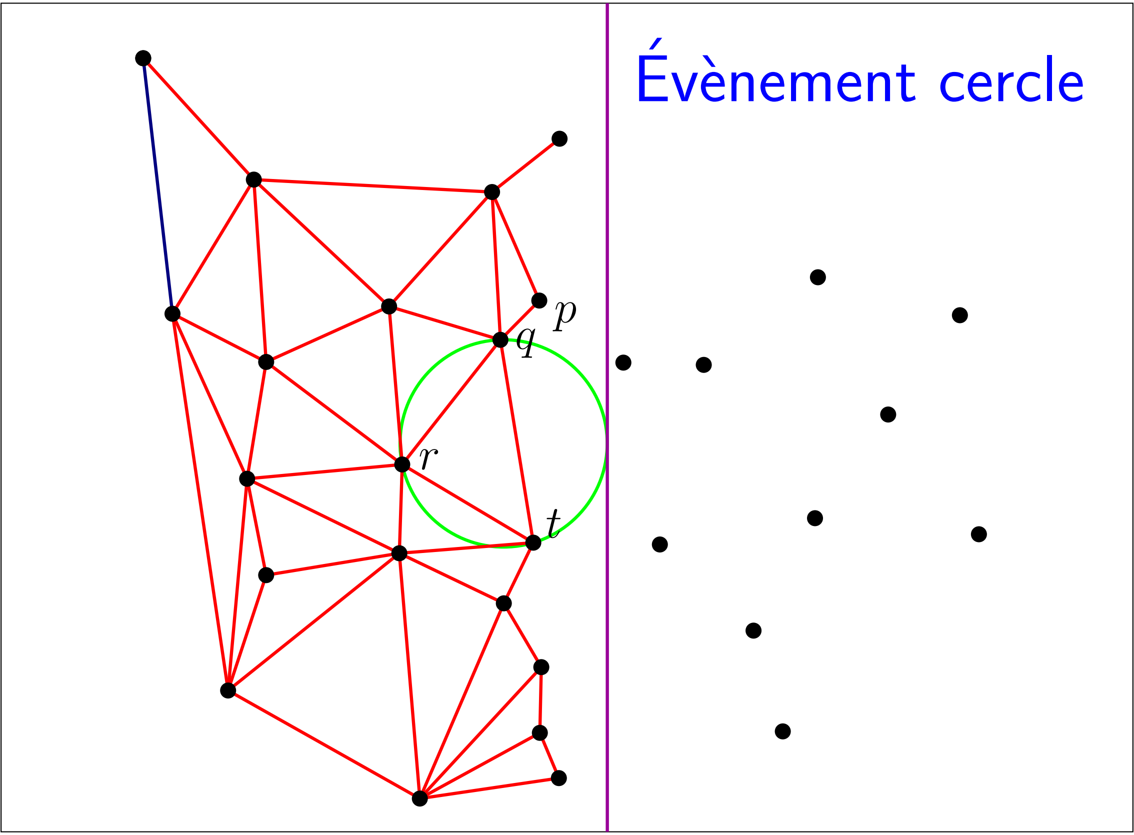
Évènements cercle



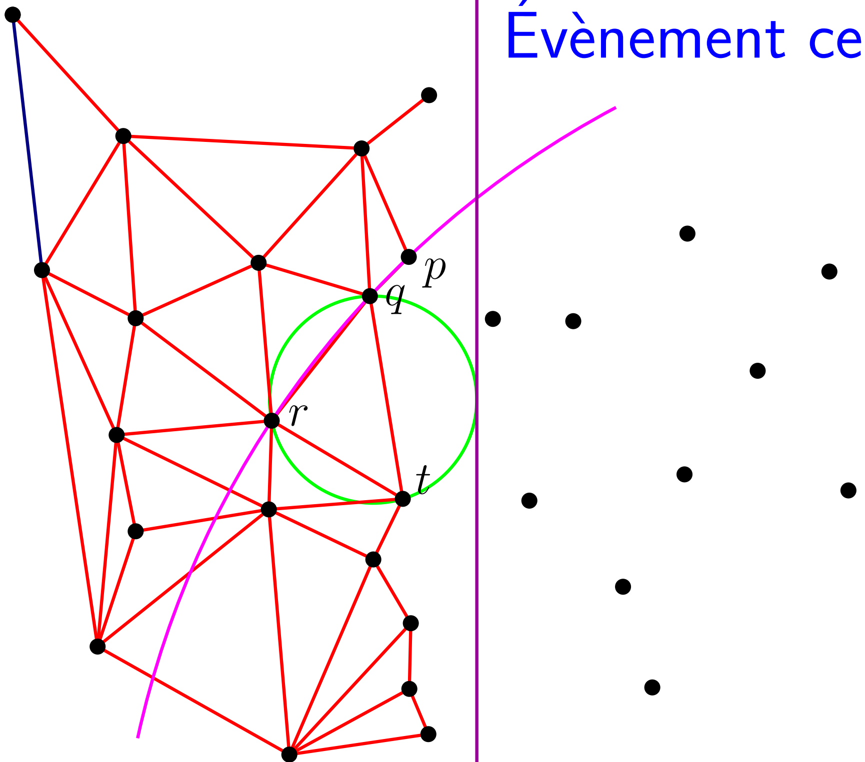

Front de balayage

Évènement cercle

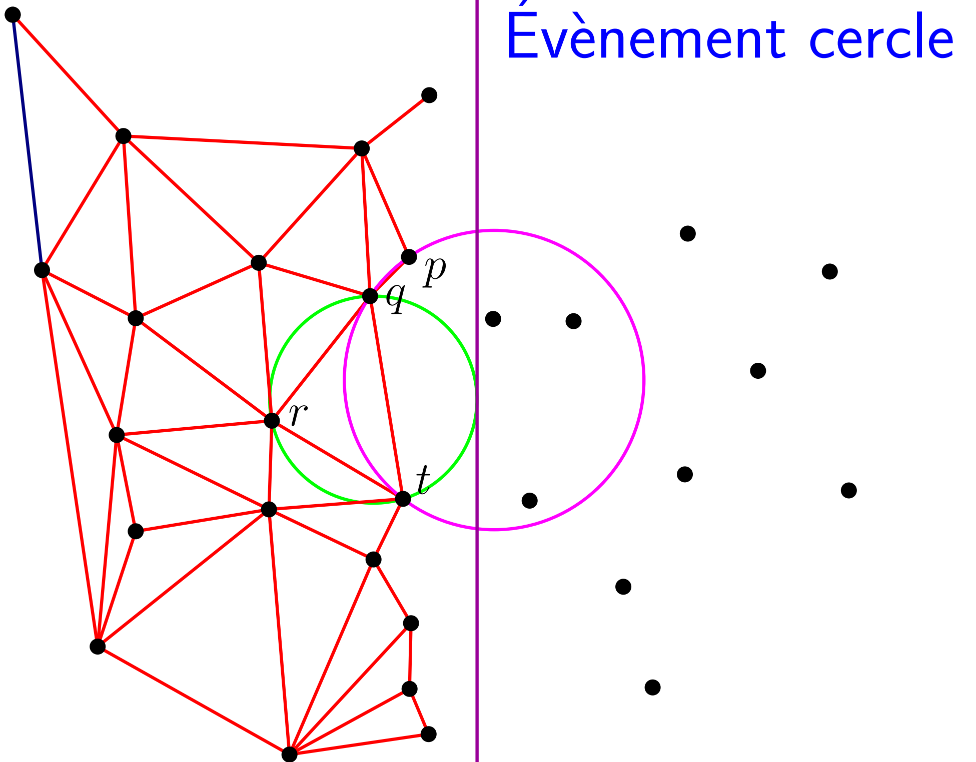


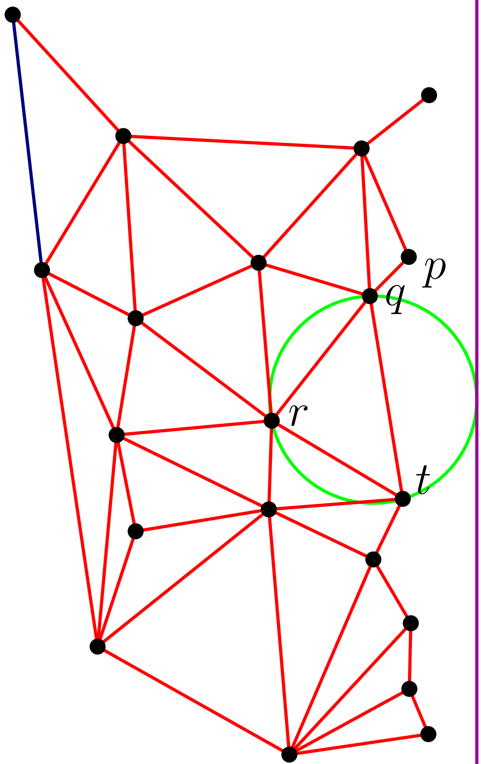

Évènement cercle
p
q
r
t

Évènement cercle



Évènement cercle





Évènement cercle

Triangulation

$+$ 1 triangle

$+$1 arête

Évenements

- 1 cercle

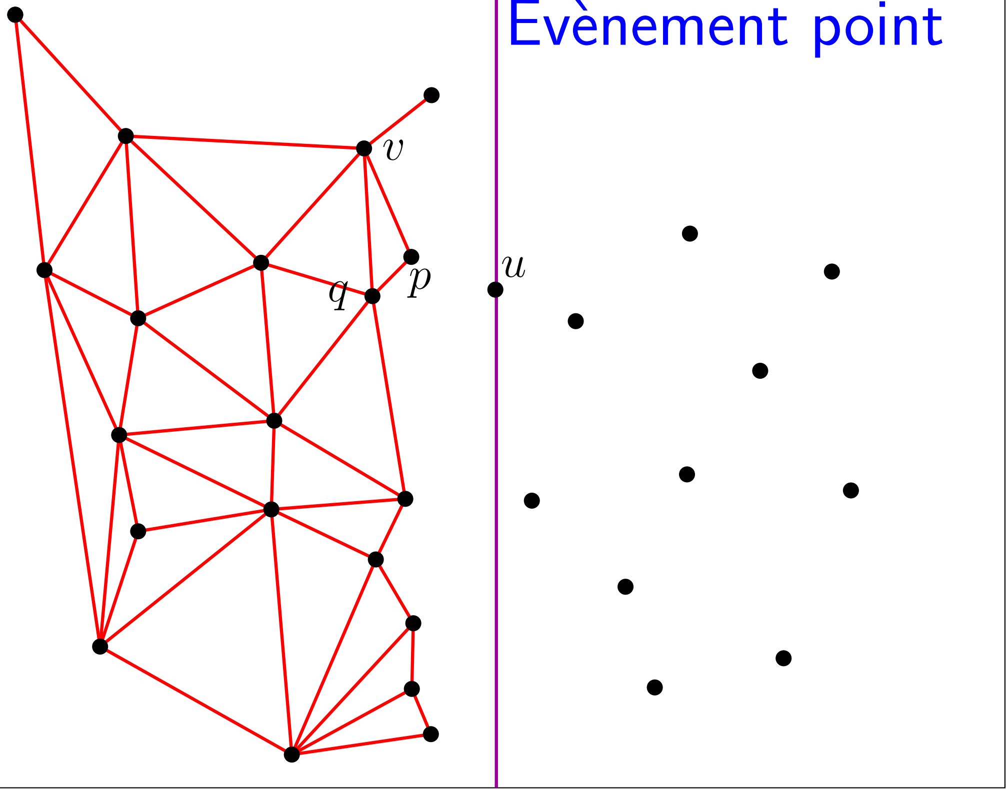
-2 cercles ?

+2 cercles ?

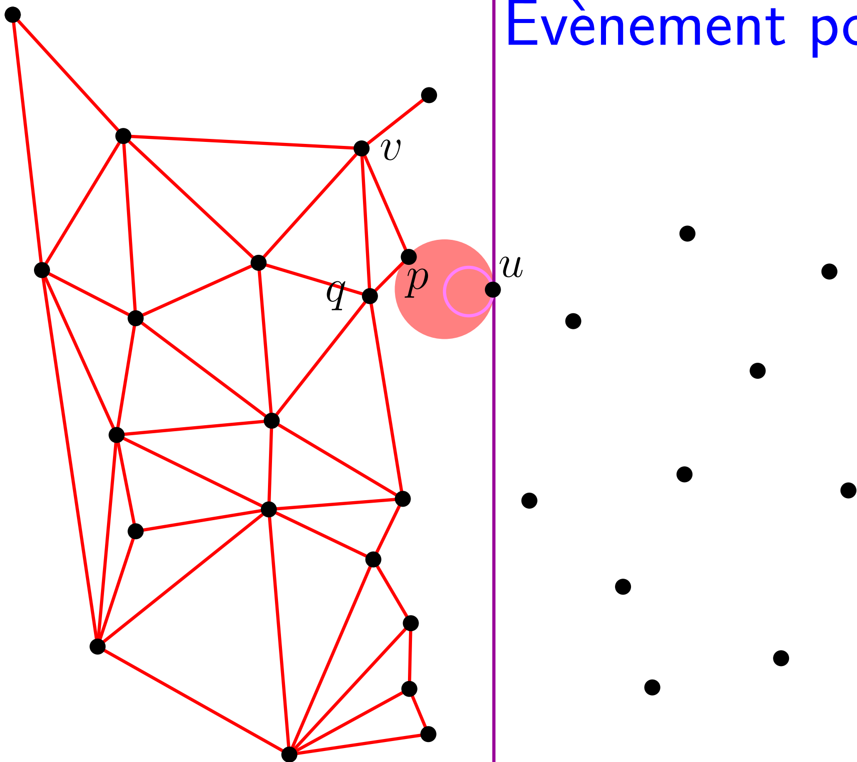
Front

2 arêtes $\rightarrow$ 1 arête

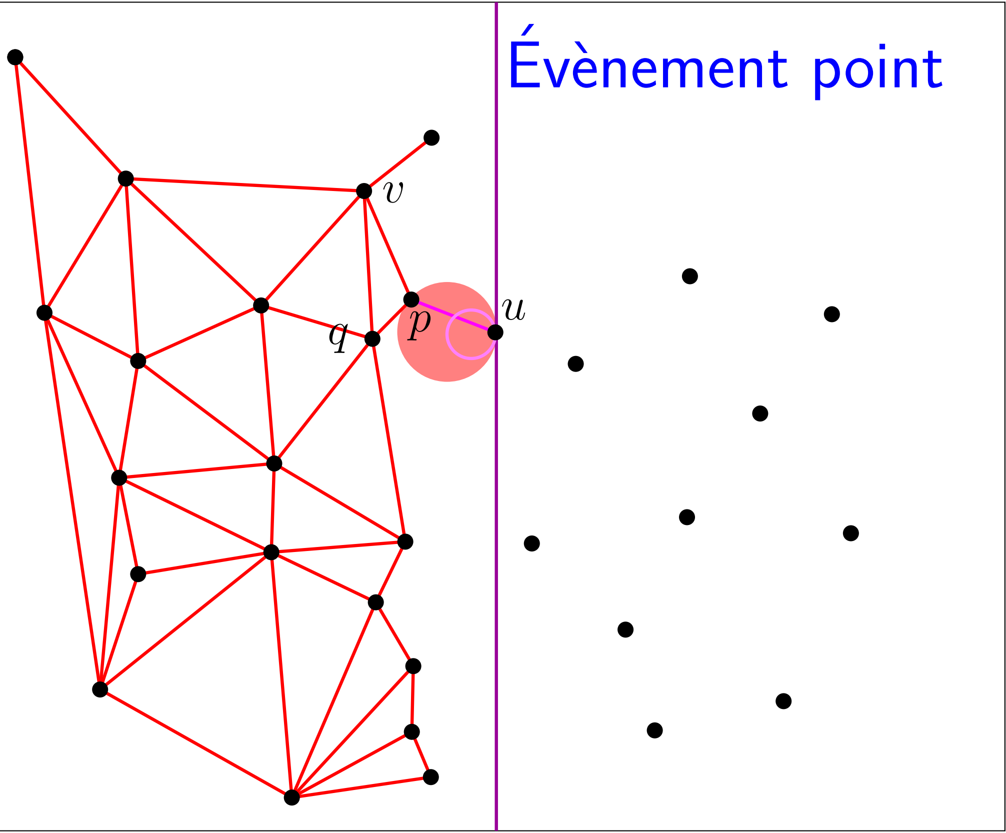
Évènement point

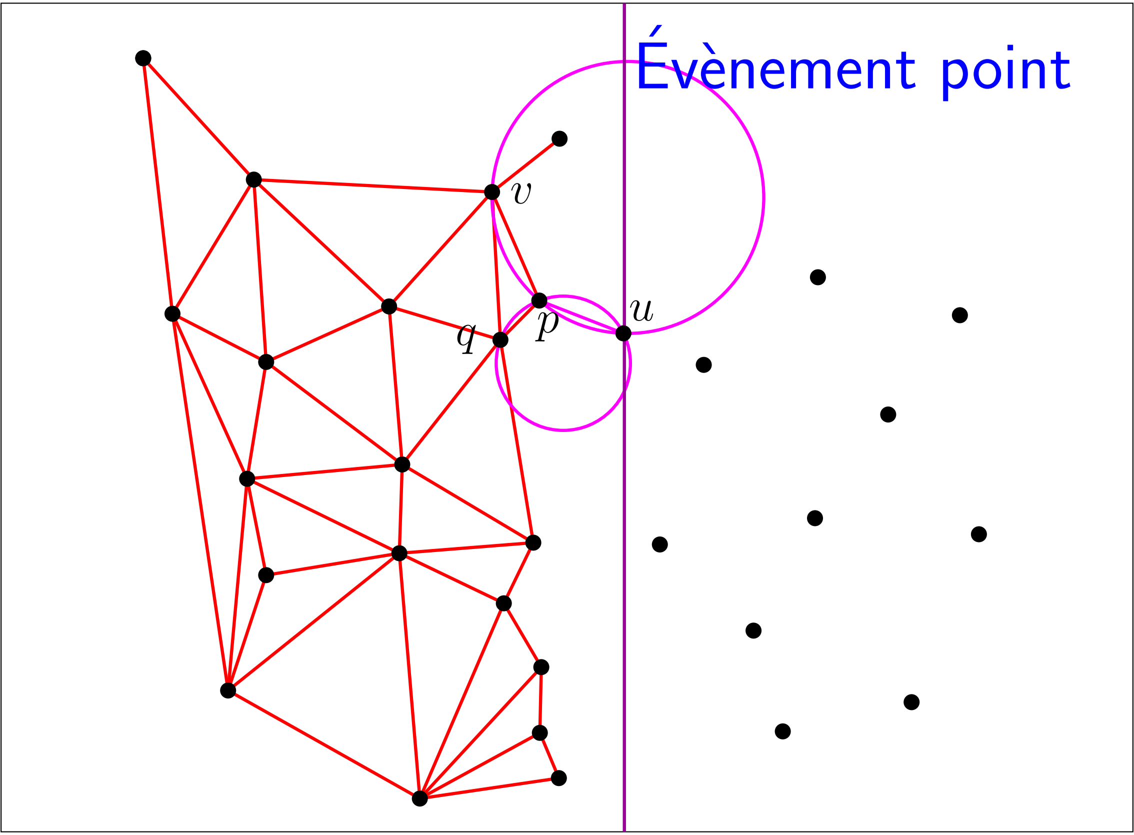


Évènement point

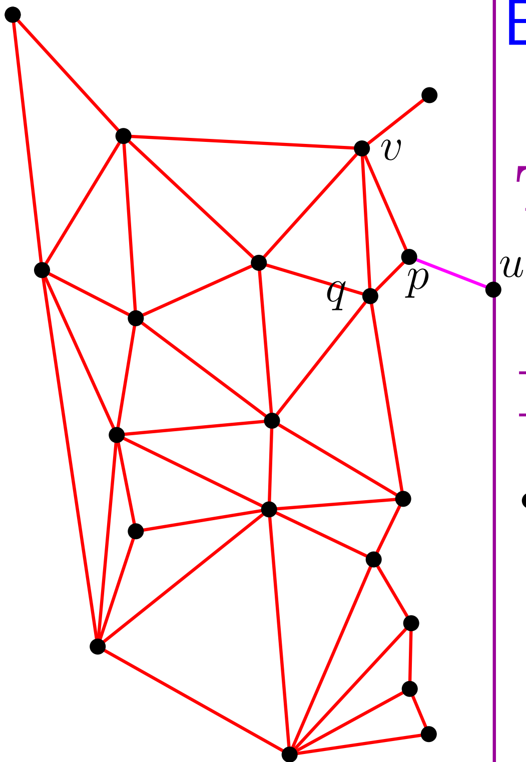


Évènement point




Évènement point
v
u
p
q

Évènement point



Triangulation

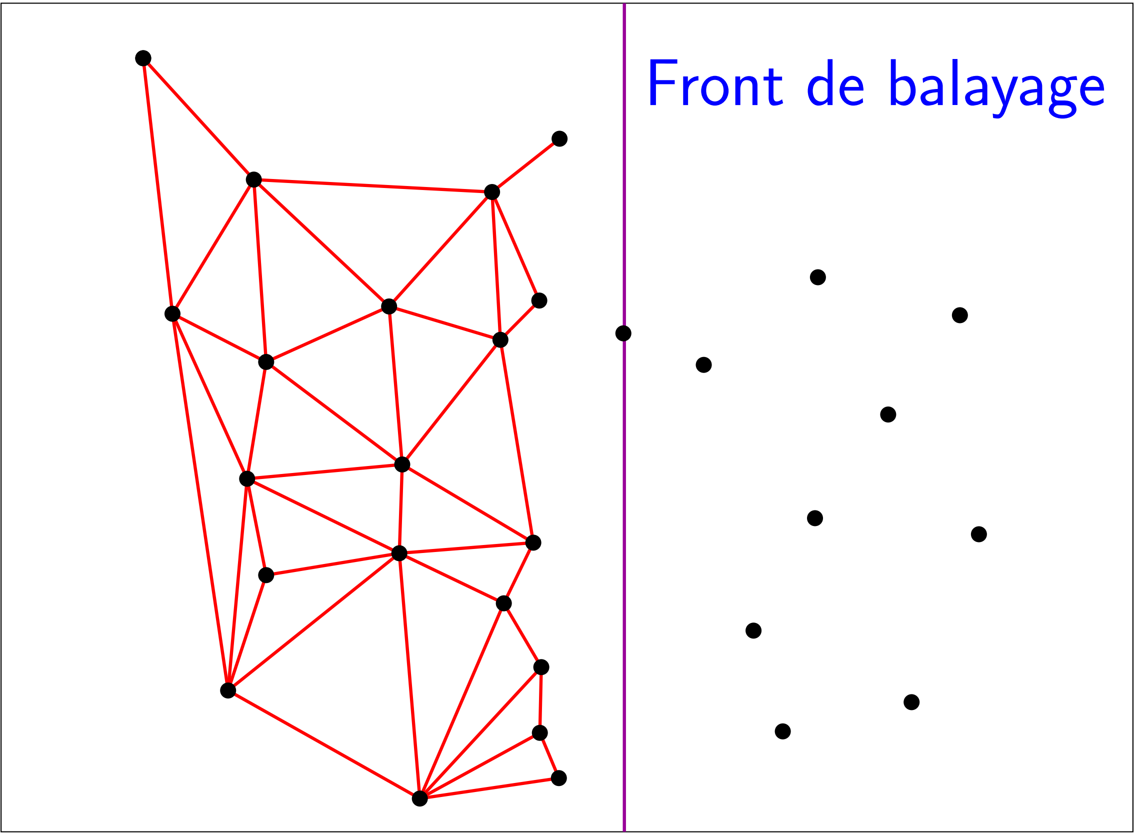
+ 1 sommet
+1 arête

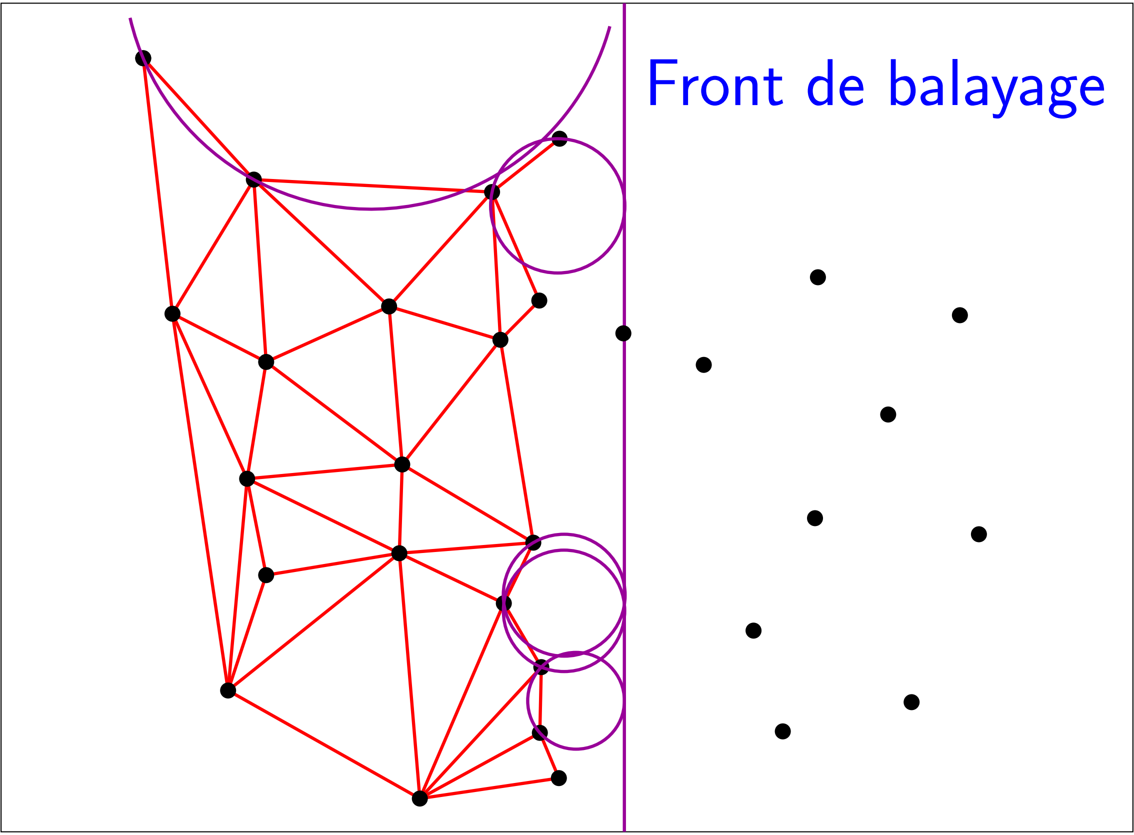
Évenements

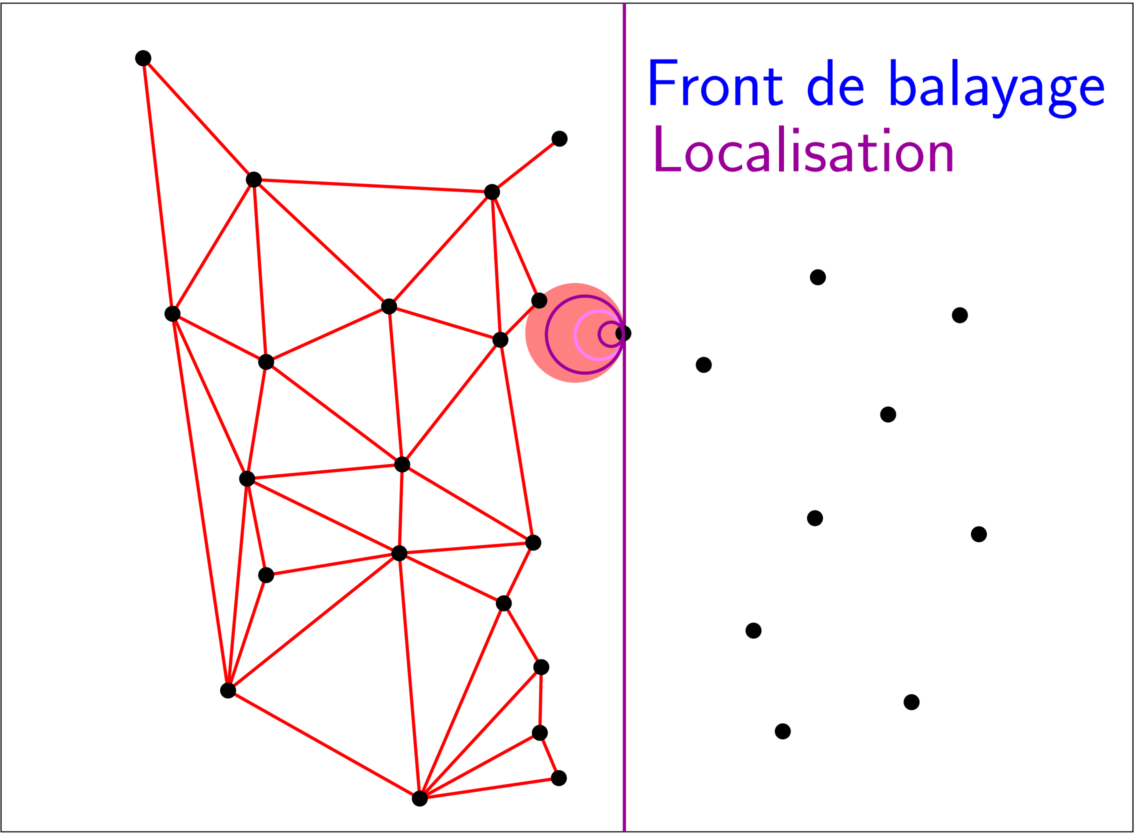
- 1 point
-1 cercle1 ?
+2 cercles ?

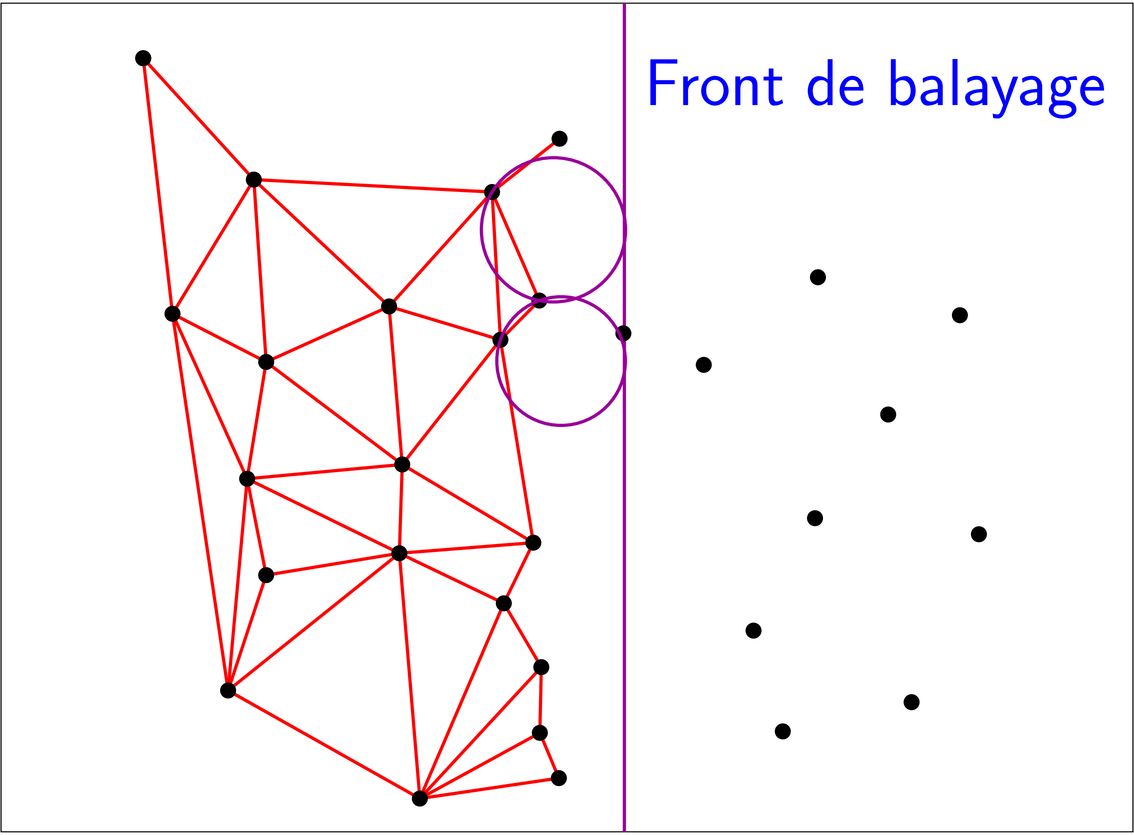
Front

+ 2 arêtes


Front de balayage


Front de balayage


Front de balayage
Localisation


Front de balayage

Algorithme de balayage

Complexité

Triangulation

$O(n)$ (taille)

Front

$O(n)$ (taille)

$O(\log n)$ insertion, suppression, requête

Évènements

$O(n)$ (taille)

$O(\log n)$ insertion, suppression, requête

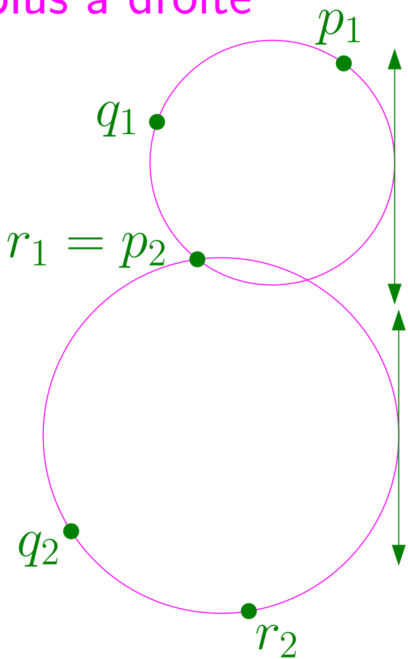
$$O(n \log n)$$

Prédicats

Prédicat : cercle le plus à droite

6 points

polynôme de degré 24

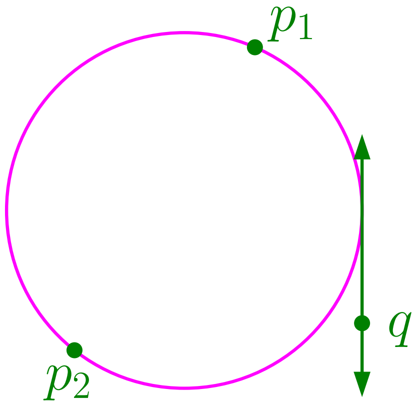


Prédicats

Prédicat : q dessus/dessous ?

3 points

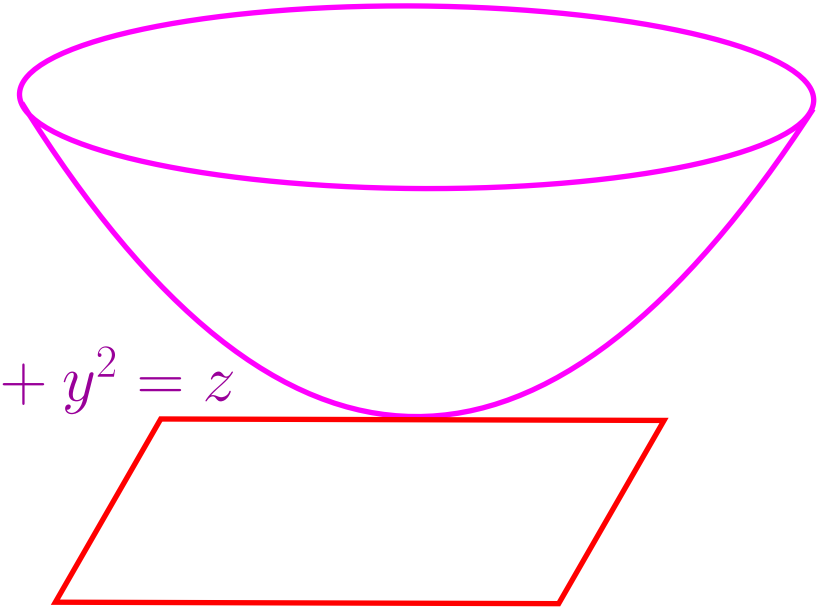
polynôme de degré ??



Algorithmes optimaux pour la triangulation de Delaunay

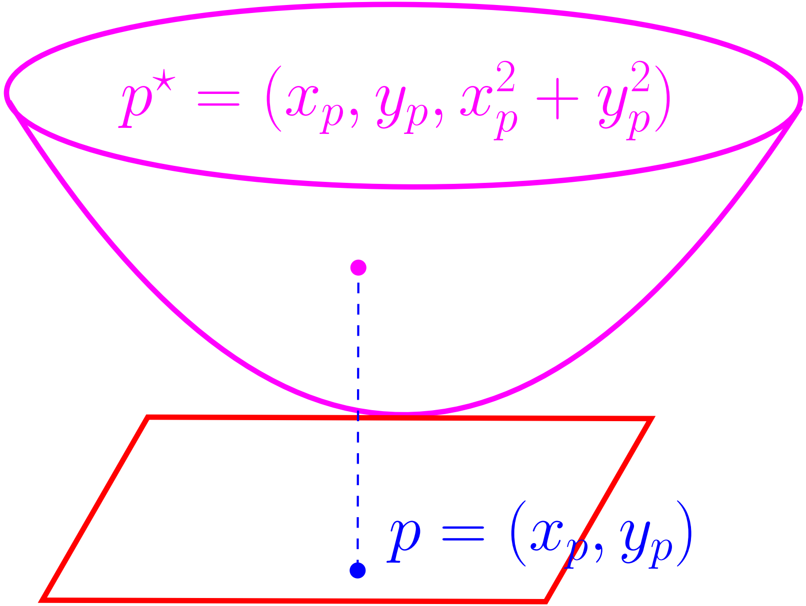
Dualité

Rappel

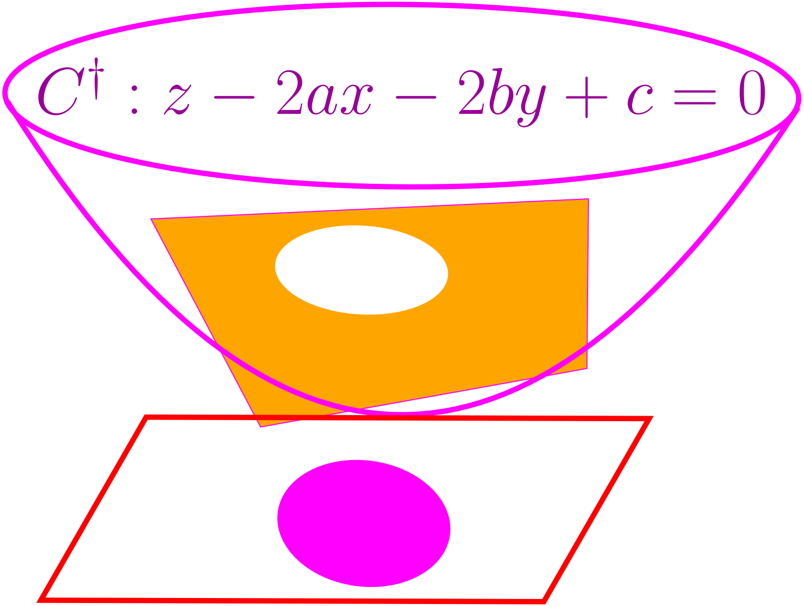


$$\Pi : x^2 + y^2 = z$$

Rappel

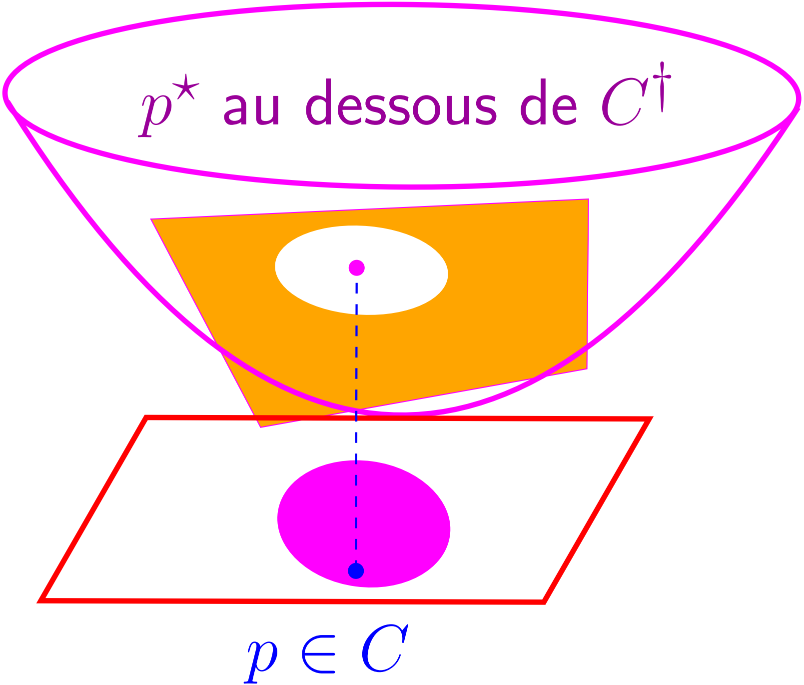


Rappel

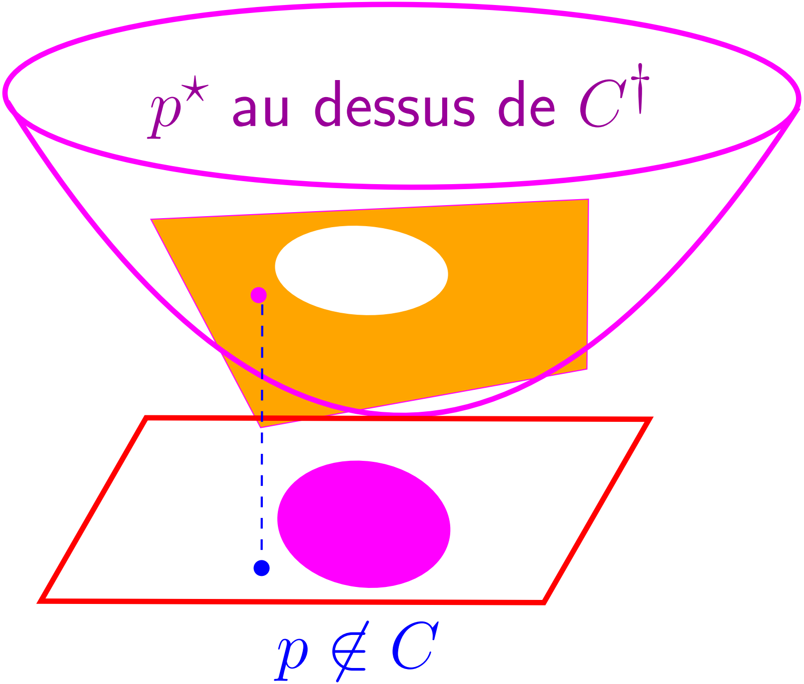


$$C : x^2 + y^2 - 2ax - 2by + c = 0$$

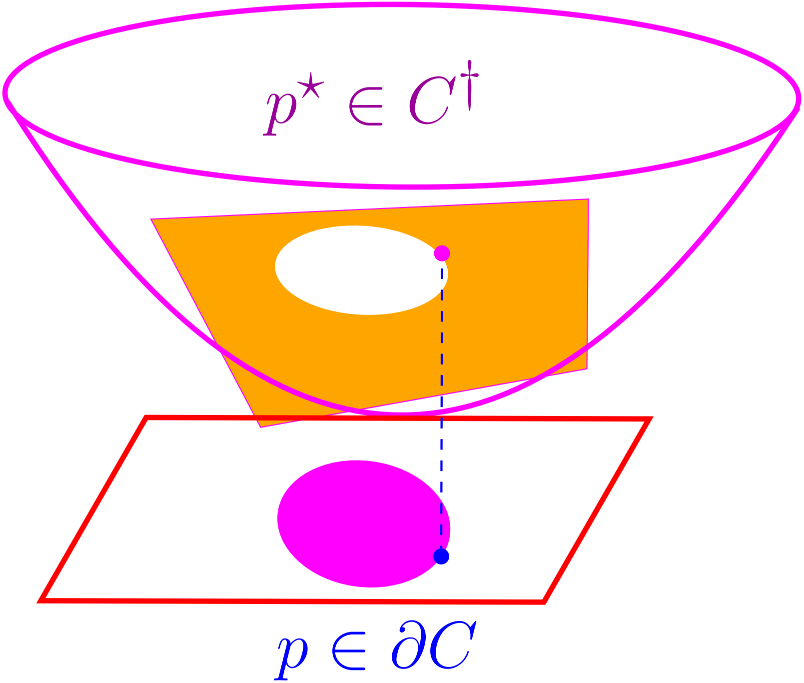
Rappel



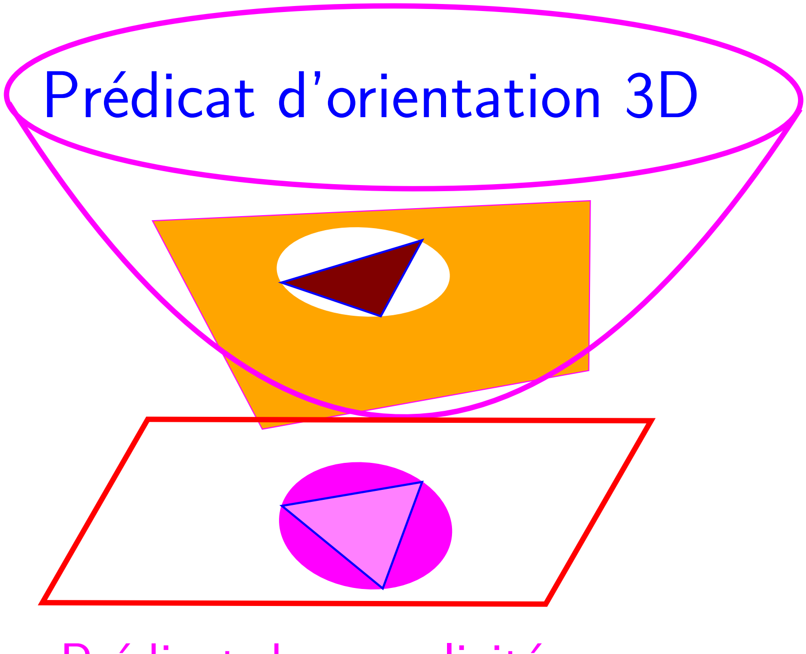
Rappel



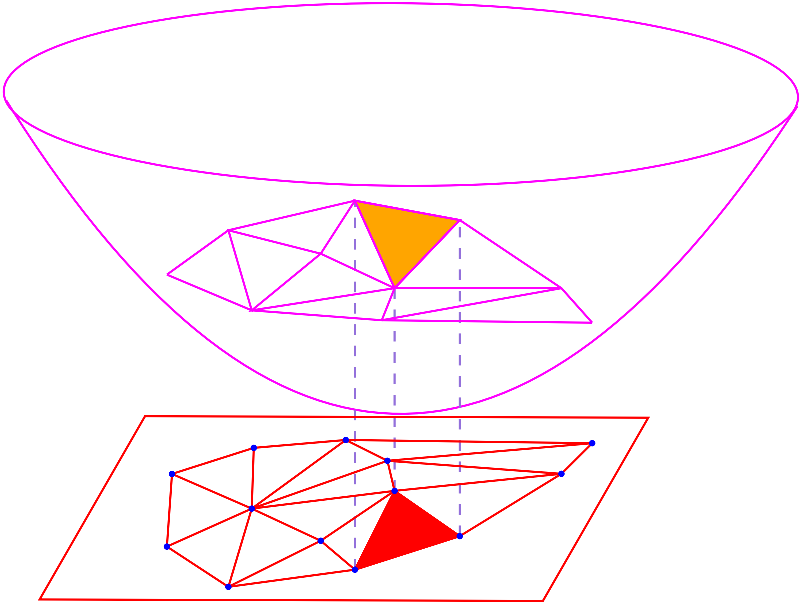
Rappel



Rappel

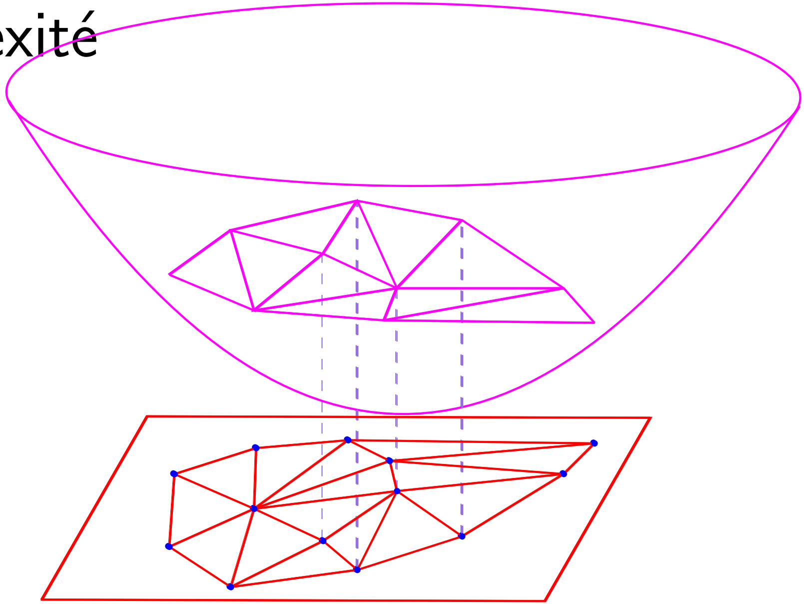


Delaunay et enveloppe convexe



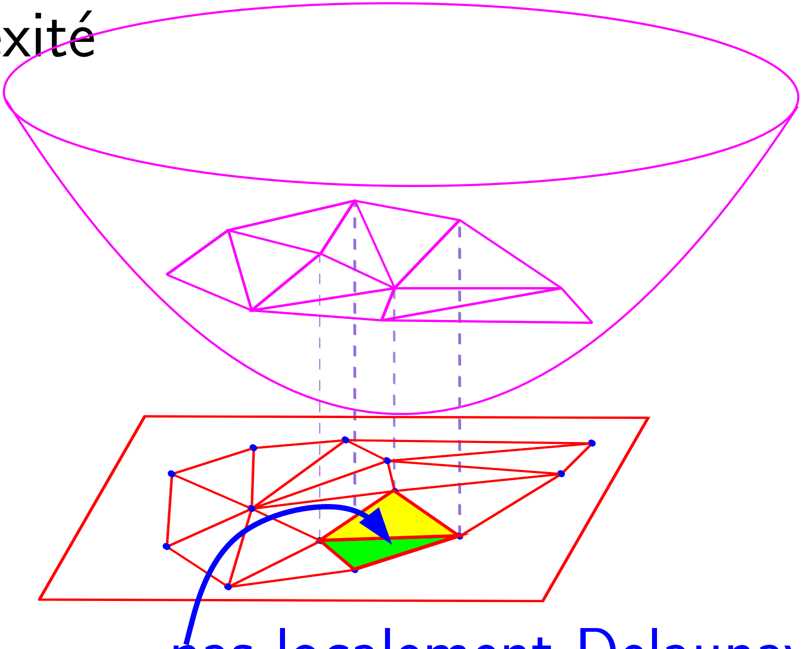
Retour sur la bascule de diagonales

Complexité



Retour sur la bascule de diagonales

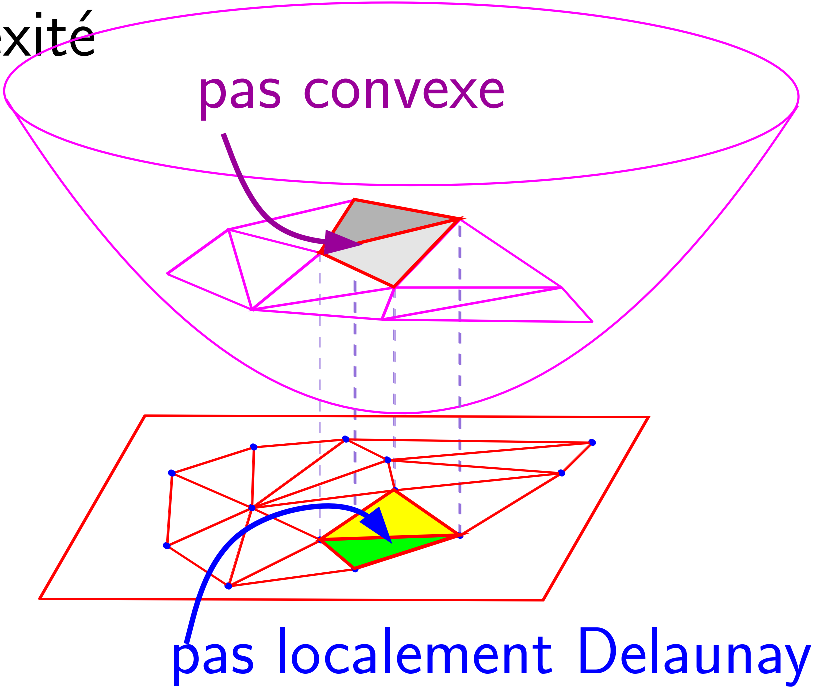
Complexité



pas localement Delaunay

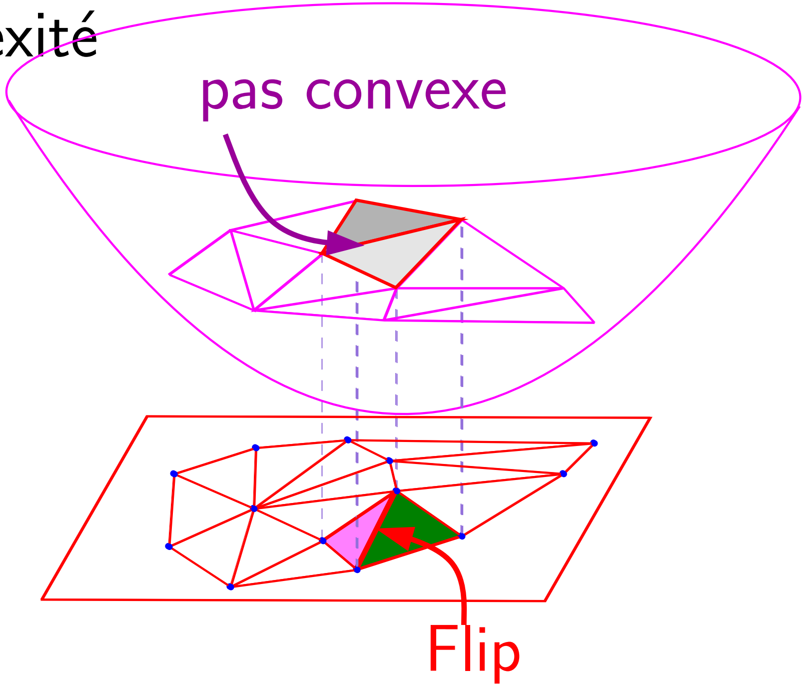
Retour sur la bascule de diagonales

Complexité



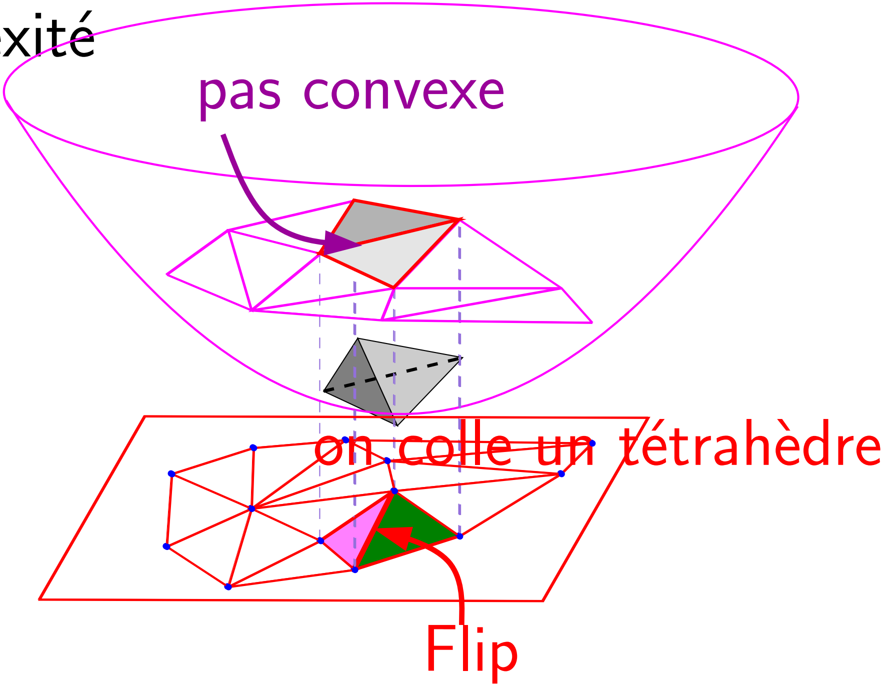
Retour sur la bascule de diagonales

Complexité



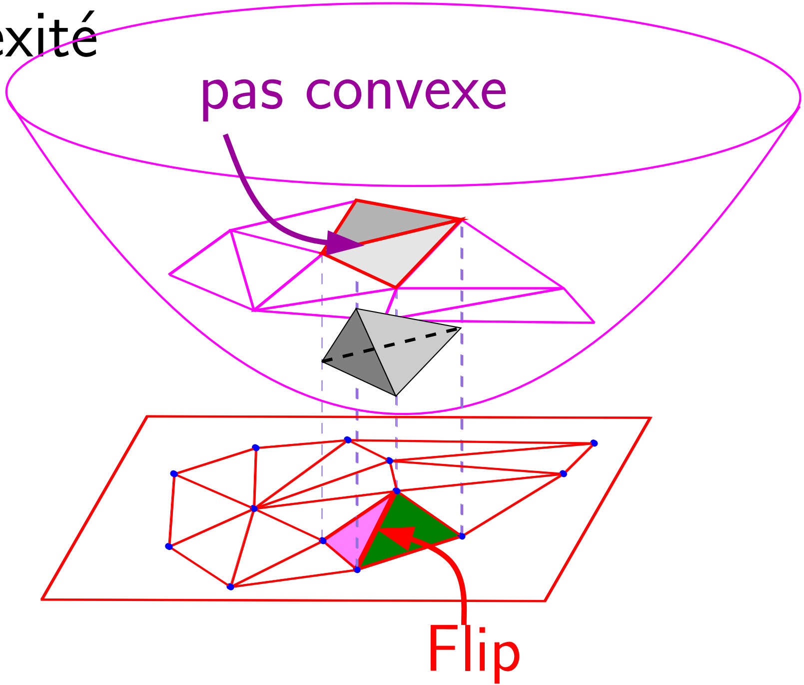
Retour sur la bascule de diagonales

Complexité



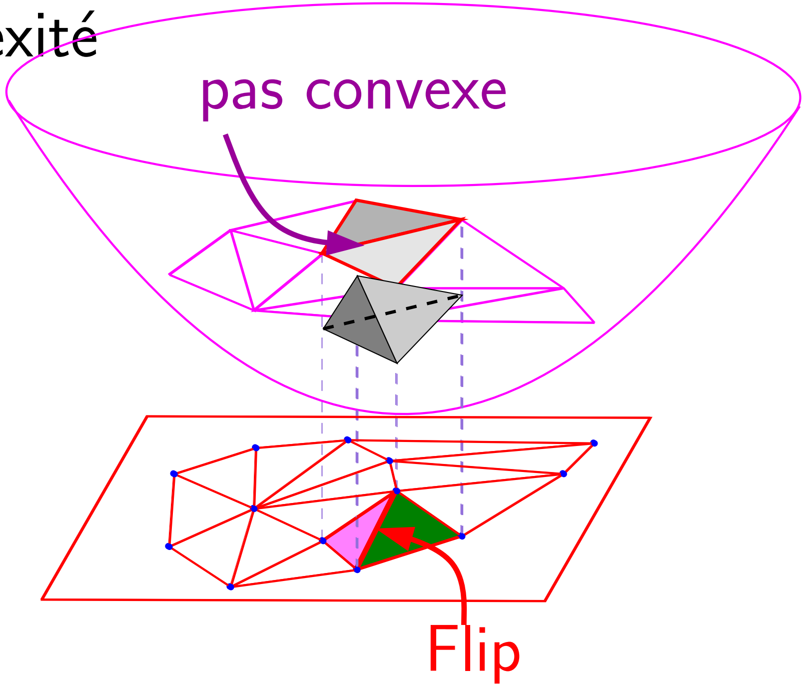
Retour sur la bascule de diagonales

Complexité



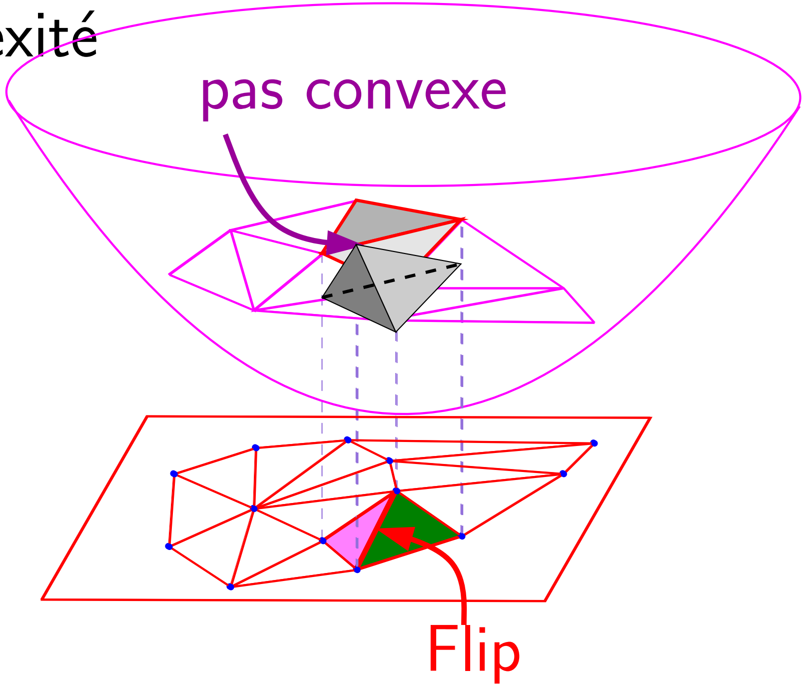
Retour sur la bascule de diagonales

Complexité



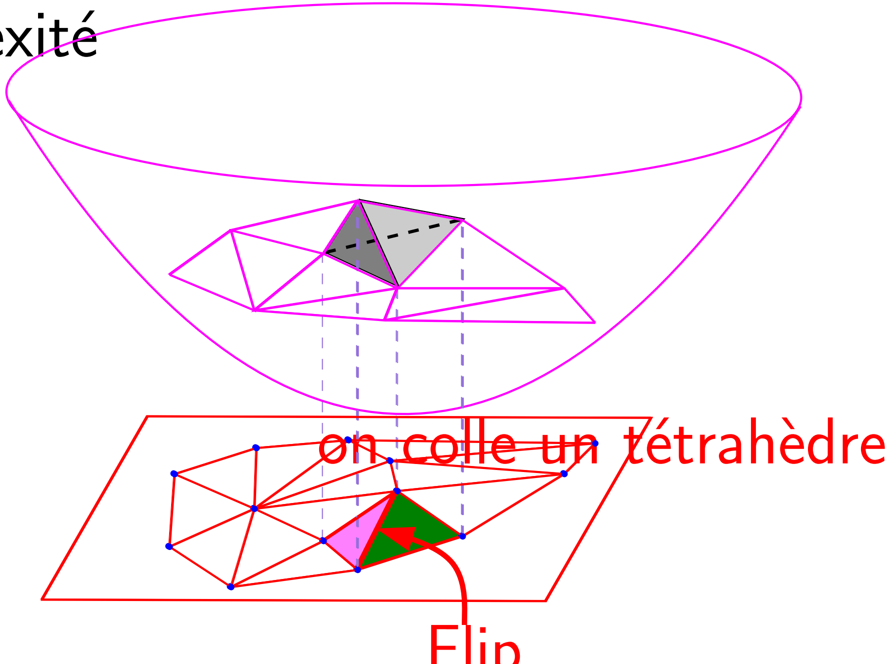
Retour sur la bascule de diagonales

Complexité



Retour sur la bascule de diagonales

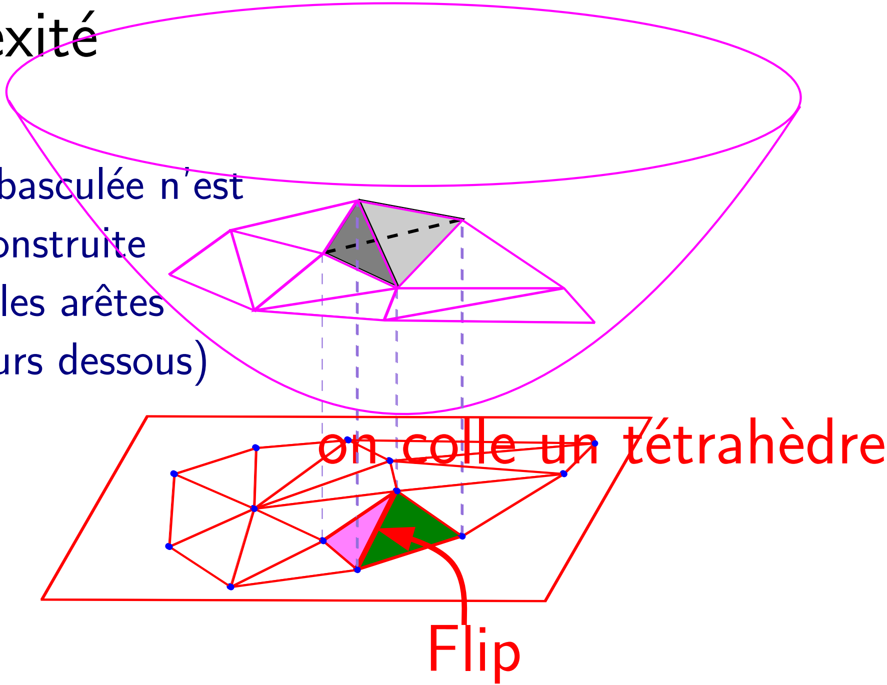
Complexité



Retour sur la bascule de diagonales

Complexité

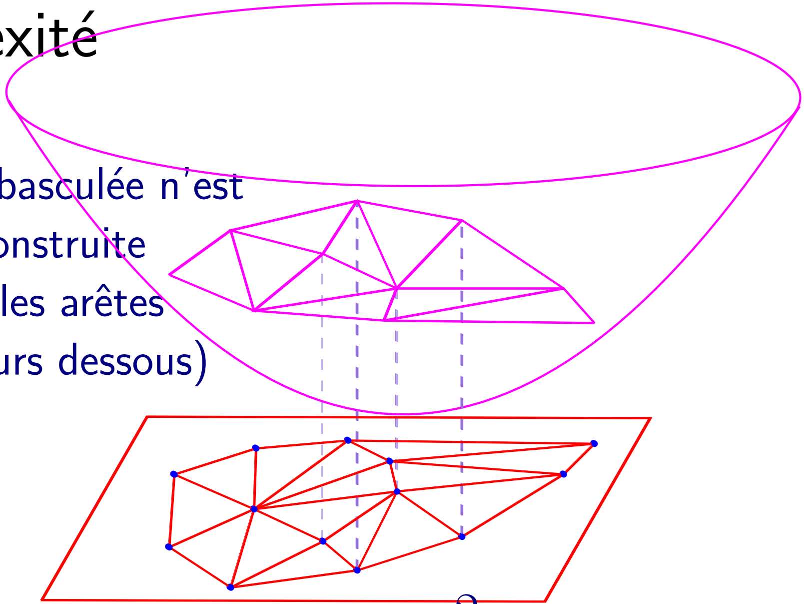
Une arête basculée n'est jamais reconstruite (les nouvelles arêtes sont toujours dessous)



Retour sur la bascule de diagonales

Complexité

Une arête basculée n'est
jamais reconstruite
(les nouvelles arêtes
sont toujours dessous)



Algorithme en $O(n^2)$

Algorithmes optimaux pour

la triangulation de Delaunay

Division-Fusion

Division-Fusion

Technique classique

exemple: tri

Problème de tade n

→ 2 sous-problèmes de taille $\frac{n}{2}$

résolution récursive

fusion

Division-Fusion

Problème de table de taille n $\quad f(n)$

$O(n)$ → 2 sous-problèmes de taille $\frac{n}{2}$

résolution récursive $\quad 2 \cdot f(\frac{n}{2})$

fusion $\quad O(n)$

Division-Fusion

$$f(n) = O(n) + 2f(\tfrac{n}{2})$$

Problème de taille n $f(n)$

$O(n)$ → 2 sous-problèmes de taille $\frac{n}{2}$

résolution récursive $2 \cdot f(\frac{n}{2})$

fusion $O(n)$

Division-Fusion

$$f(n) = O(n) + 2f\left(\frac{n}{2}\right)$$

$$f(n) =$$

$$n + 2f\left(\frac{n}{2}\right)$$

$$n + 2\left(\frac{n}{2} + 2f\left(\frac{n}{4}\right)\right)$$

$$n + 2\left(\frac{n}{2} + 2\left(\frac{n}{4} + 2f\left(\frac{n}{8}\right)\right)\right)$$

$$\underbrace{n + 2\frac{n}{2} + 2.2\frac{n}{4} + \dots}_{\log_2 n}$$

$$f(n) = O(n \log n)$$

Division

Fusion facile !

Partition équilibrée

$O(n)$

Division

Fusion facile !

Partition par une droite

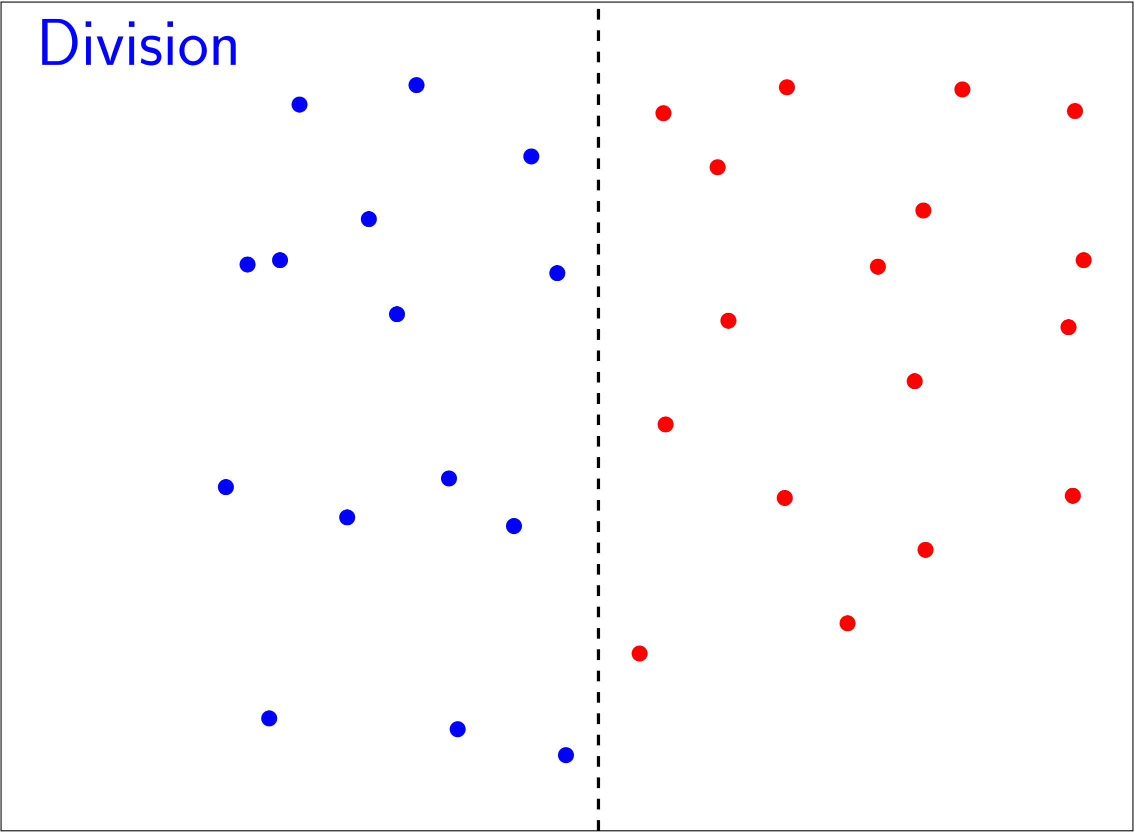
Partition équilibrée

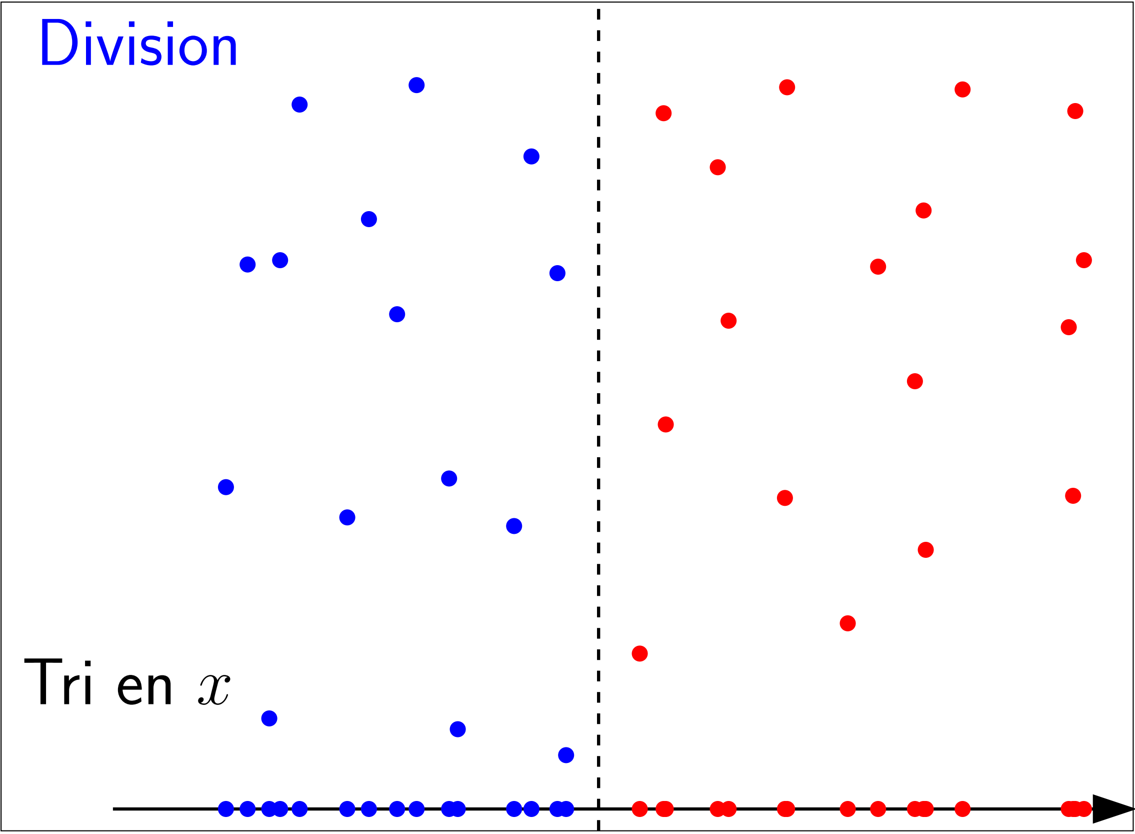
Droite médiane

$O(n)$

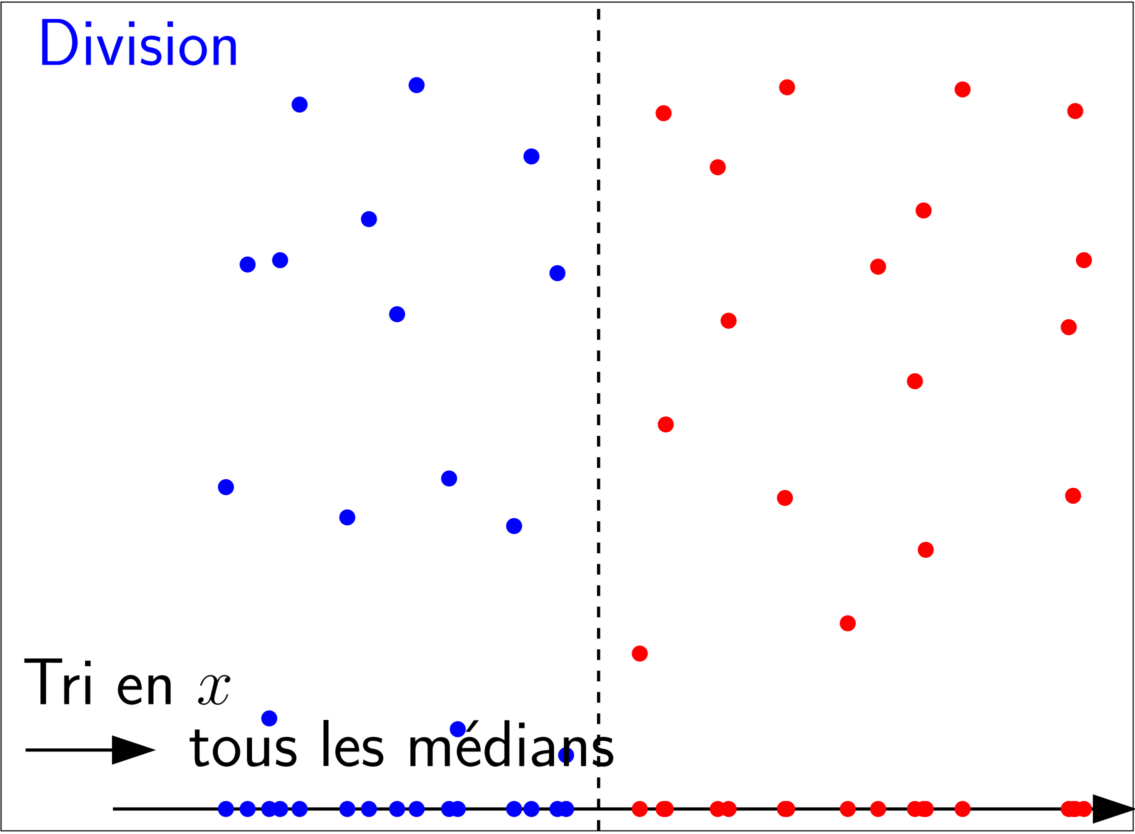
Médian linéaire ?

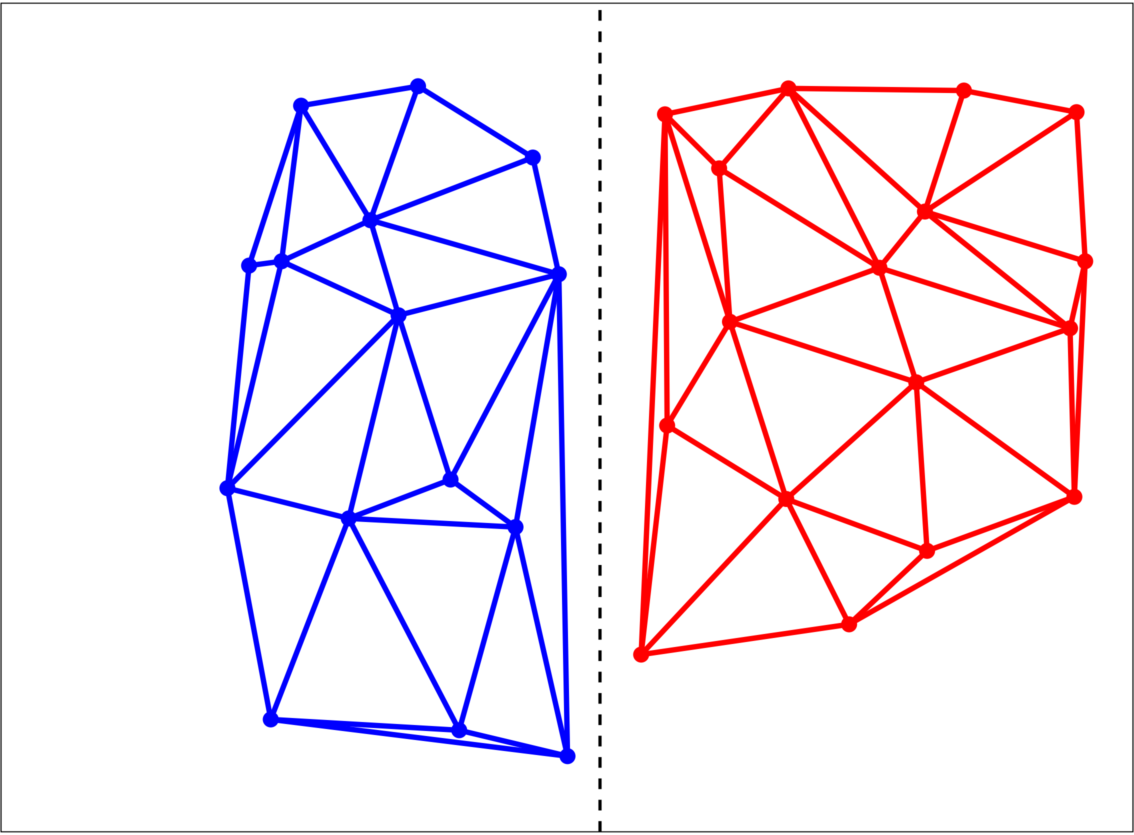
Prétraitement

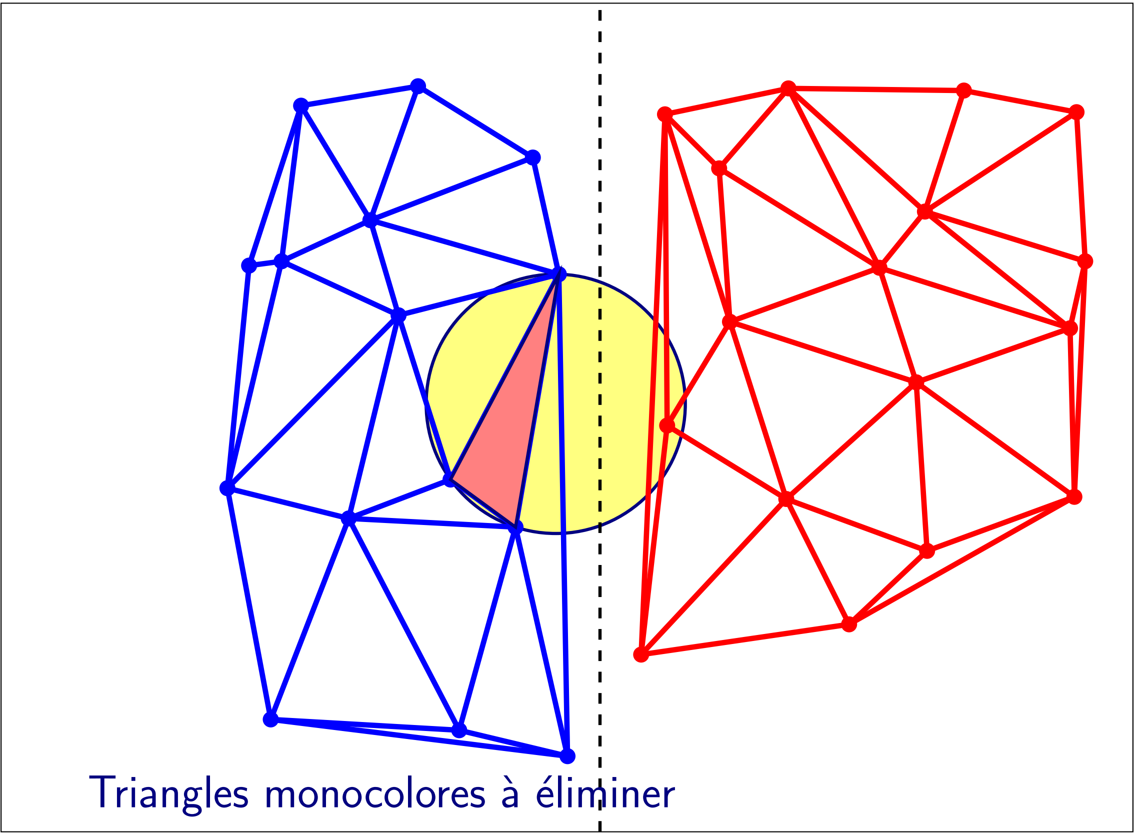

Division

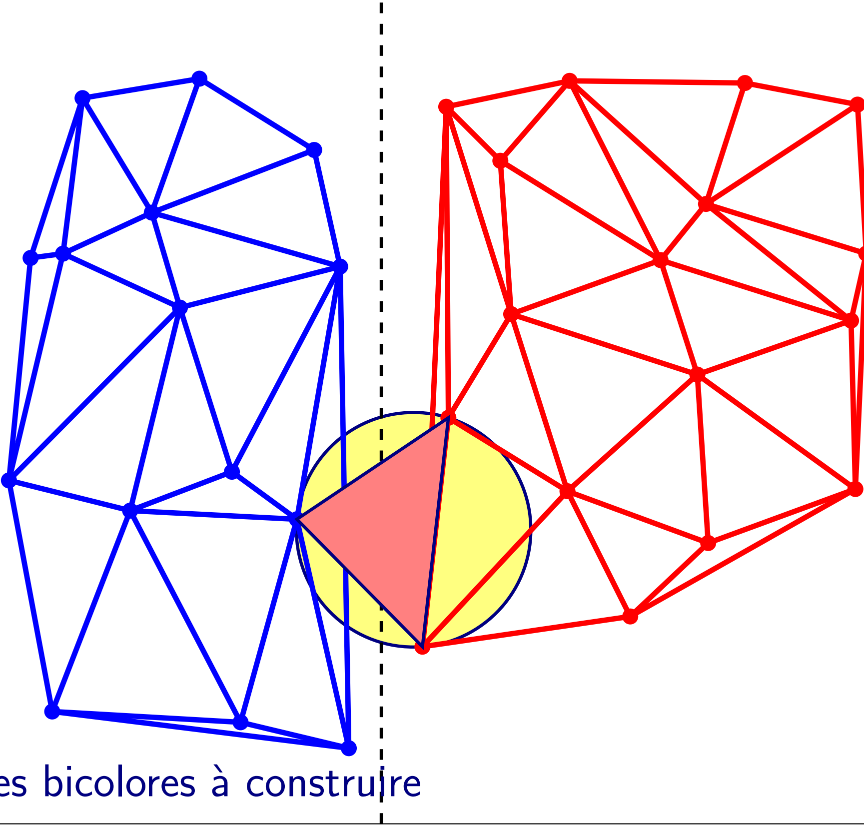


Division
Tri en x

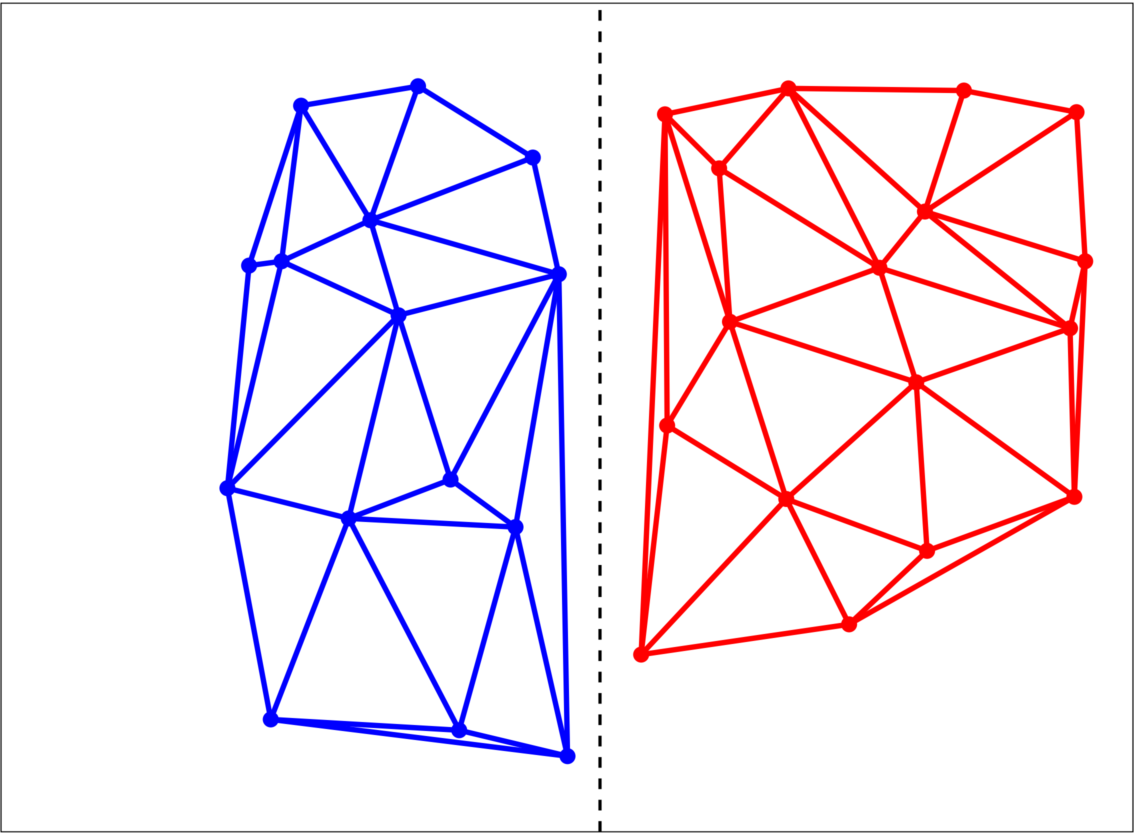

Division
Tri en x
tous les médians




Triangles monocolores à éliminer

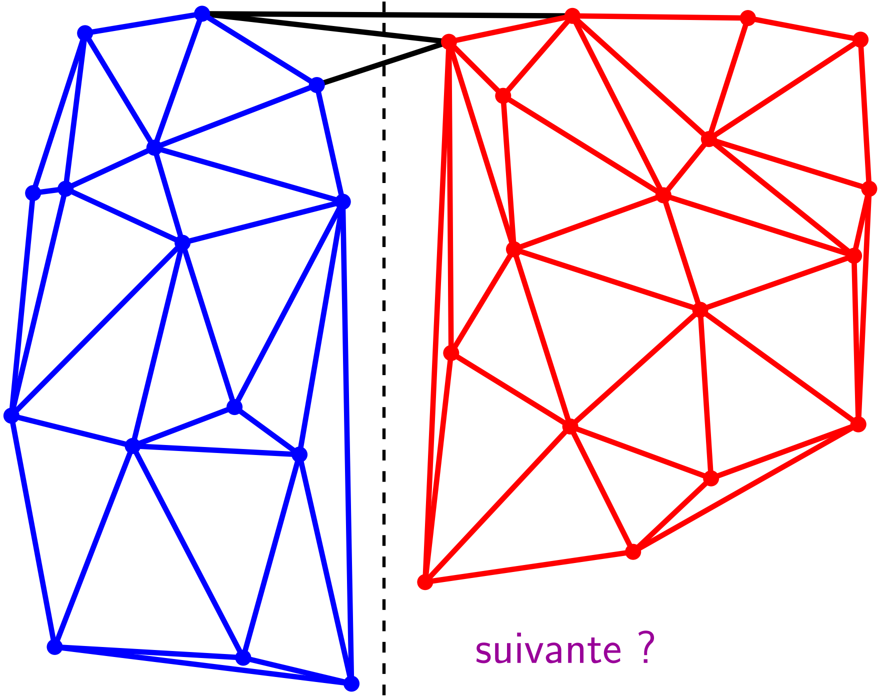


Triangles bicolores à construire



Construction des arêtes bicolores

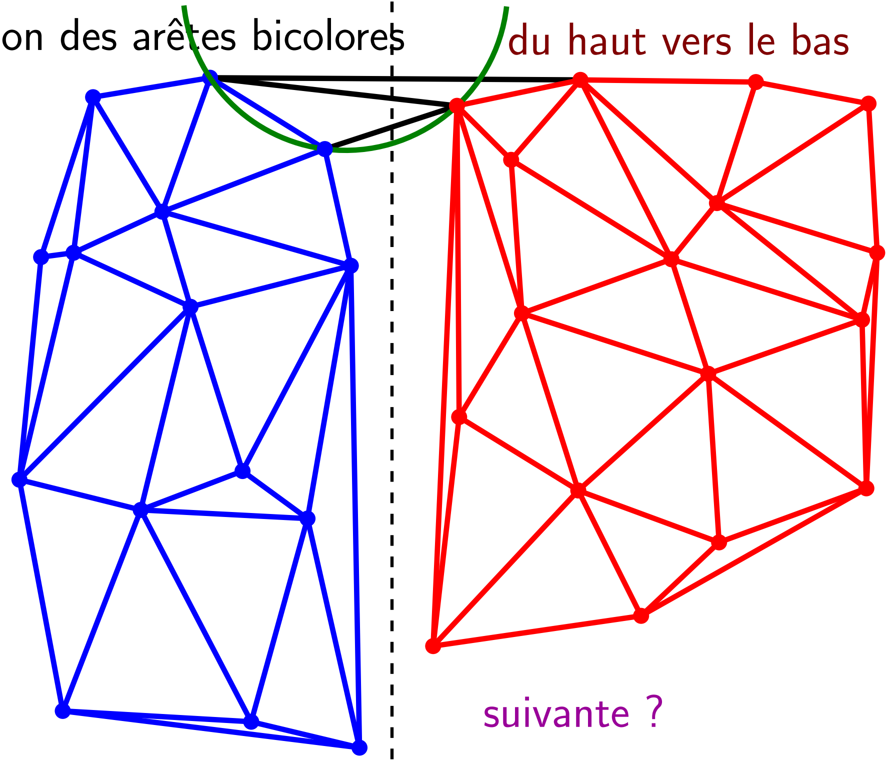
du haut vers le bas



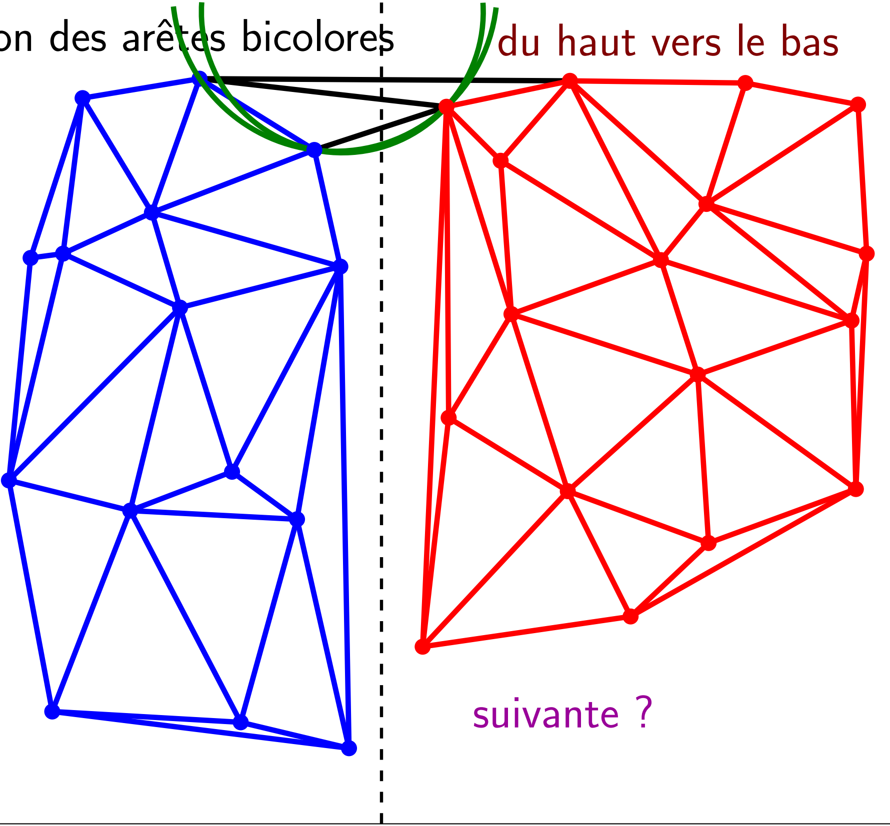
suivante ?

Construction des arêtes bicolores

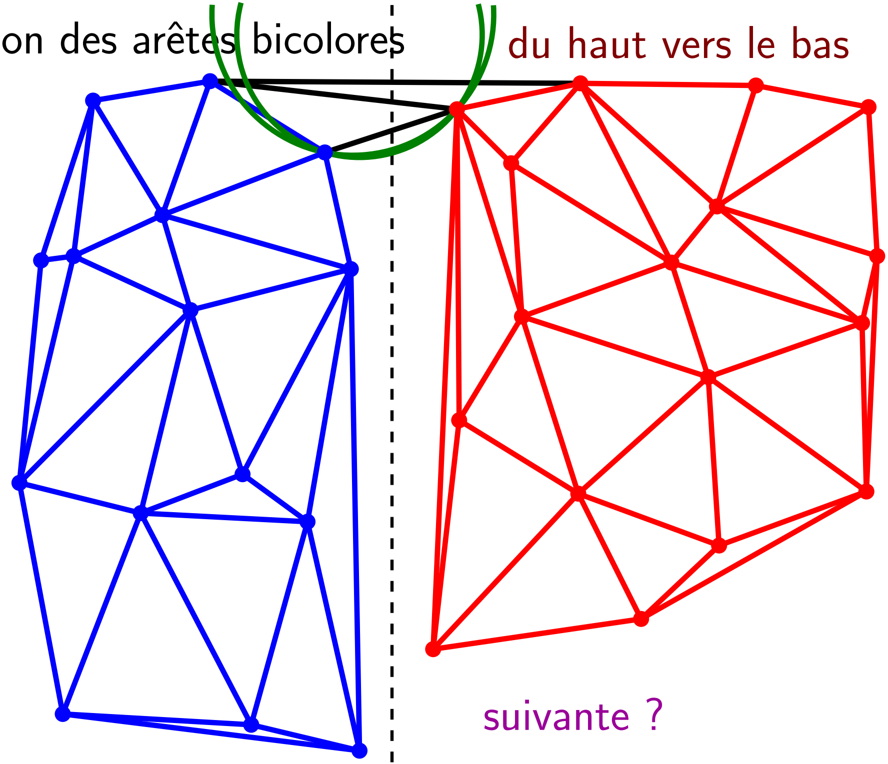
du haut vers le bas



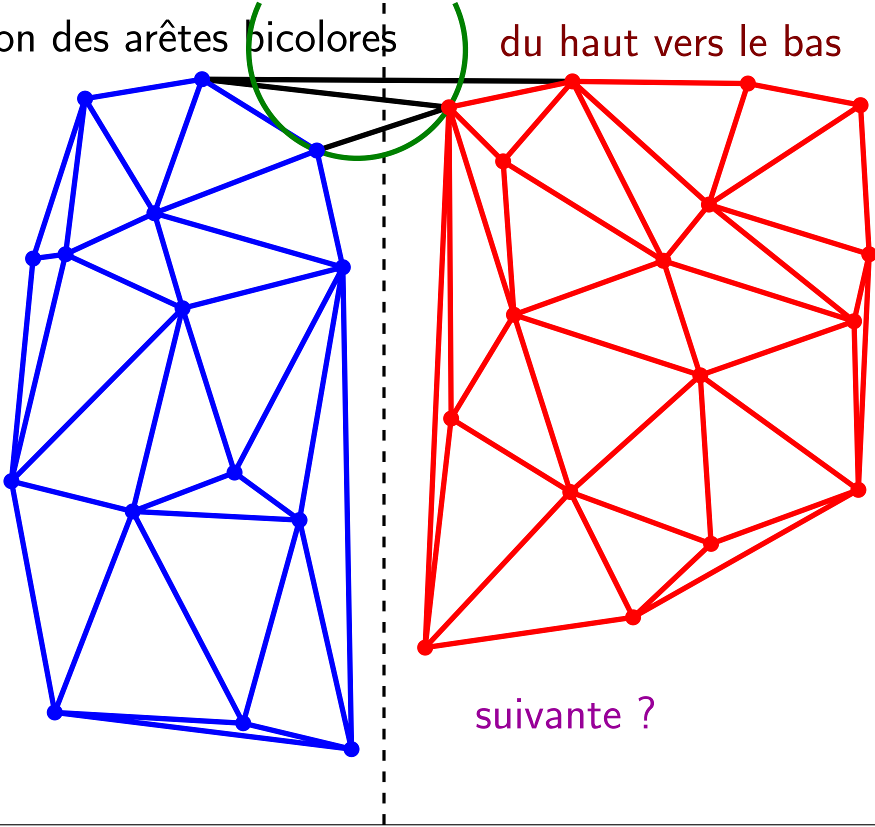
Construction des arêtes bicolores



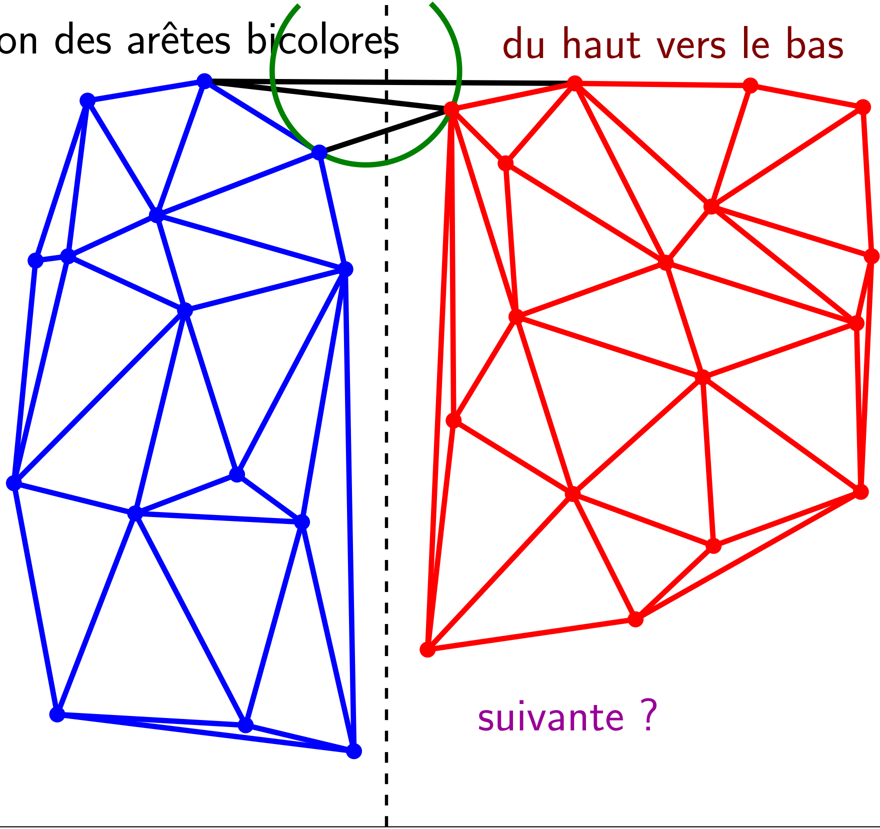
Construction des arêtes bicolores



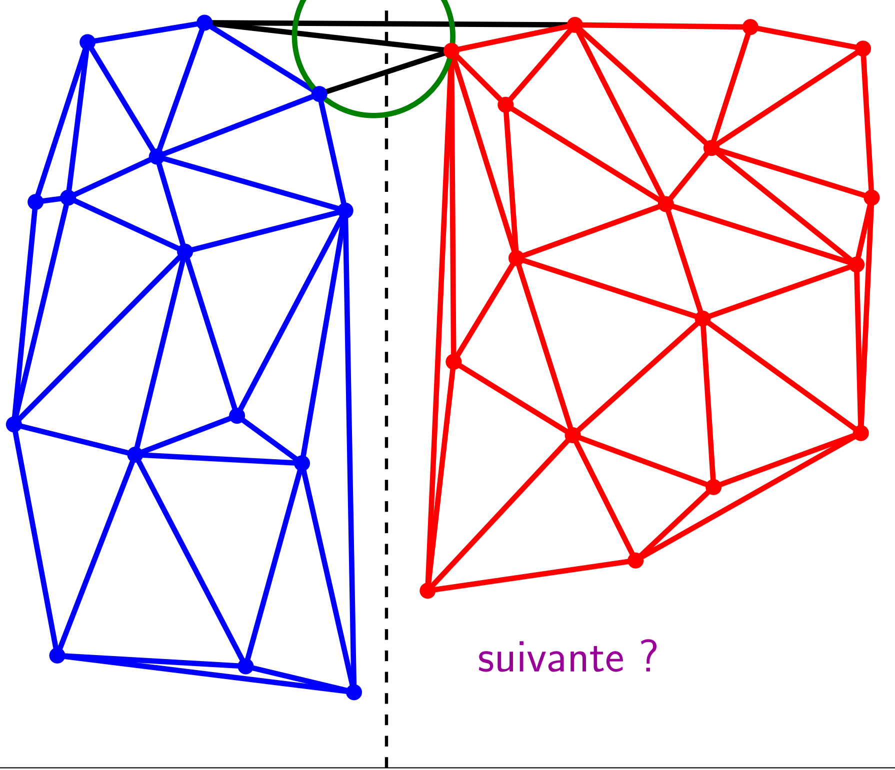
Construction des arêtes bicolores du haut vers le bas



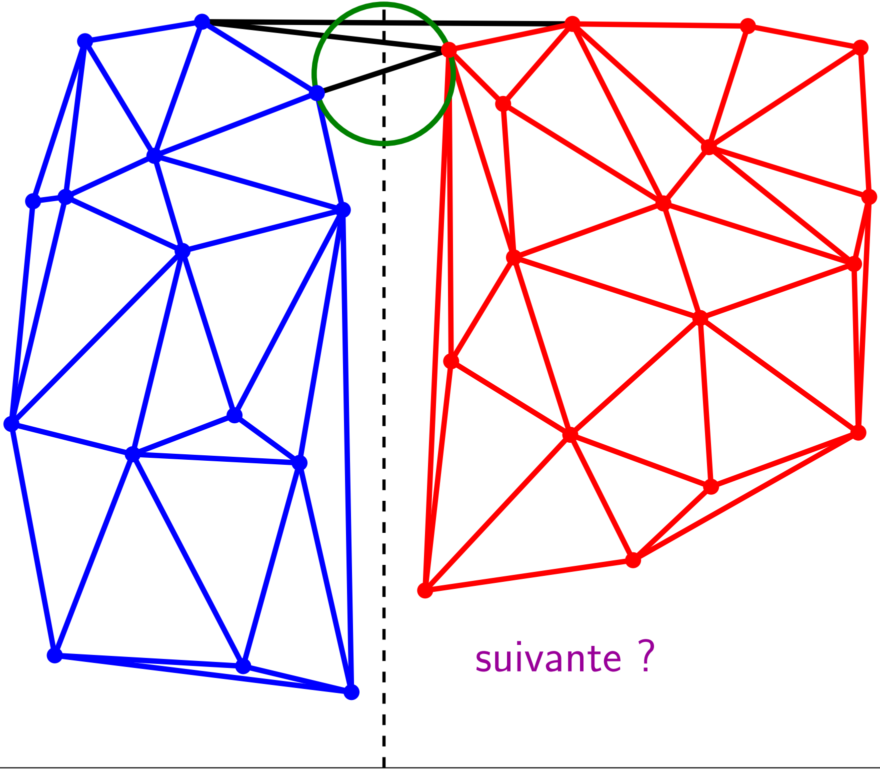
Construction des arêtes bicolores



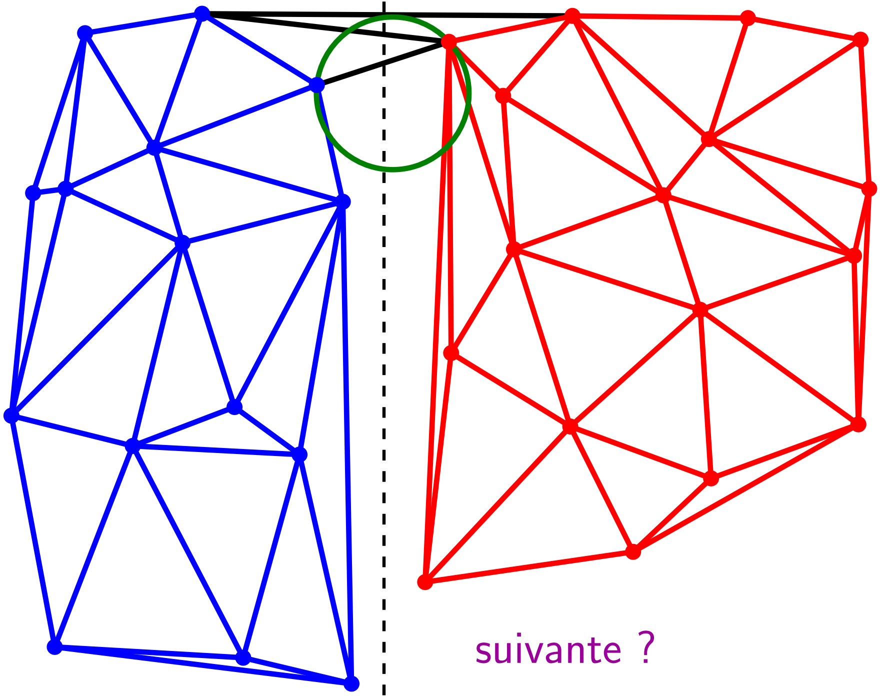
Construction des arêtes bicolores



Construction des arêtes bicolores du haut vers le bas

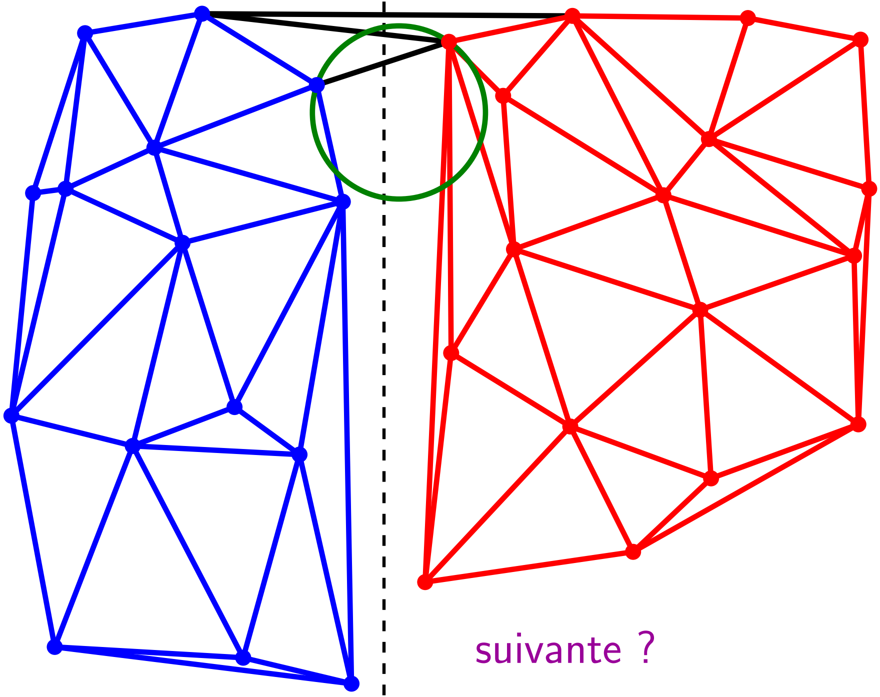


Construction des arêtes bicolores du haut vers le bas

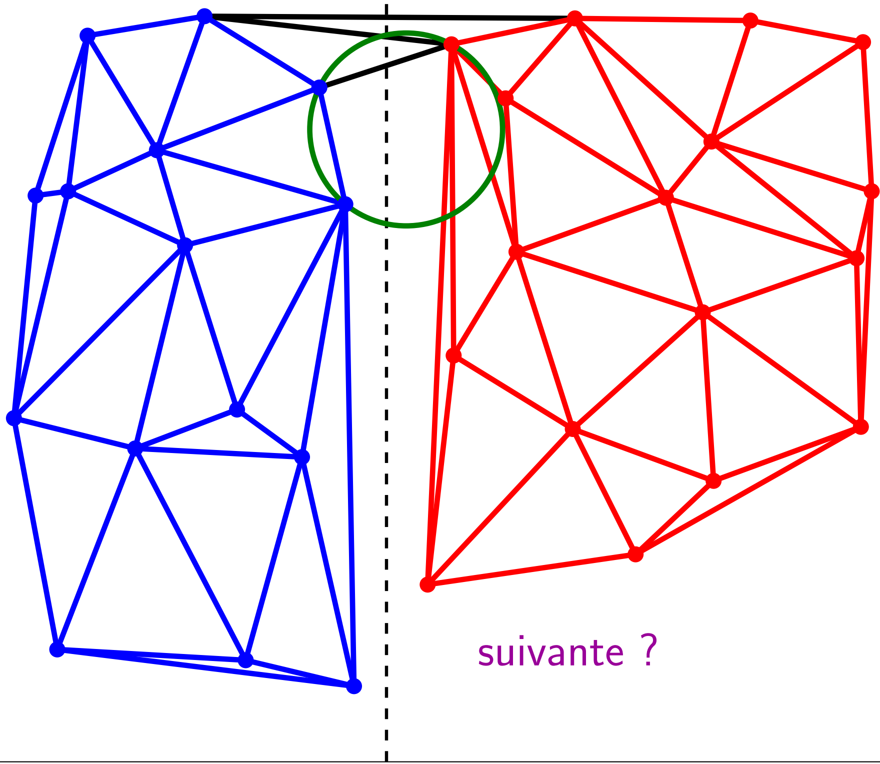


Construction des arêtes bicolores

du haut vers le bas

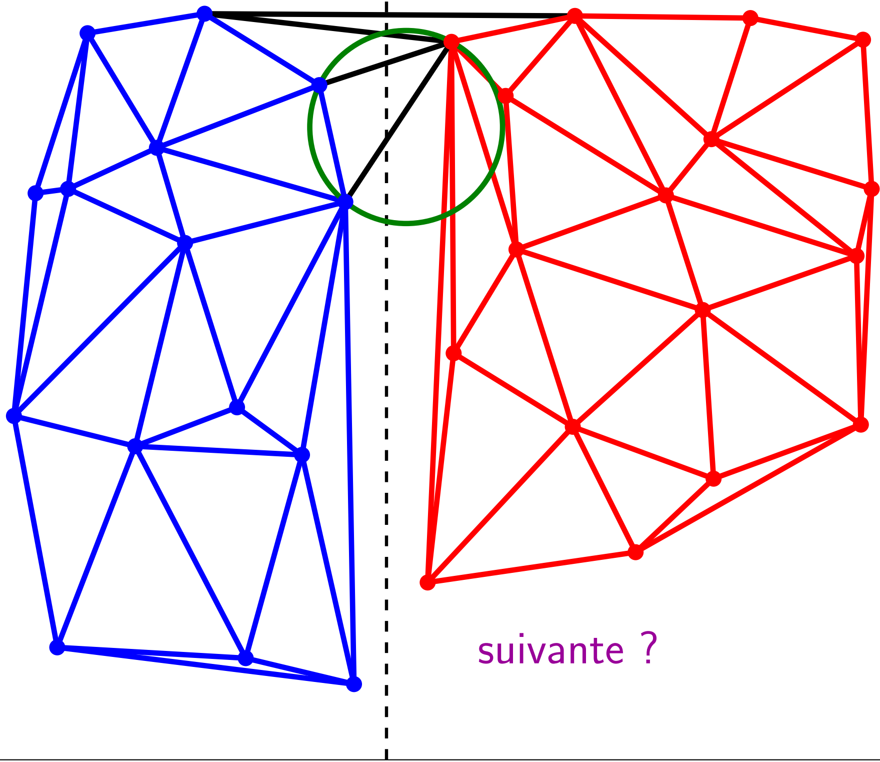


Construction des arêtes bicolores

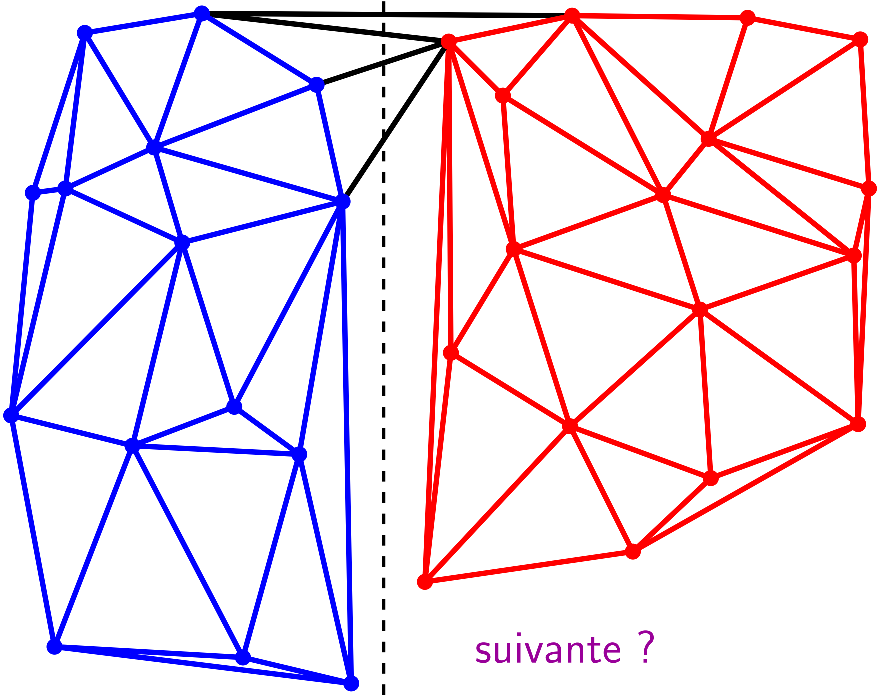


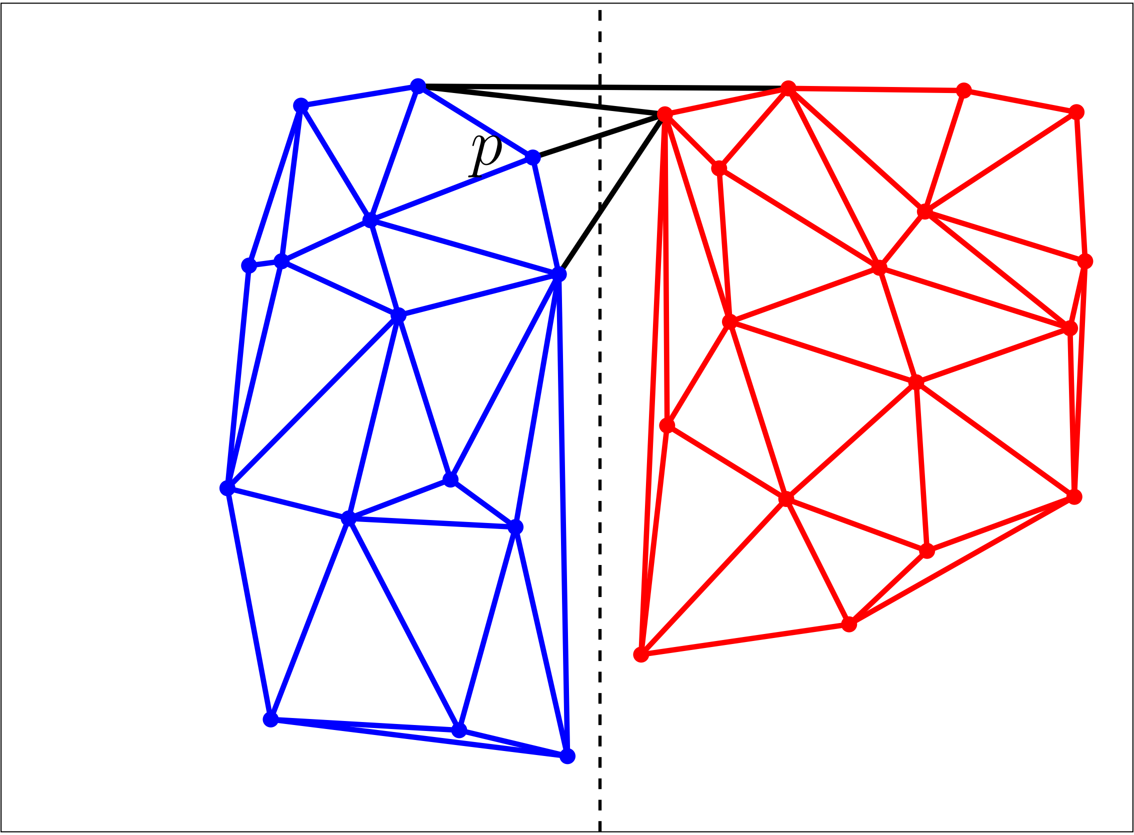
Construction des arêtes bicolores

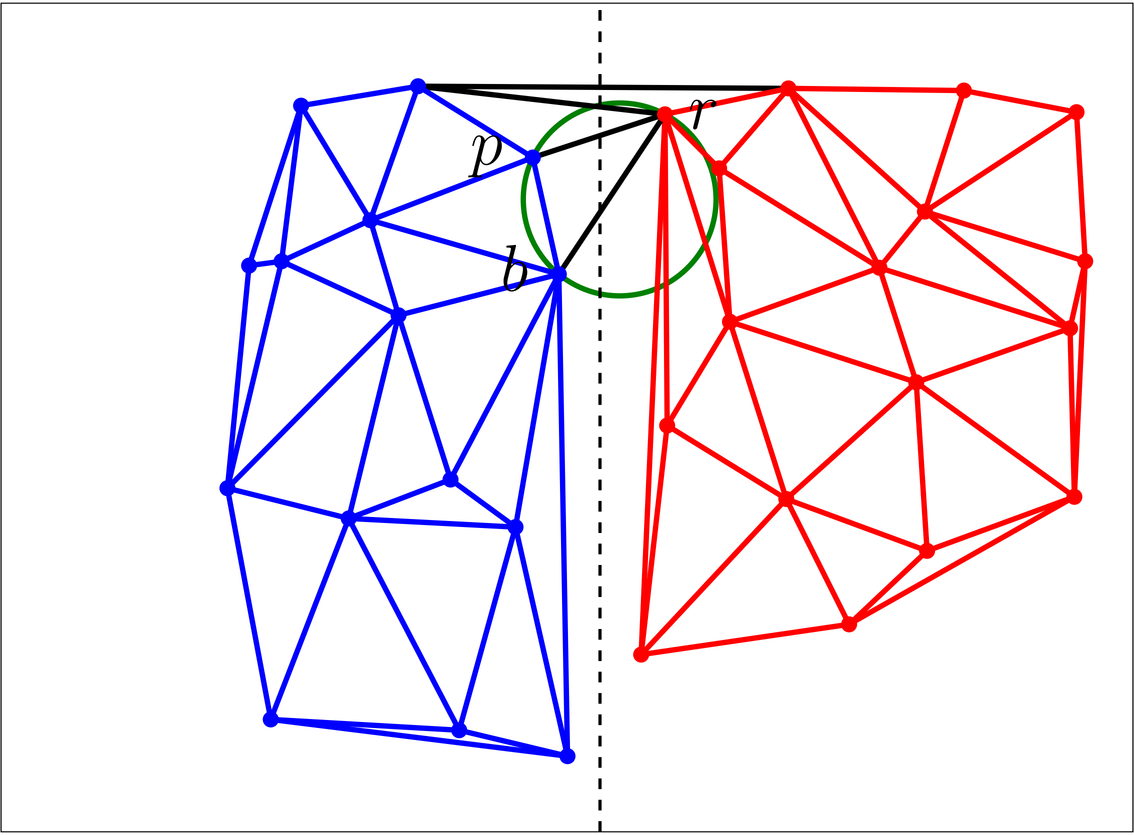
du haut vers le bas

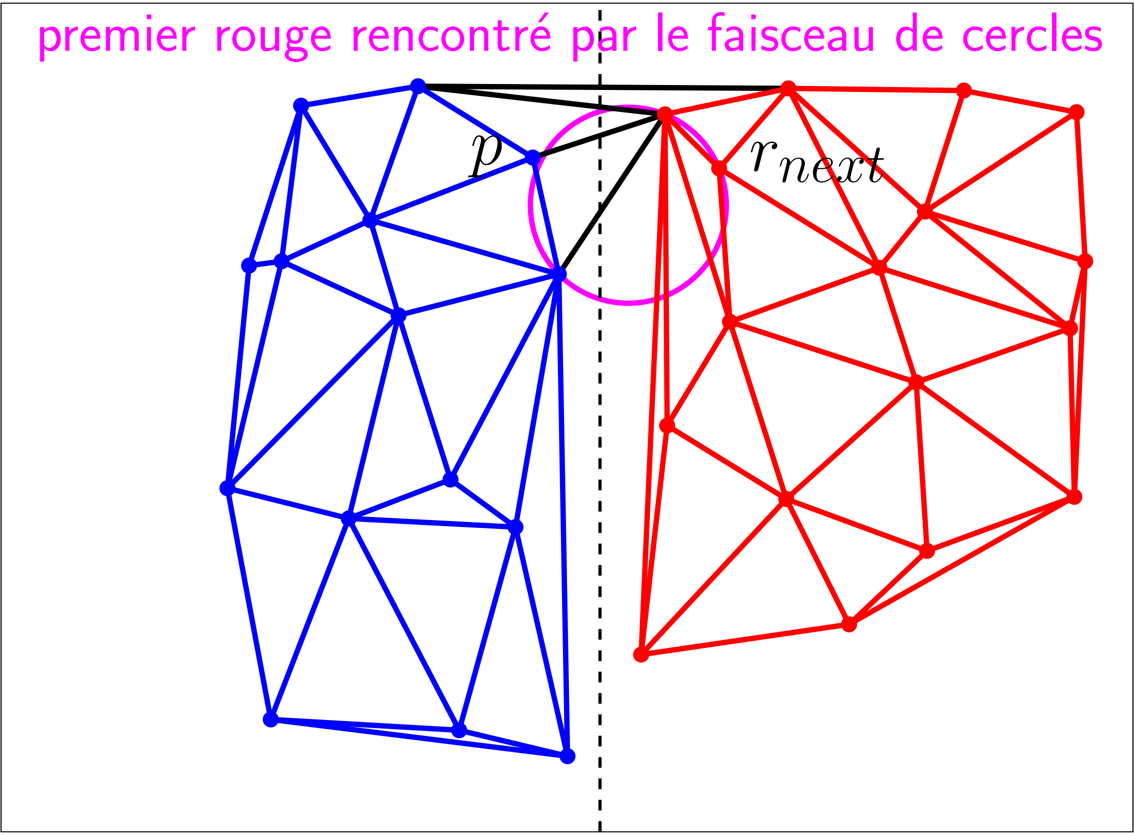


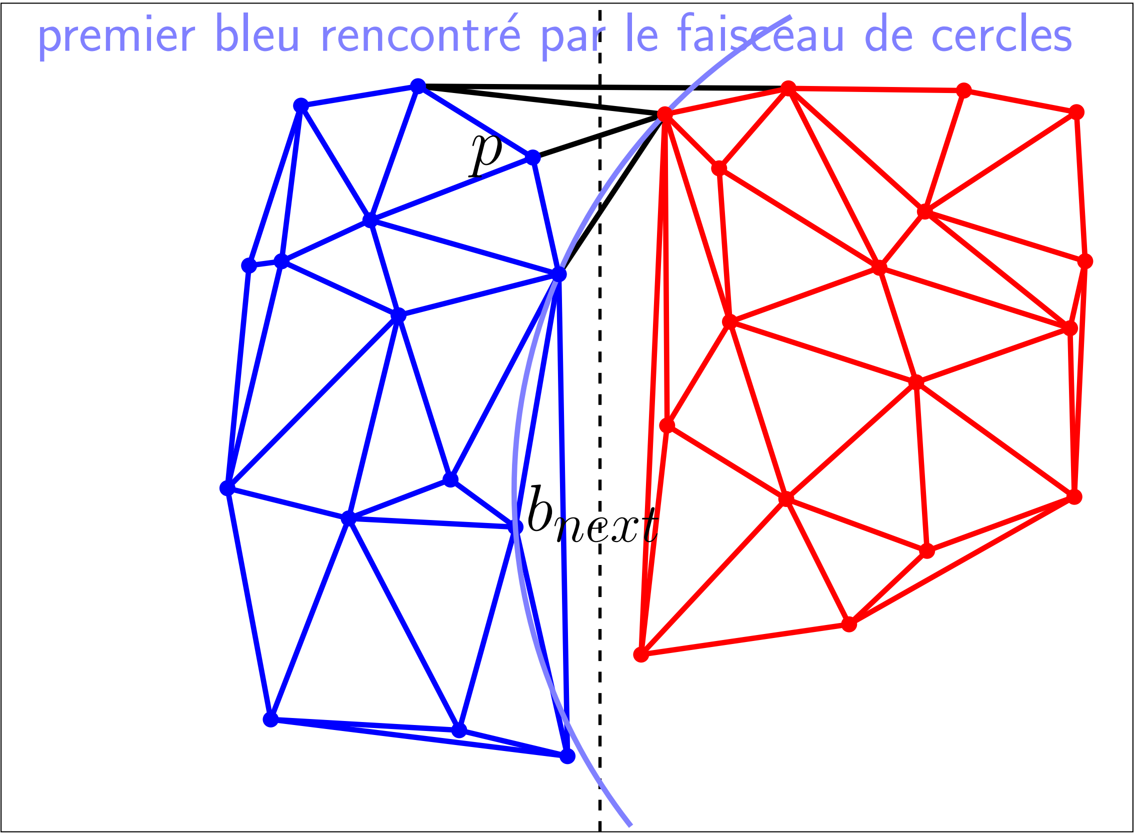
Construction des arêtes bicolores du haut vers le bas



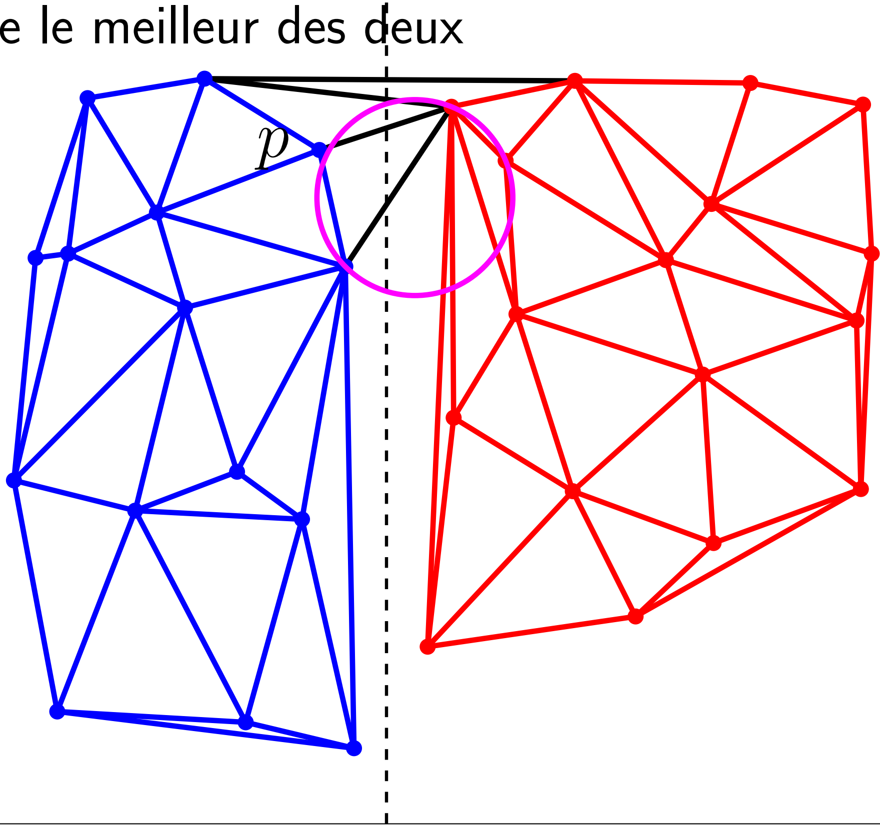

p


p
b
r

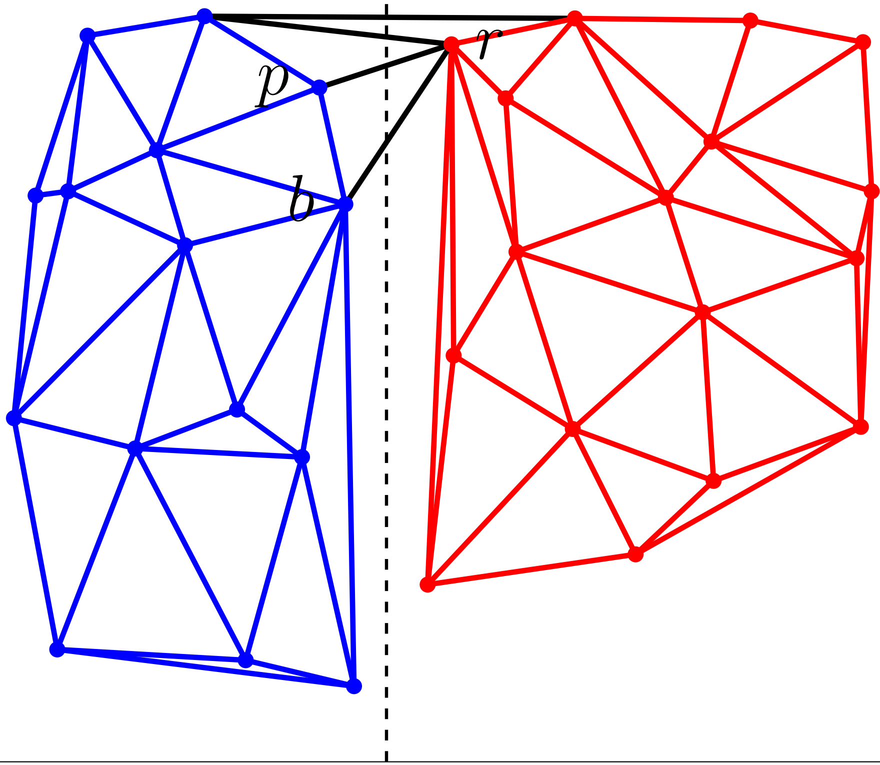

premier rouge rencontré par le faisceau de cercles
p
r_{next}

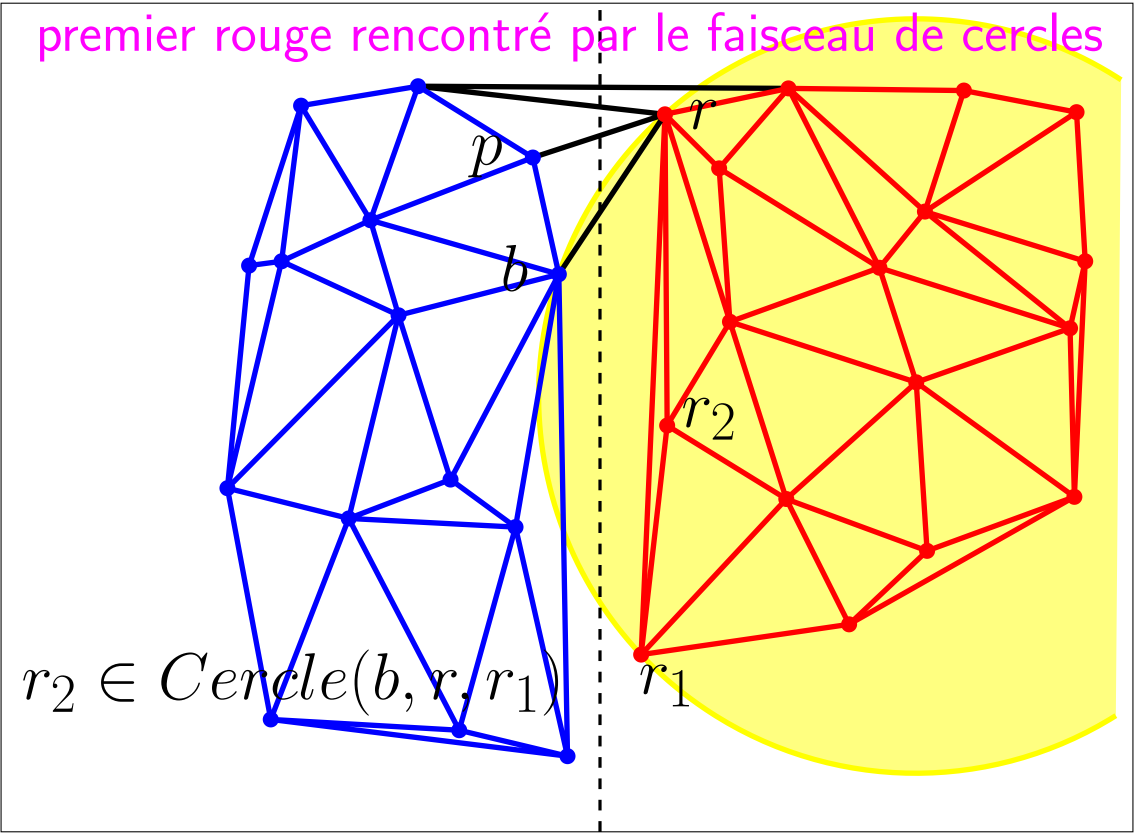

premier bleu rencontré par le faisceau de cercles
p
b_{next}

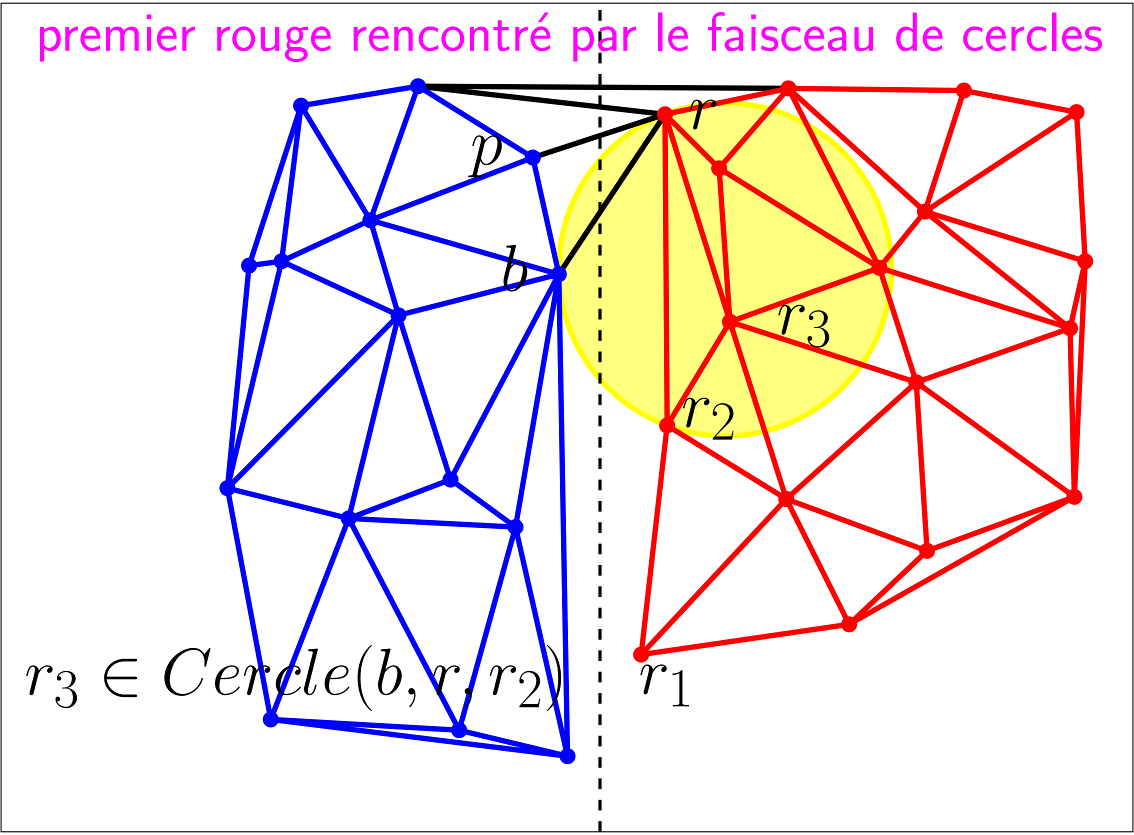
On garde le meilleur des deux



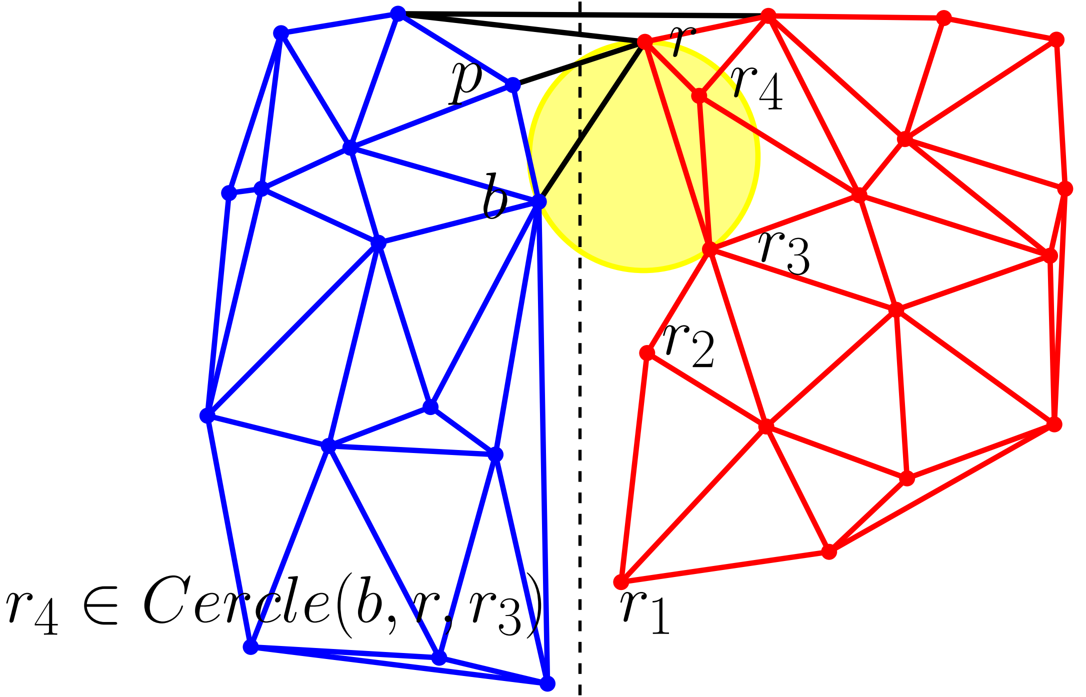
premier rouge rencontré par le faisceau de cercles

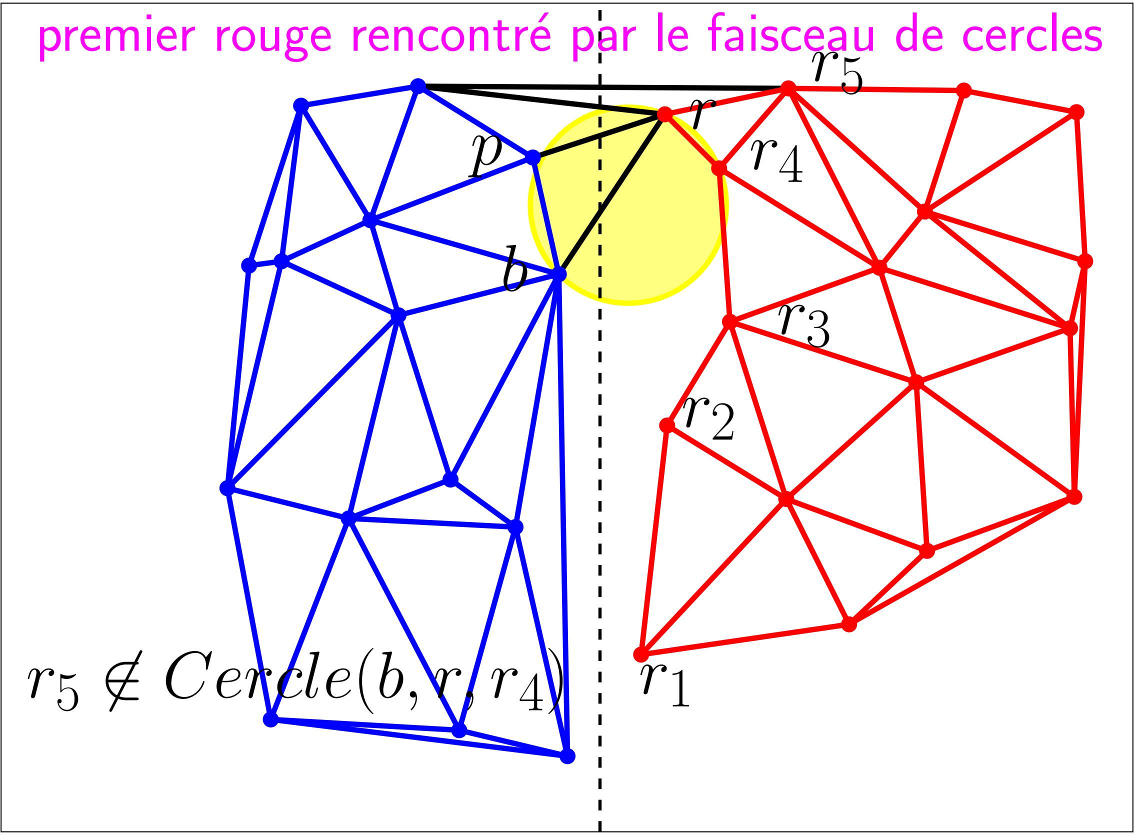


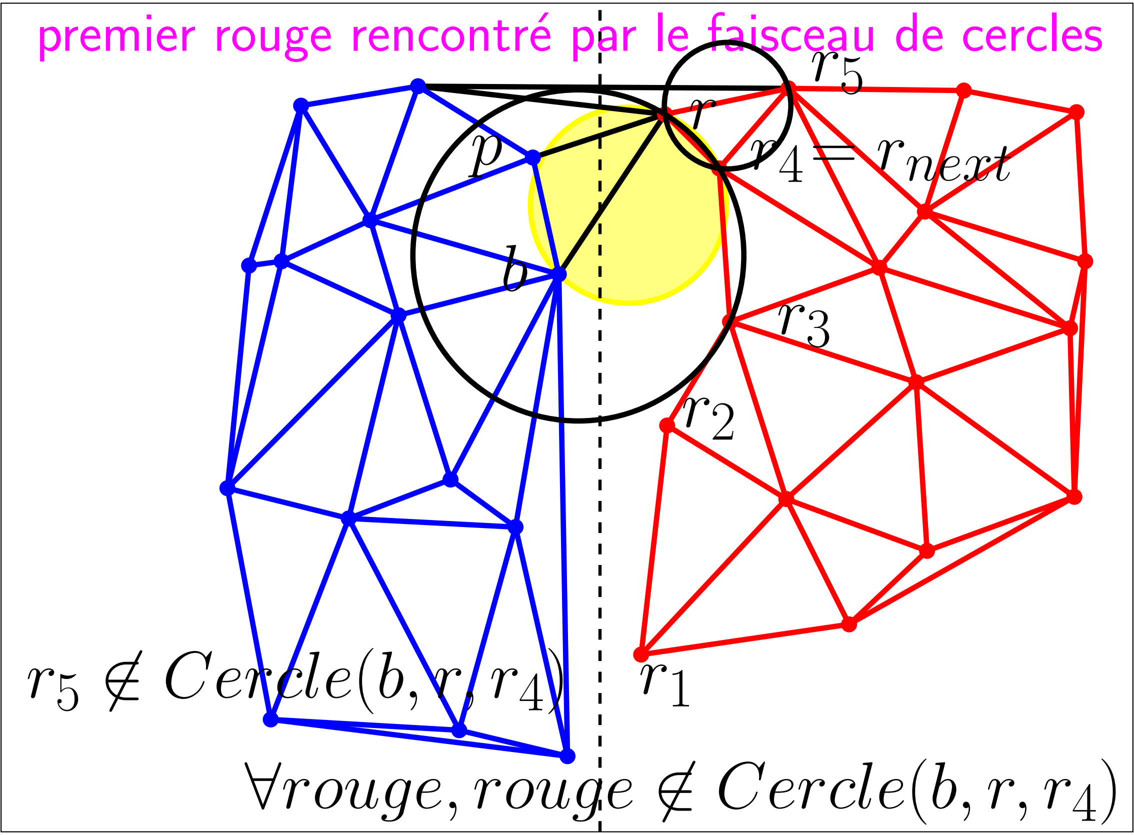

premier rouge rencontré par le faisceau de cercles
p
r
b
r2
r1
r2 ∈ Cercle(b, r, r1)


premier rouge rencontré par le faisceau de cercles
p
r
b
r3
r2
r3 ∈ Cercle(b, r, r2)
r1

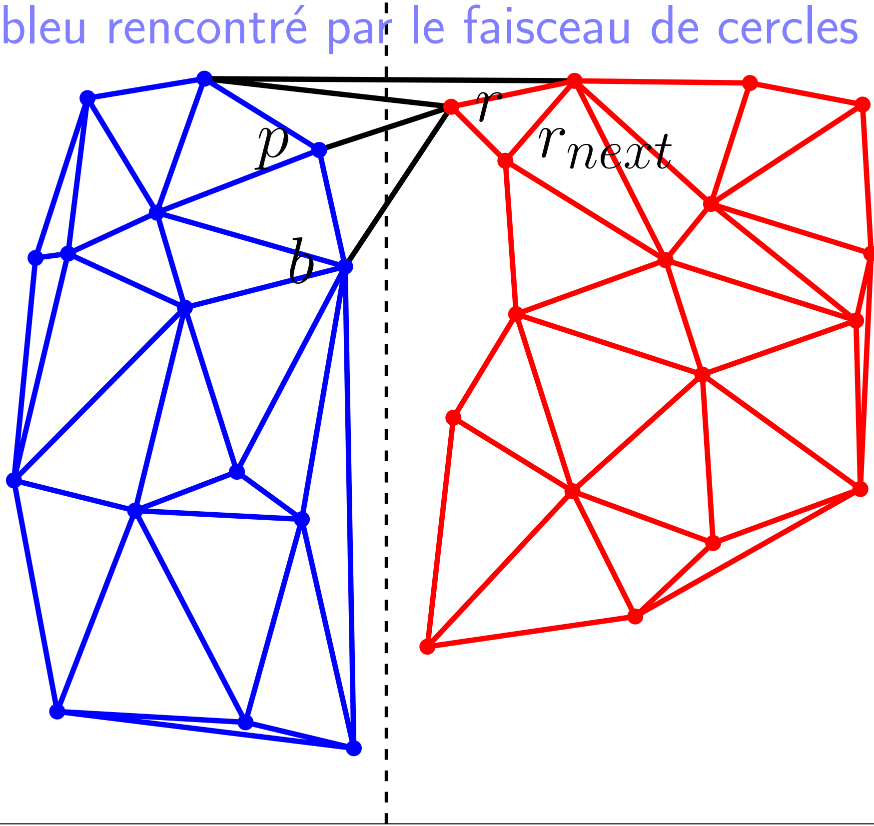
premier rouge rencontré par le faisceau de cercles

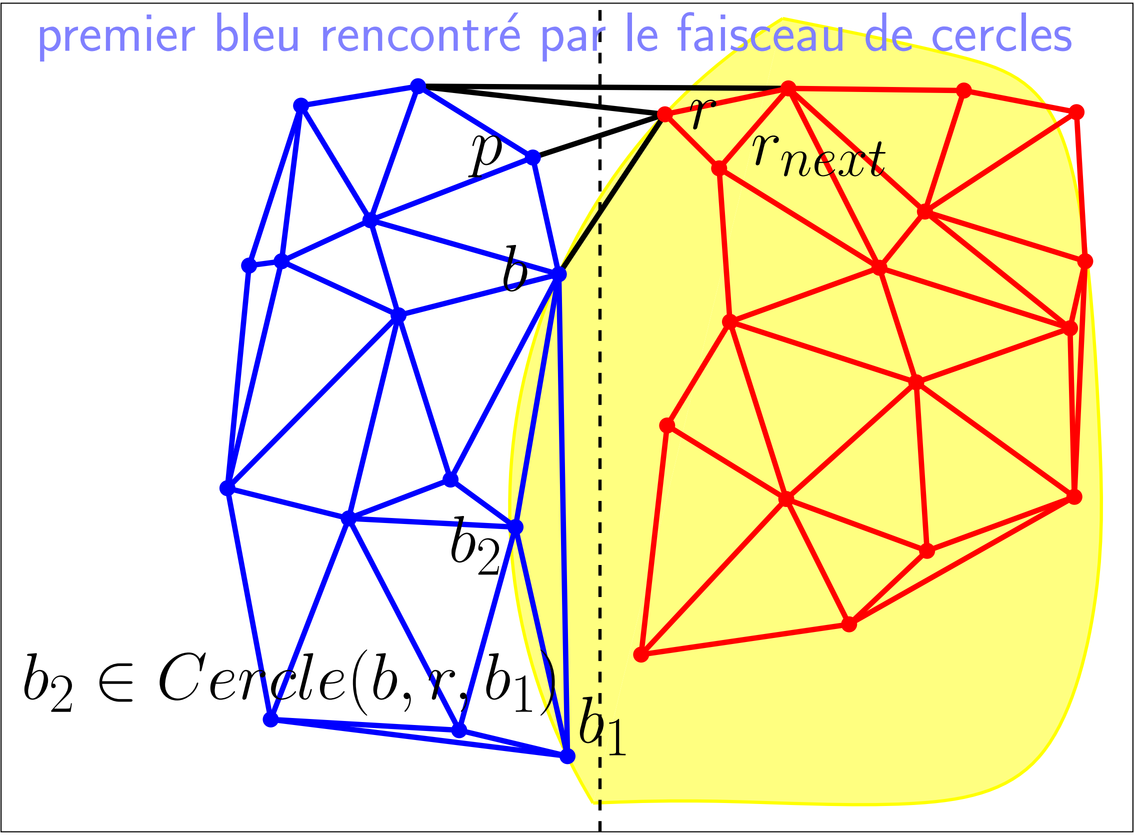


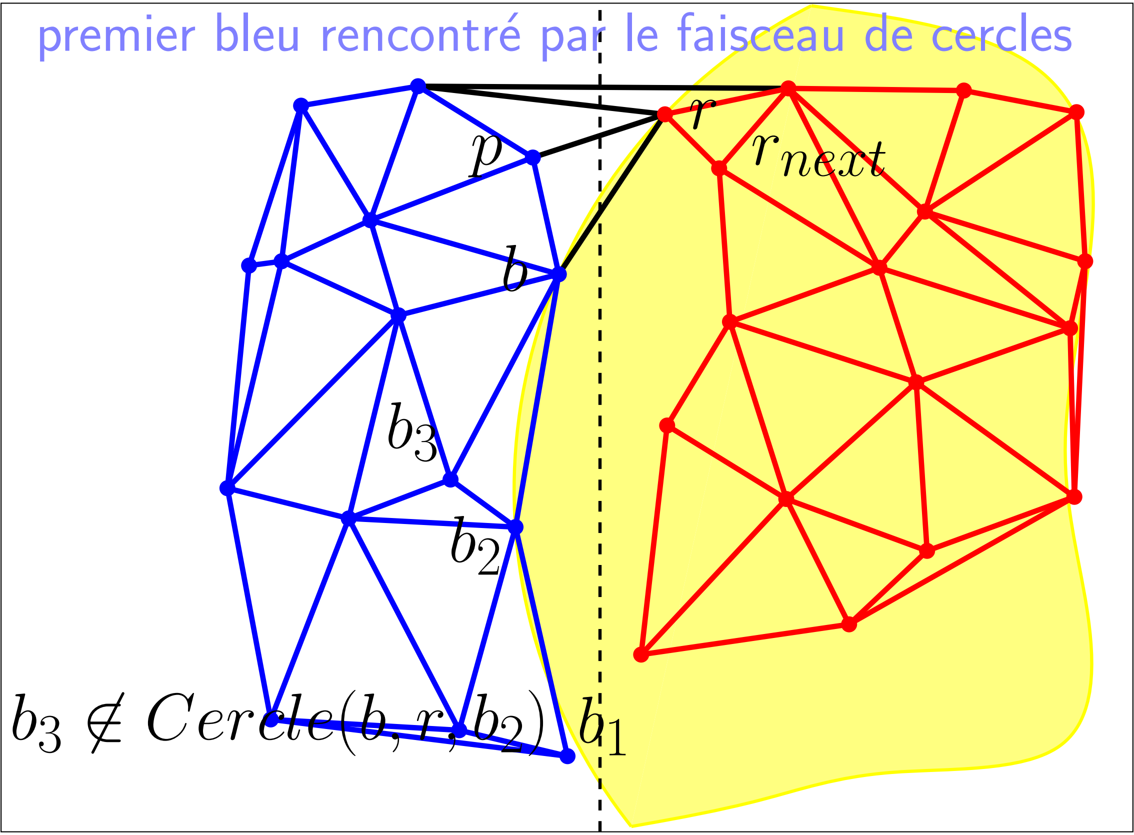

premier rouge rencontré par le faisceau de cercles
r_5
r
p
r_4
b
r_3
r_2
$r_5 \notin Cercle(b, r, r_4)$
r_1

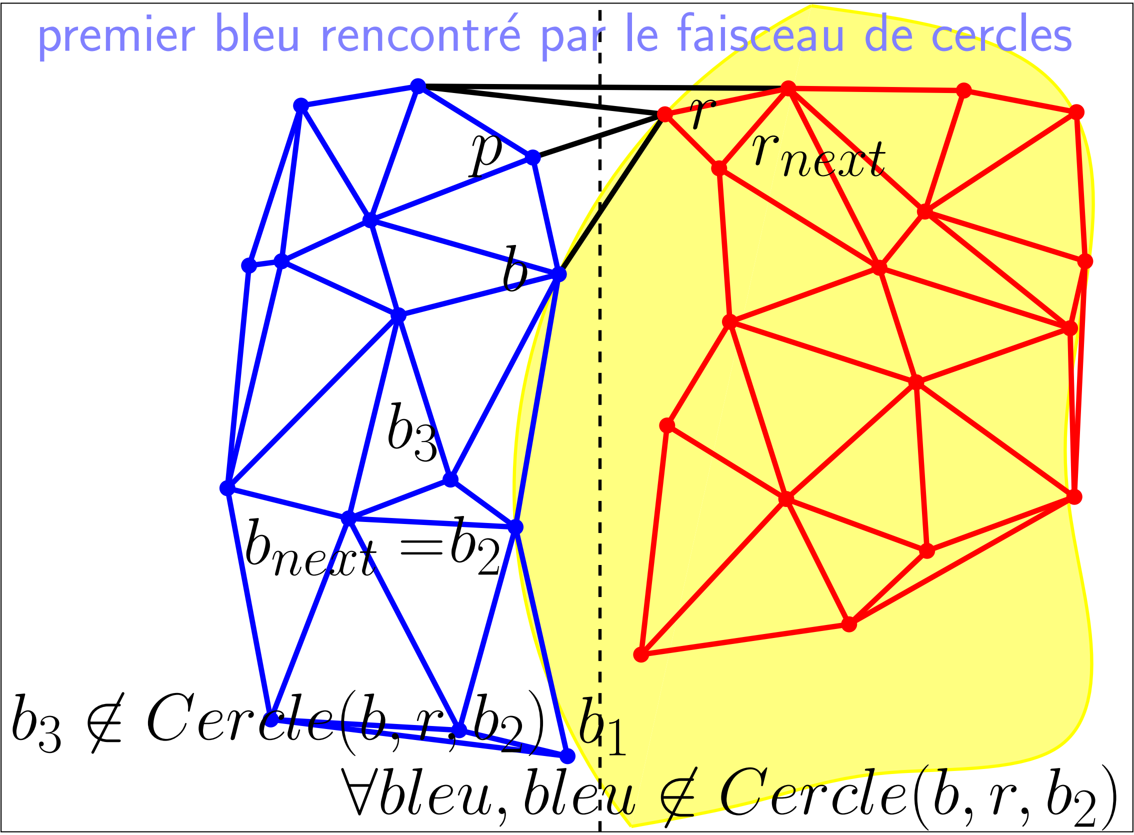

premier rouge rencontré par le faisceau de cercles
r_5
r
p
$r_4 = r_{next}$
b
r_3
r_2
r_1
$r_5 \notin Cercle(b, r, r_4)$
$\forall rouge, rouge \notin Cercle(b, r, r_4)$

premier bleu rencontré par le faisceau de cercles

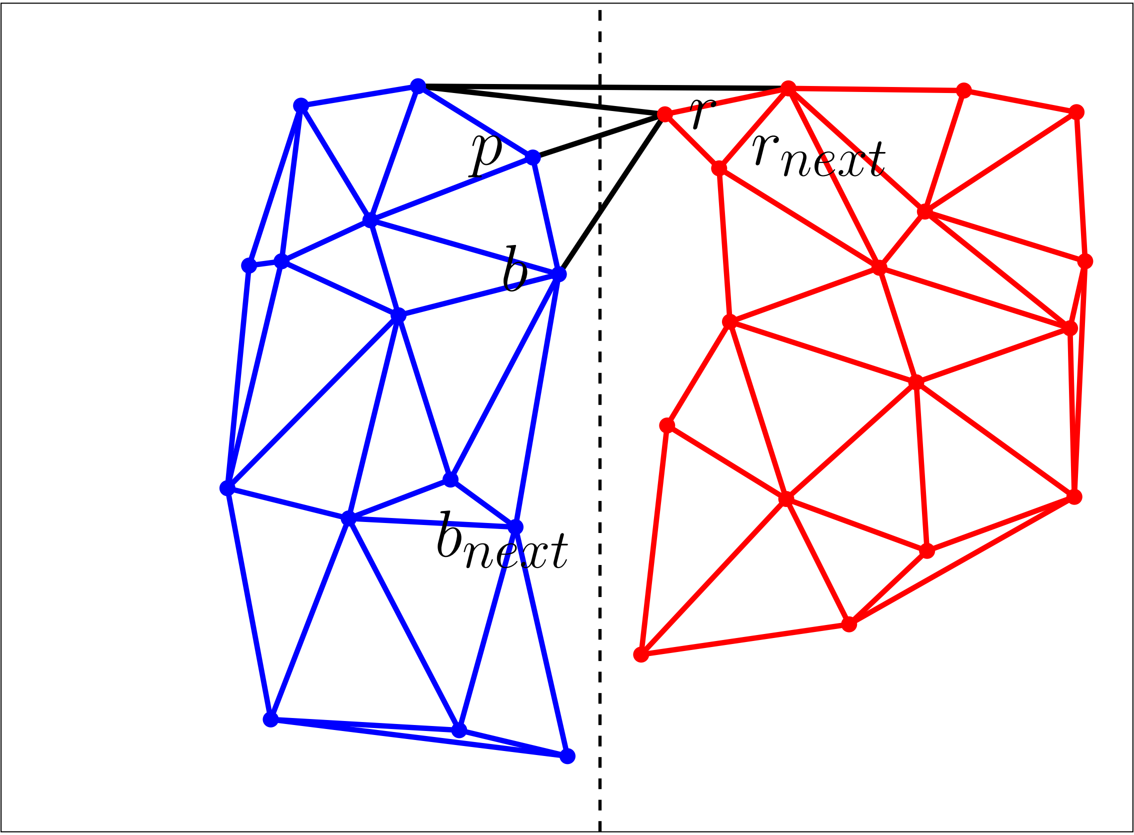



premier bleu rencontré par le faisceau de cercles
r
r_{next}
p
b
b_2
$b_2 \in Cercle(b, r, b_1)$
b_1

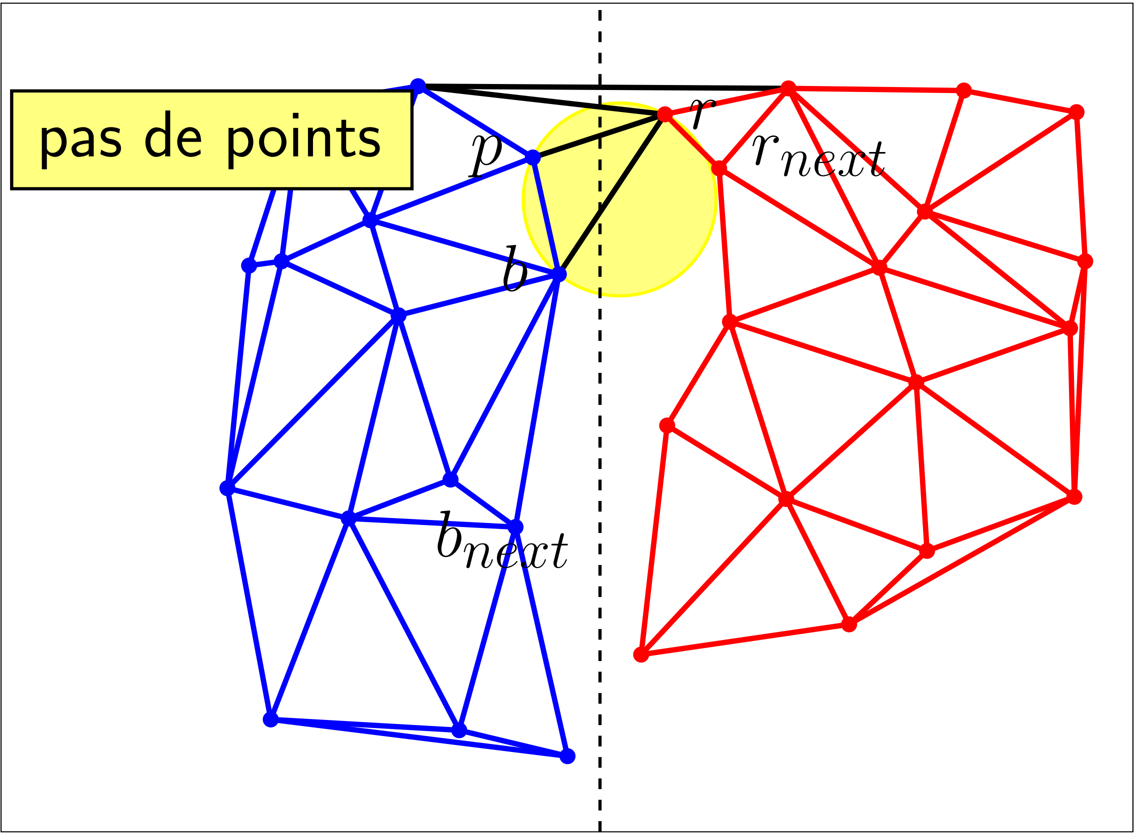

premier bleu rencontré par le faisceau de cercles
r
r_{next}
p
b
b_3
b_2
$b_3 \notin Cercle(b, r, b_2)$
b_1

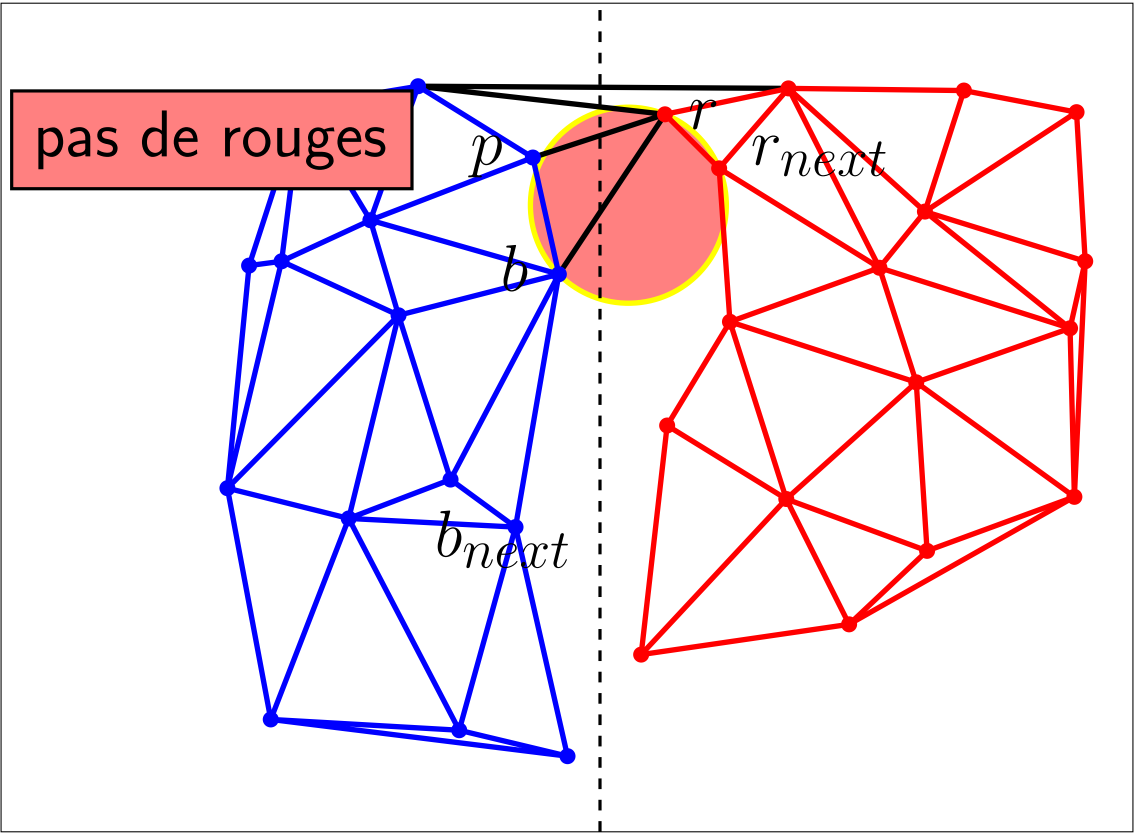


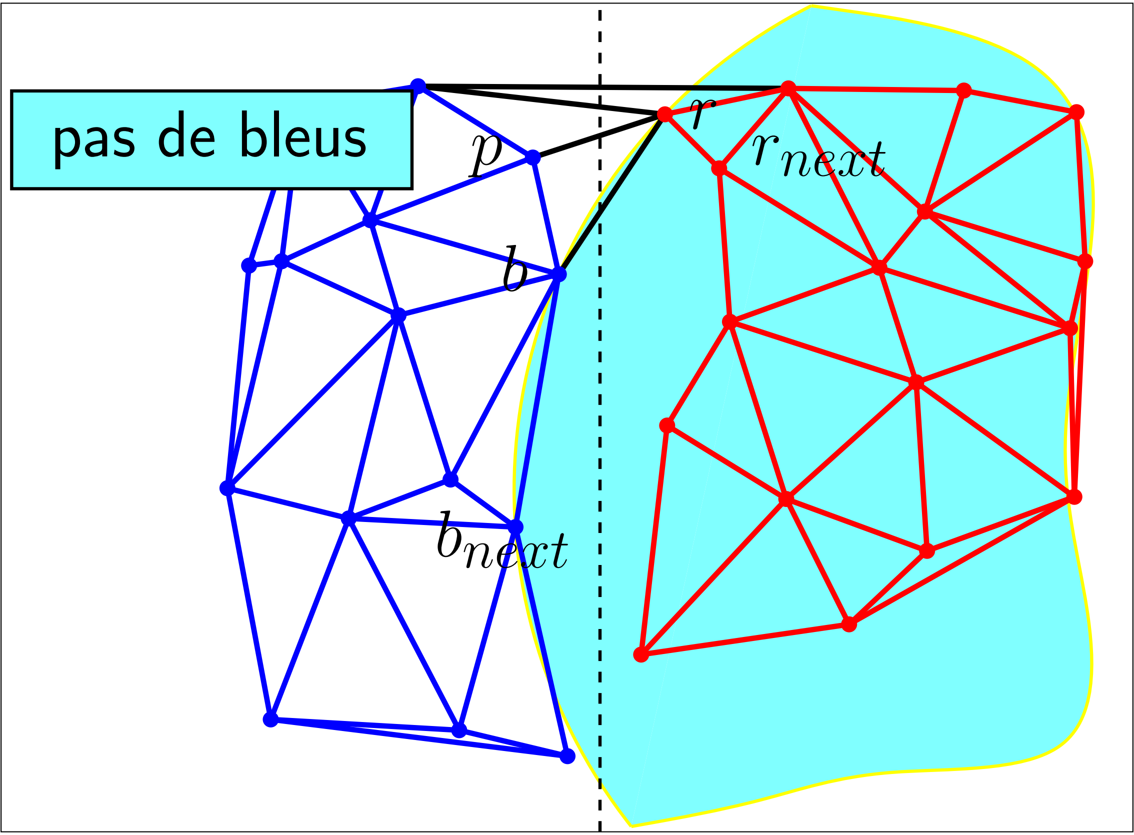
premier bleu rencontré par le faisceau de cercles
r
r_{next}
p
b
b_3
$b_{next} = b_2$
$b_3 \notin Cercle(b, r, b_2)$
b_1
$\forall bleu, bleu \notin Cercle(b, r, b_2)$



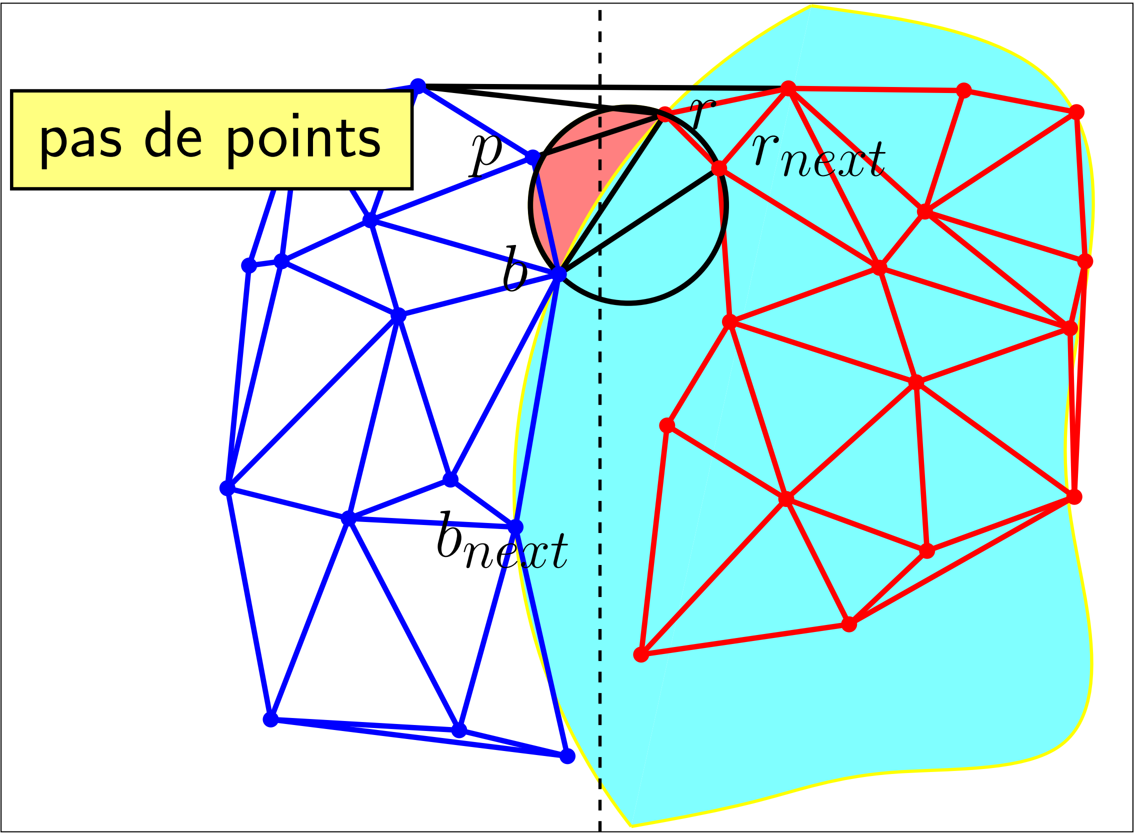
r
r_{next}
p
b
b_{next}

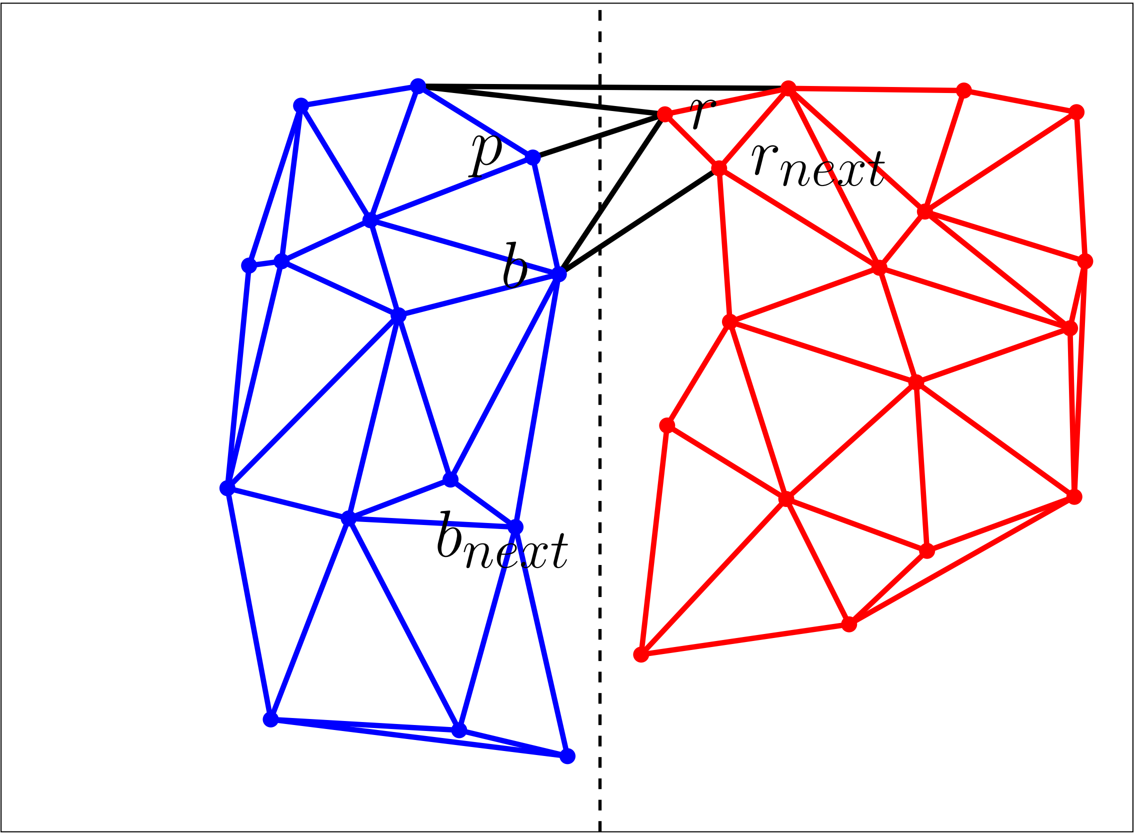

pas de points
p
r
r_{next}
b
b_{next}


pas de rouges
p
r
r_{next}
b
b_{next}

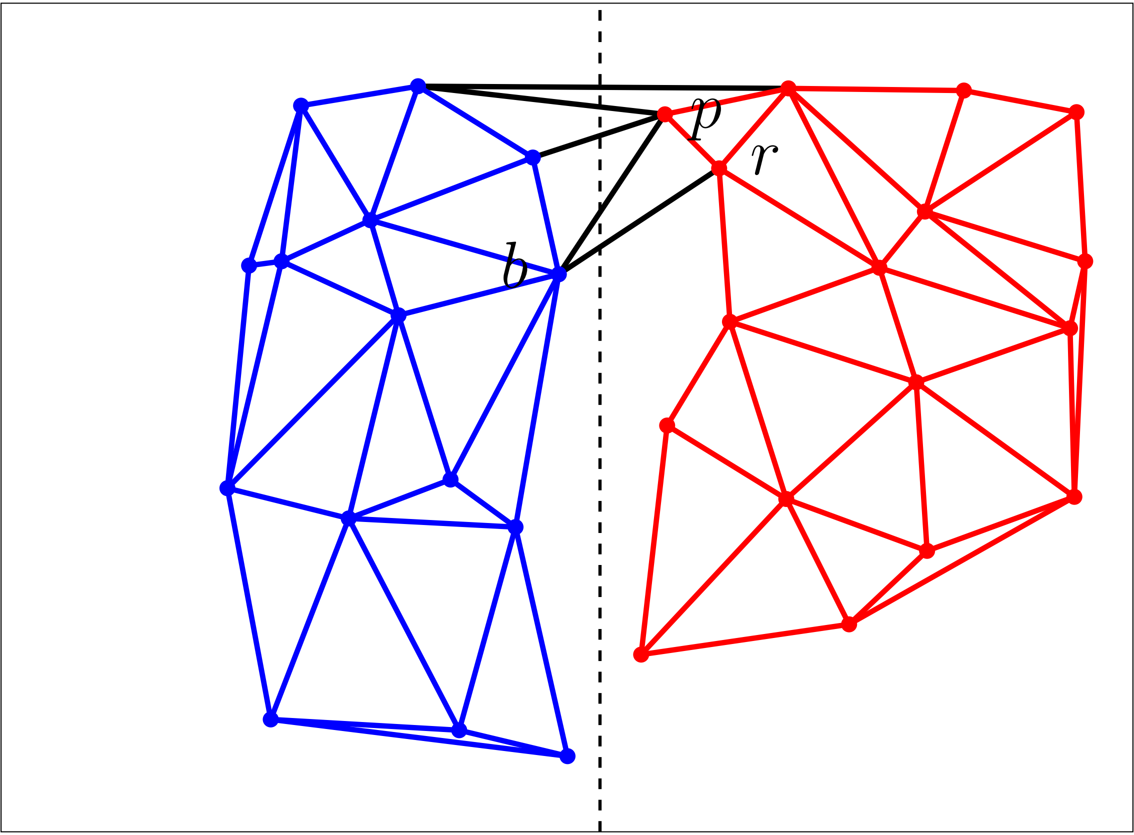


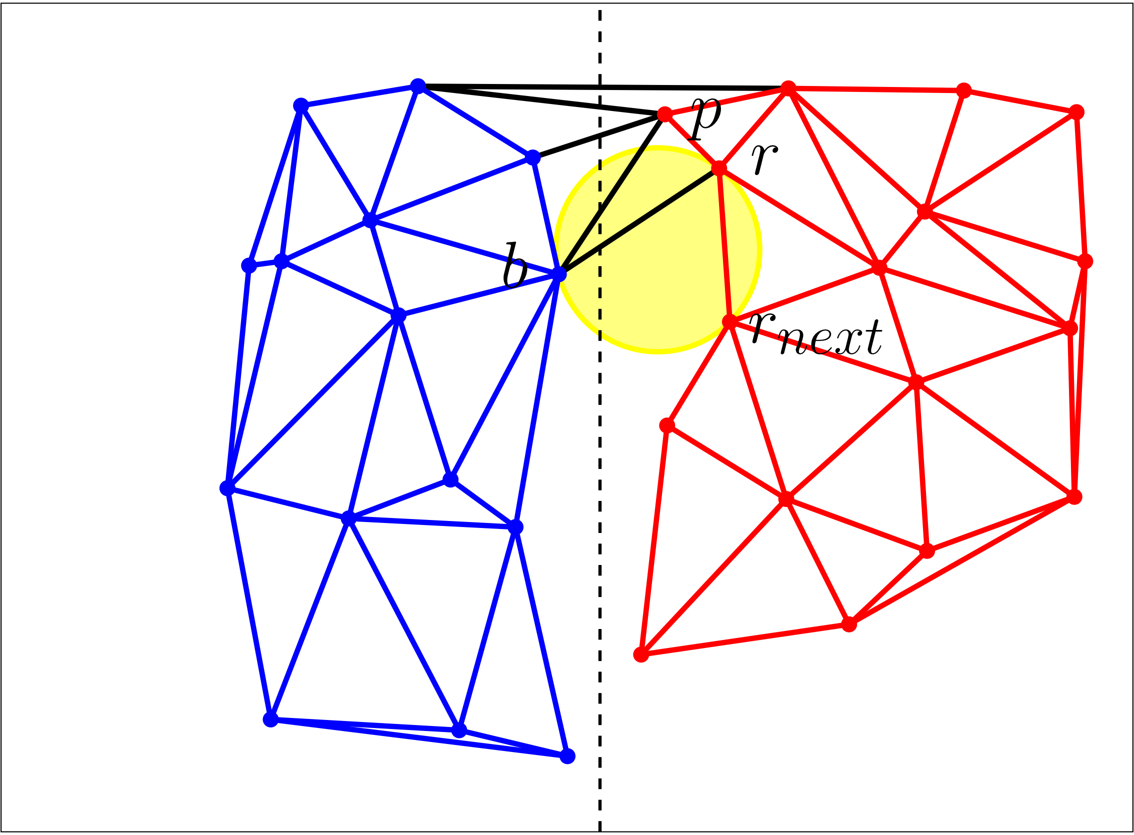
pas de bleus
p
r
r_{next}
b
b_{next}

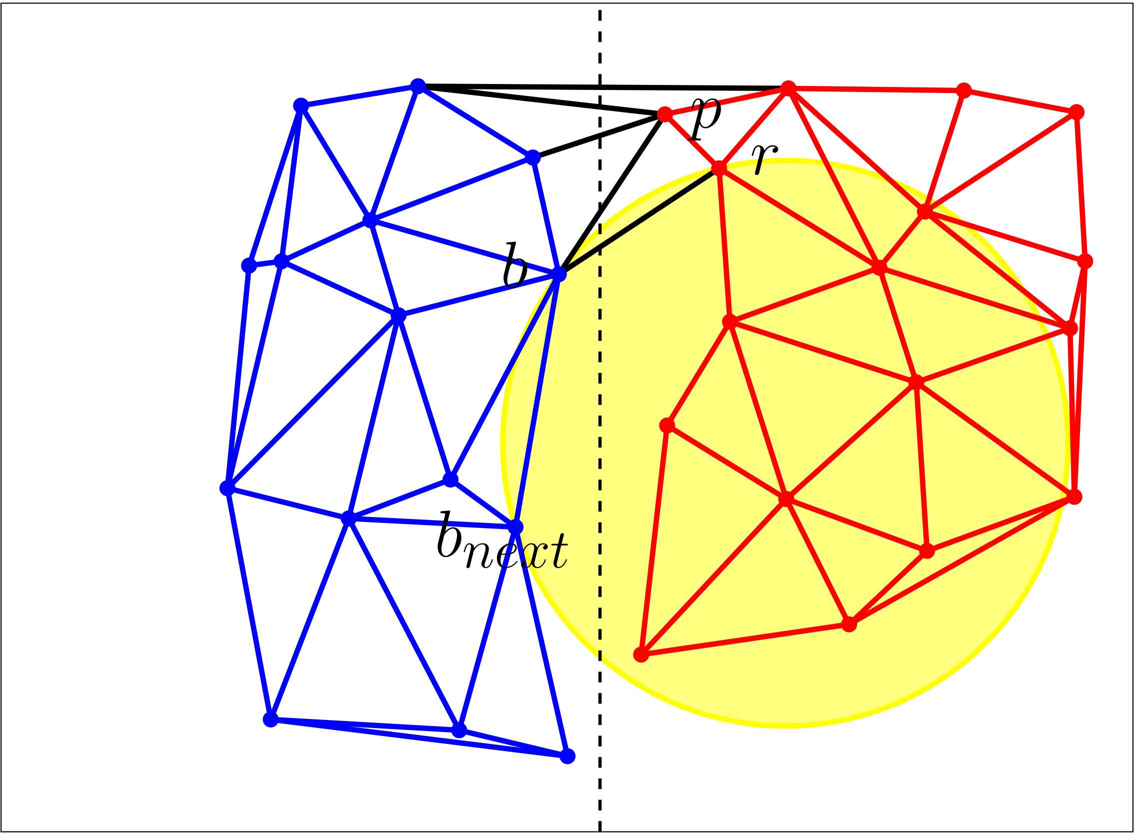

pas de points
p
r
r_{next}
b
b_{next}

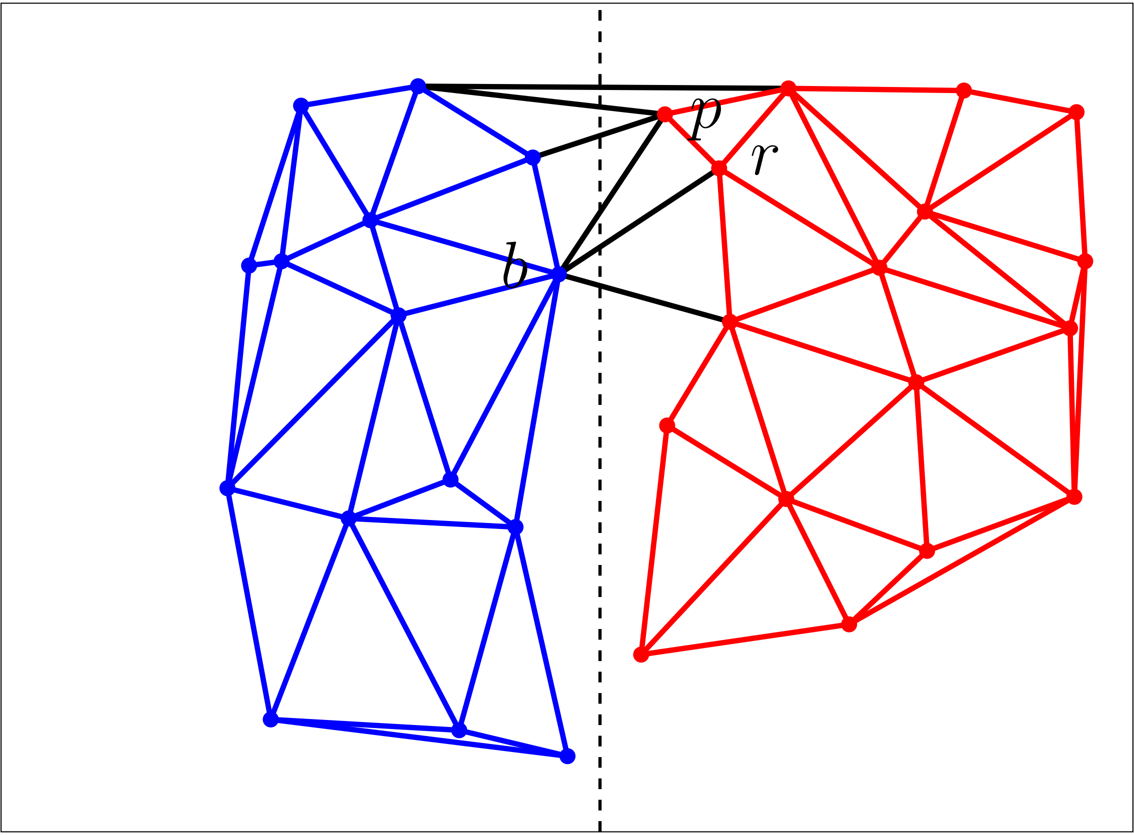


p
r
r_{next}
b
b_{next}


p
r
b


p
r
b
r_next


p
r
b
b_{next}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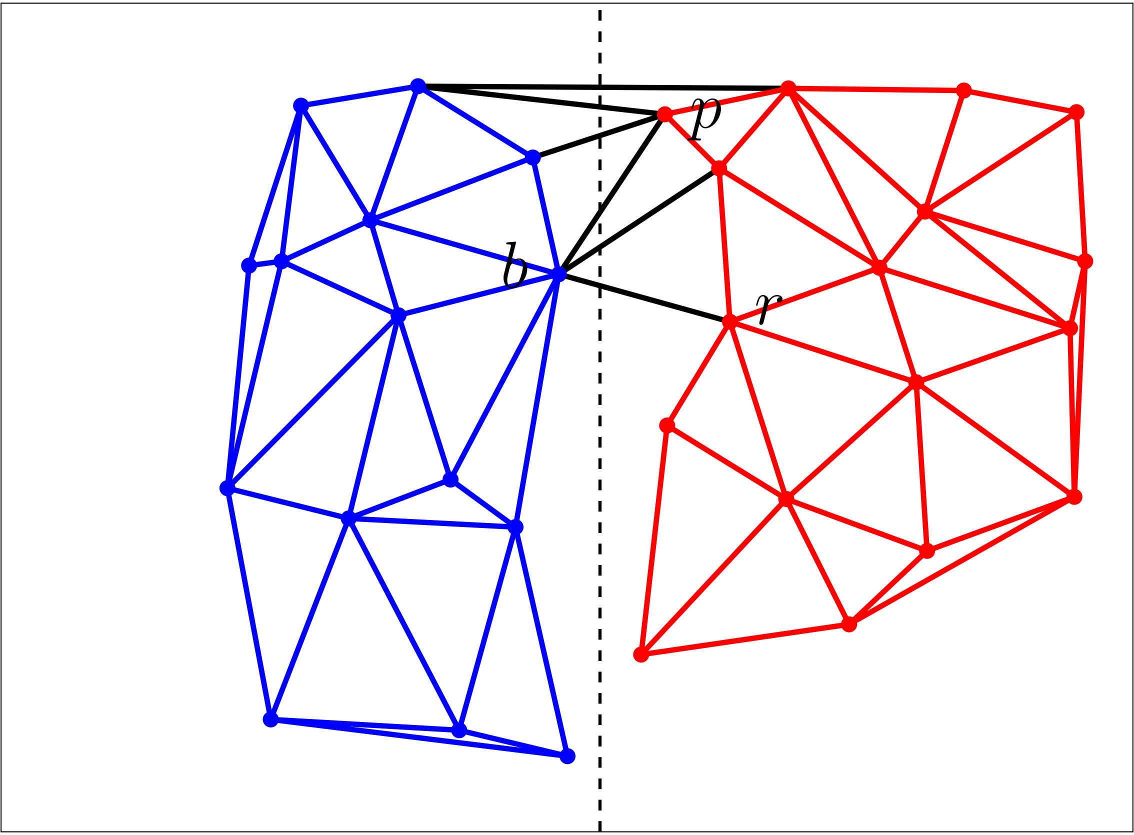
r
b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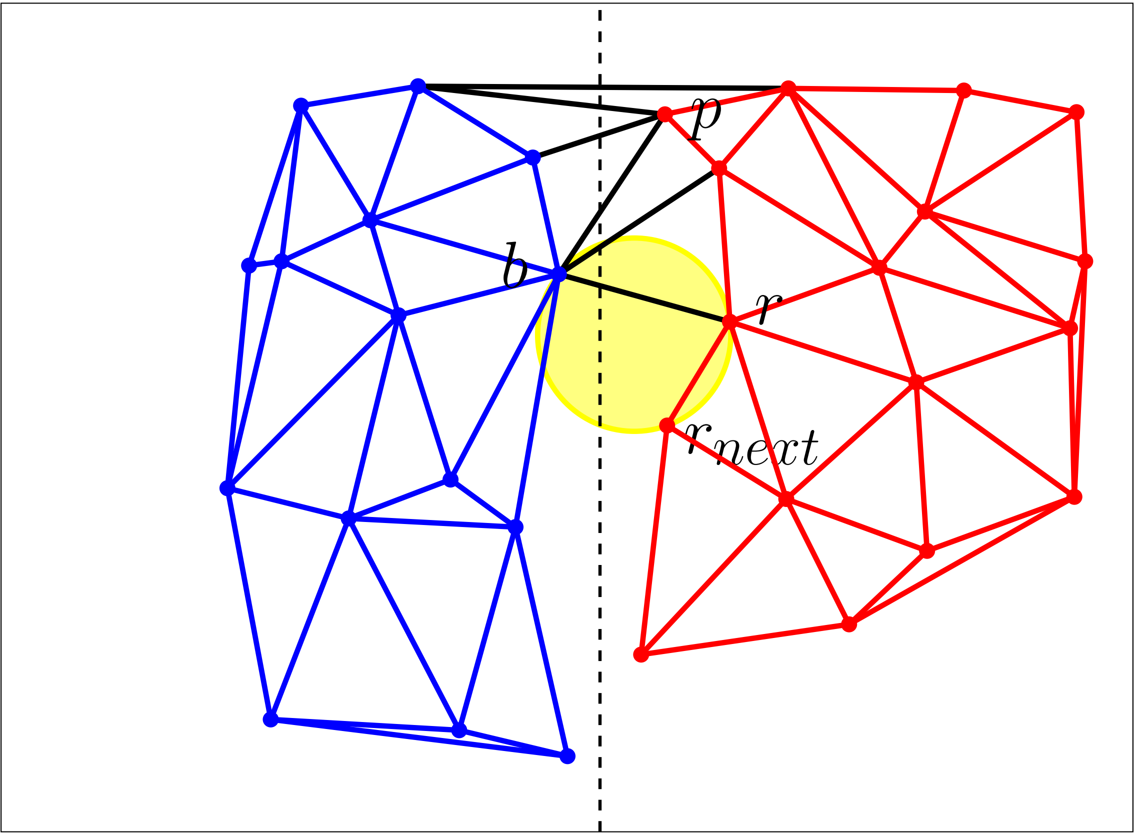
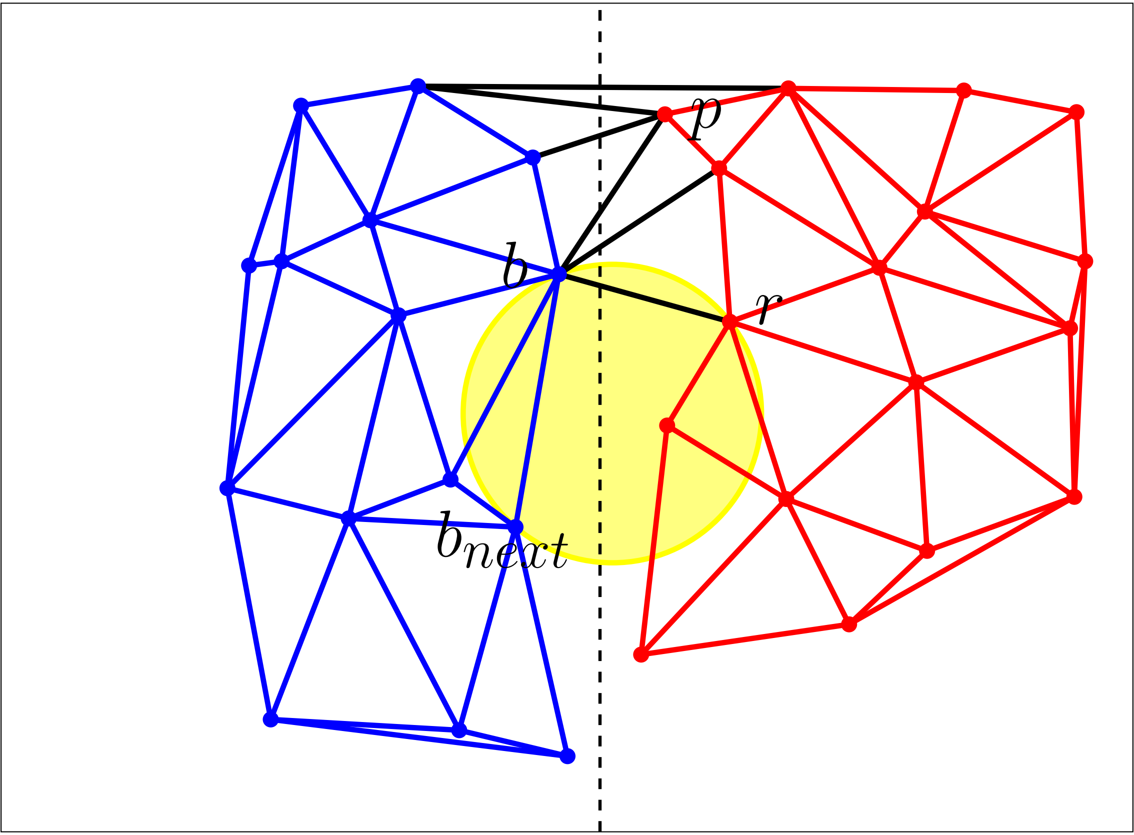
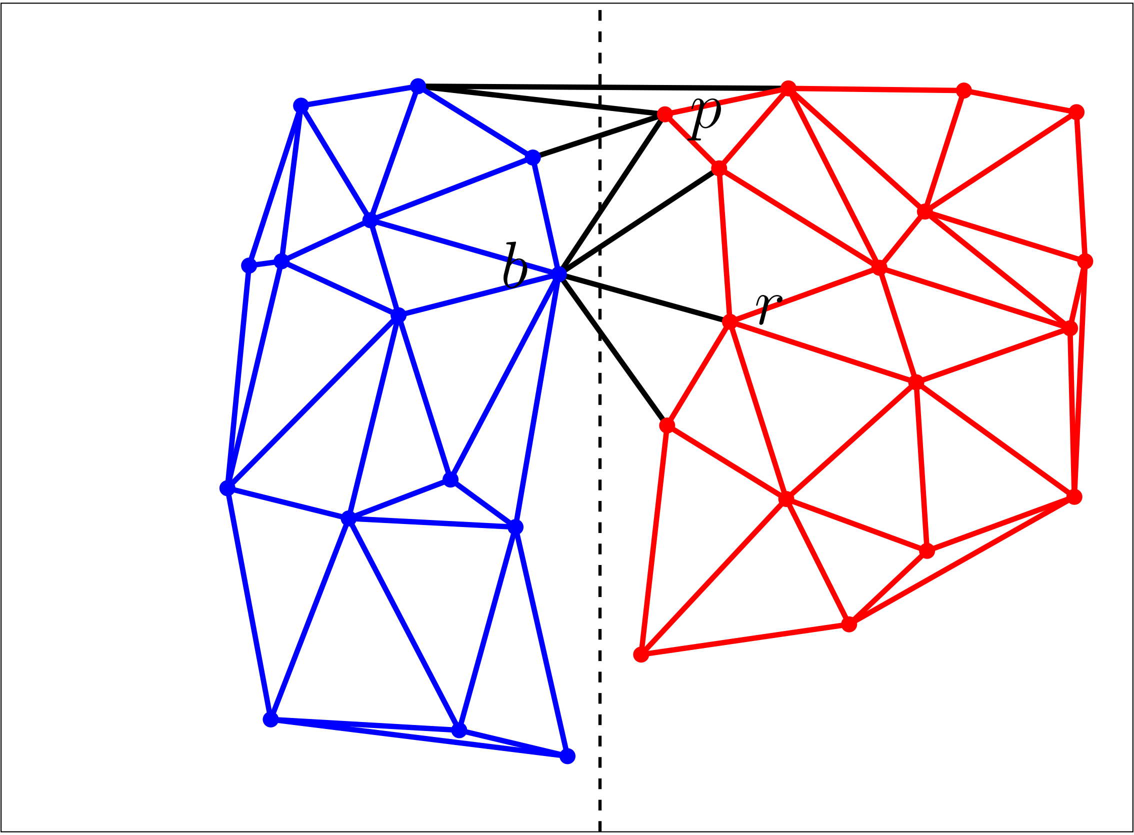
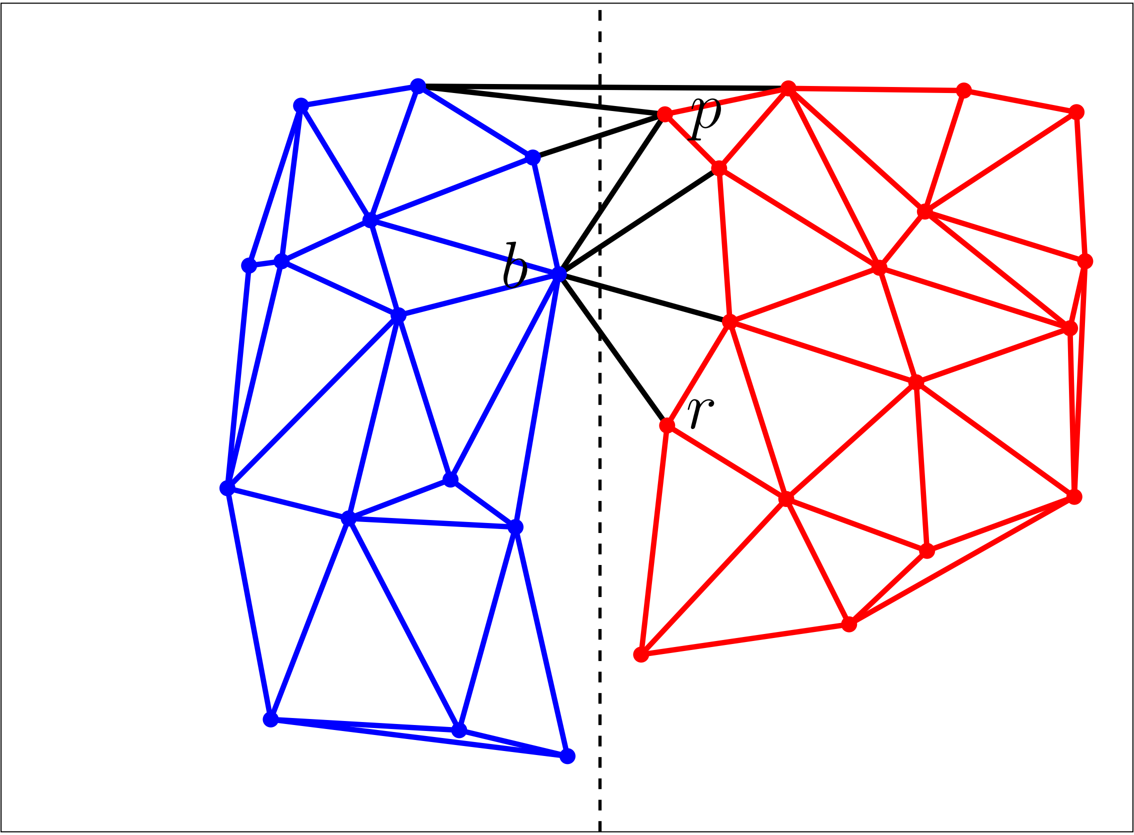
b
r

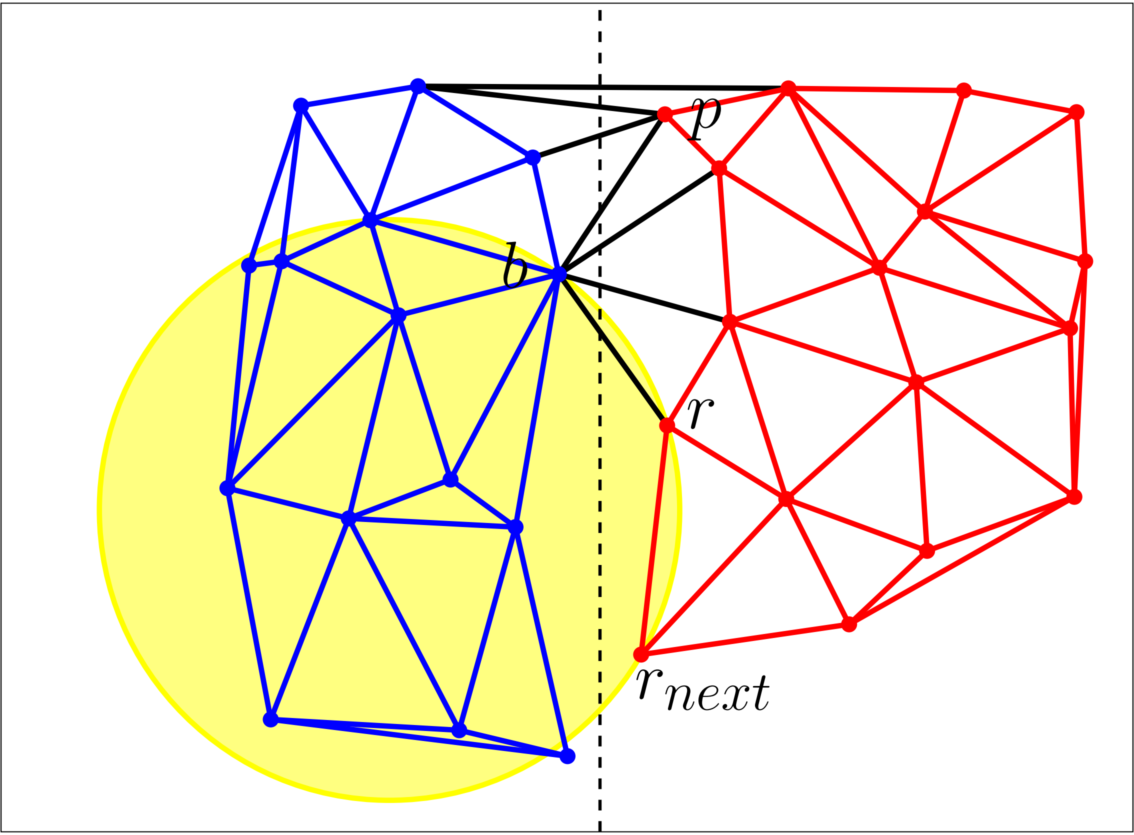

p
b
r
r_next

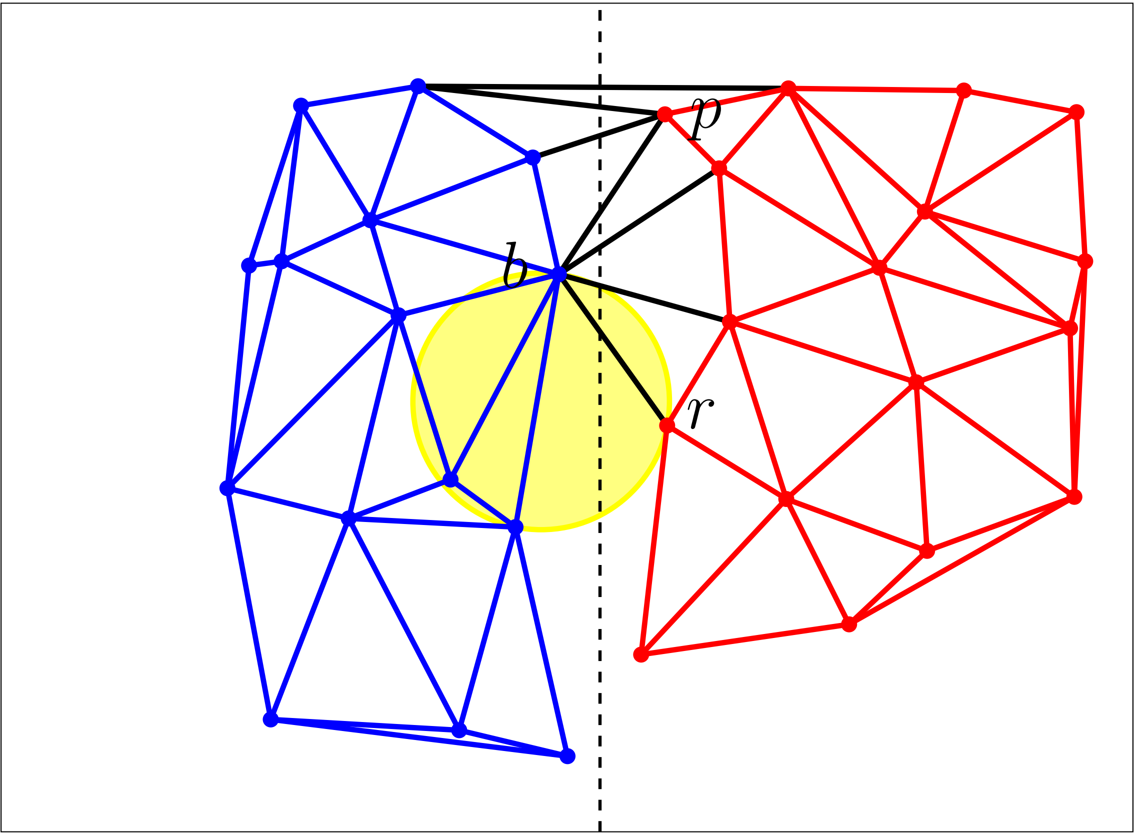

p
b
r
b_{next}

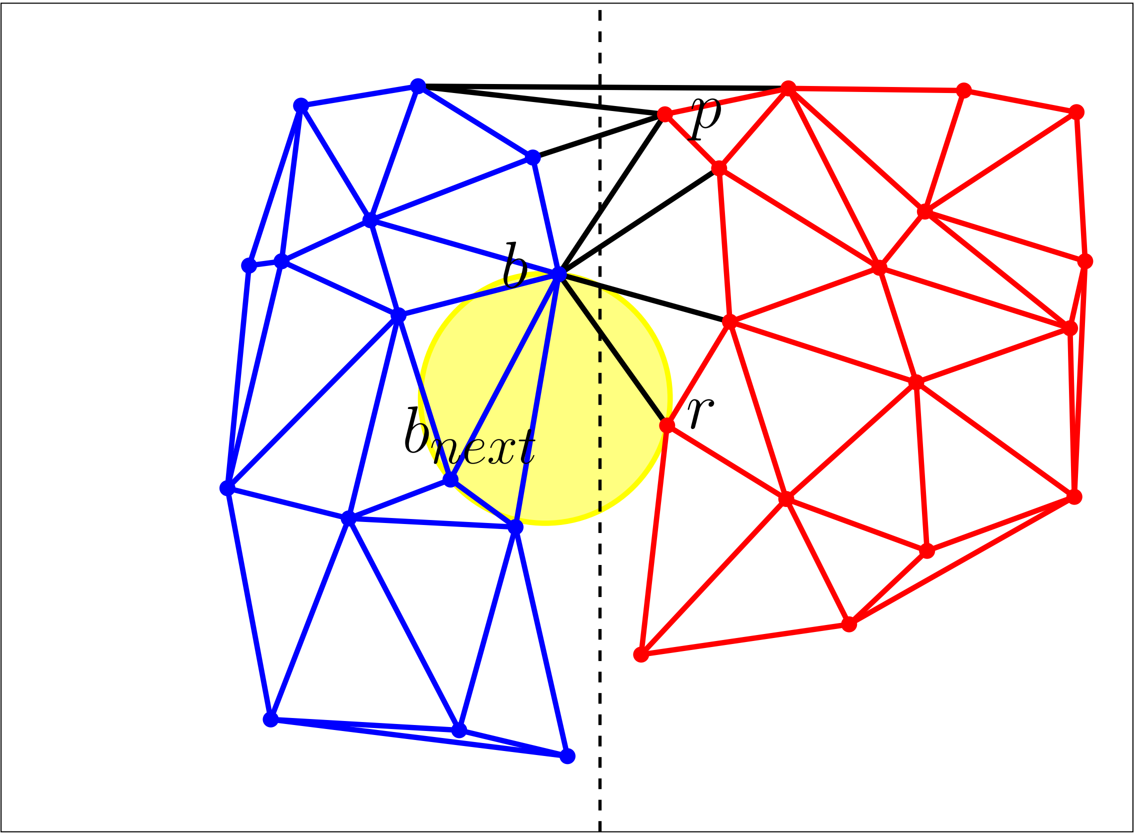


p
b
r

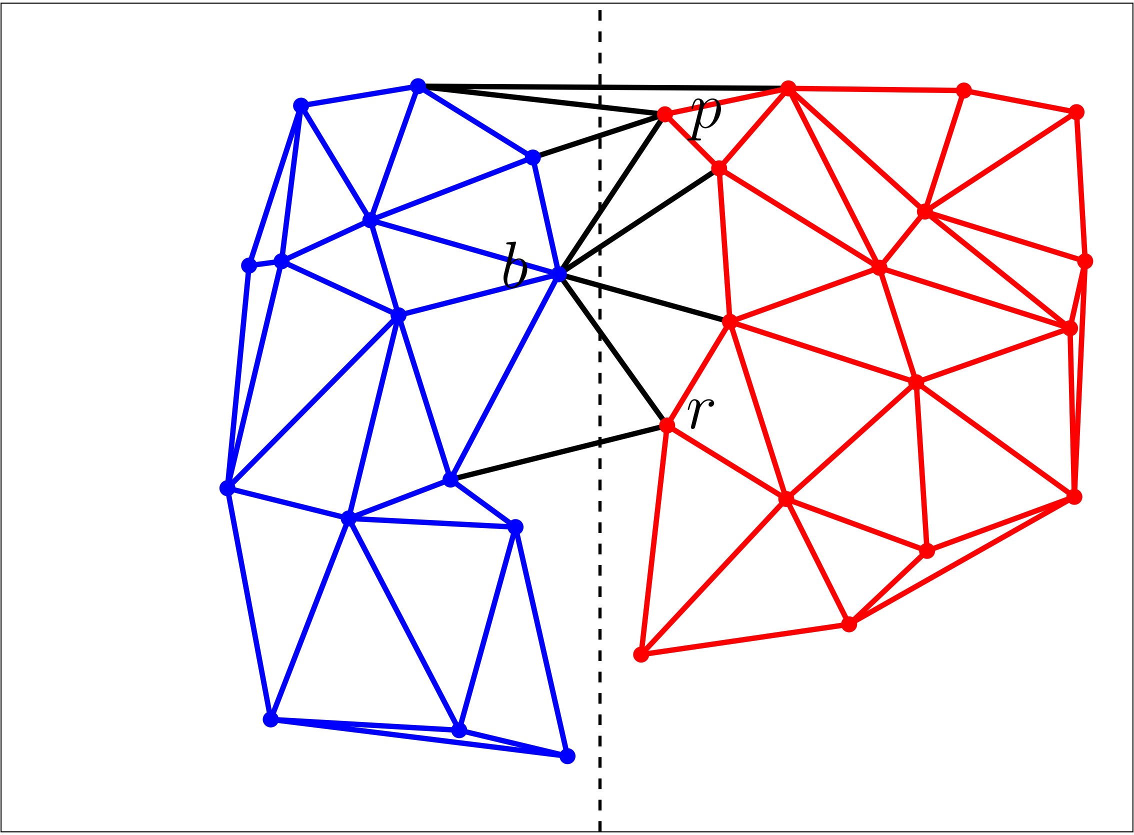

p
b
r

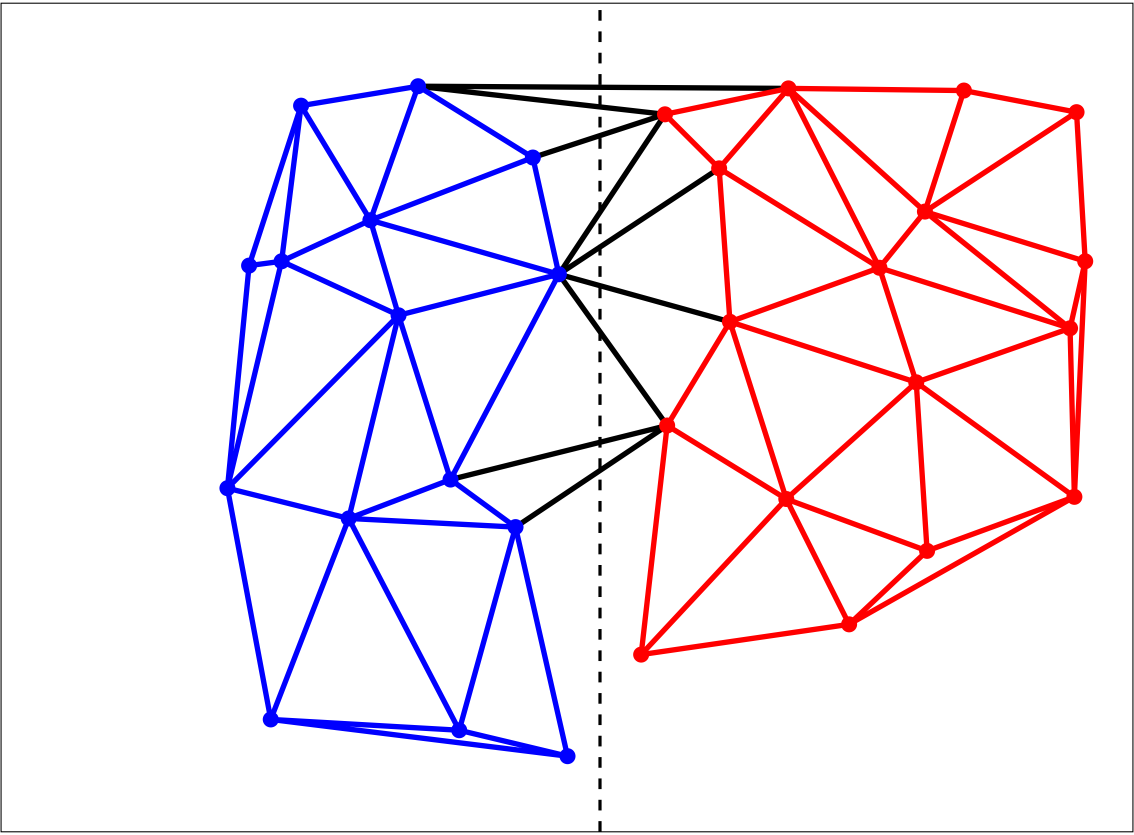

p
b
r
r_next

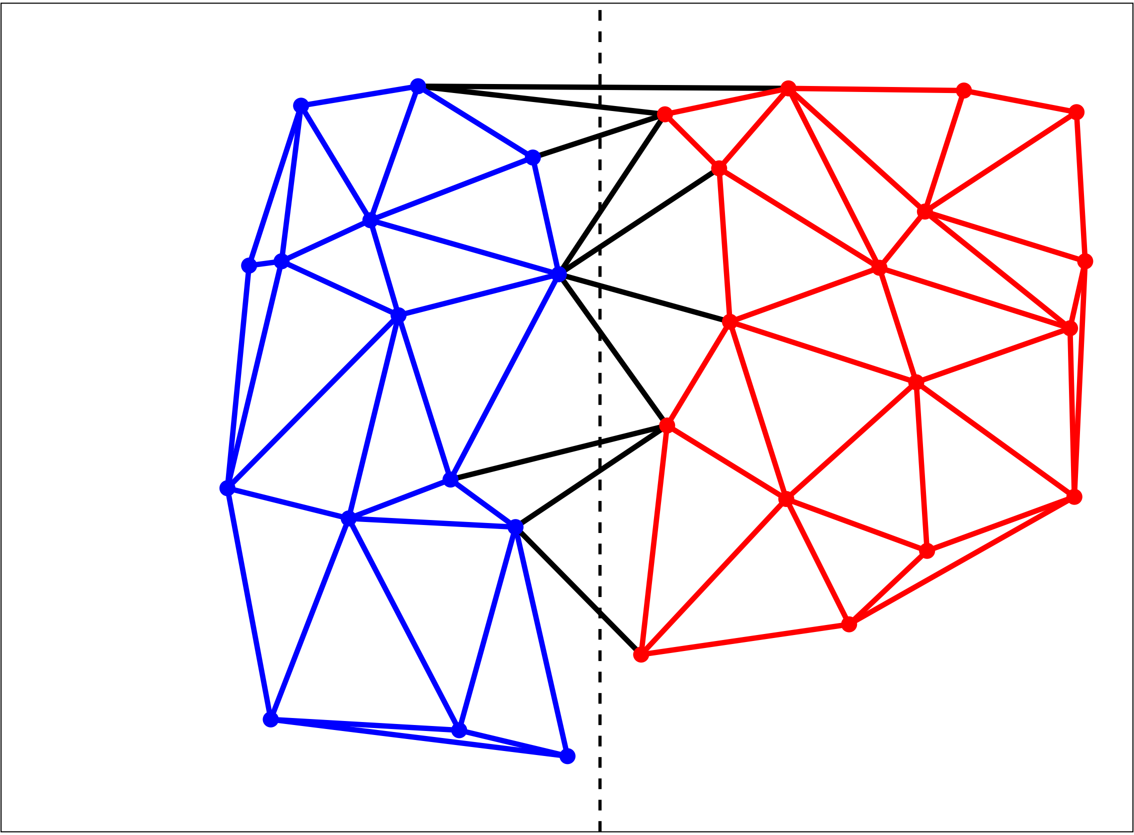

p
b
r

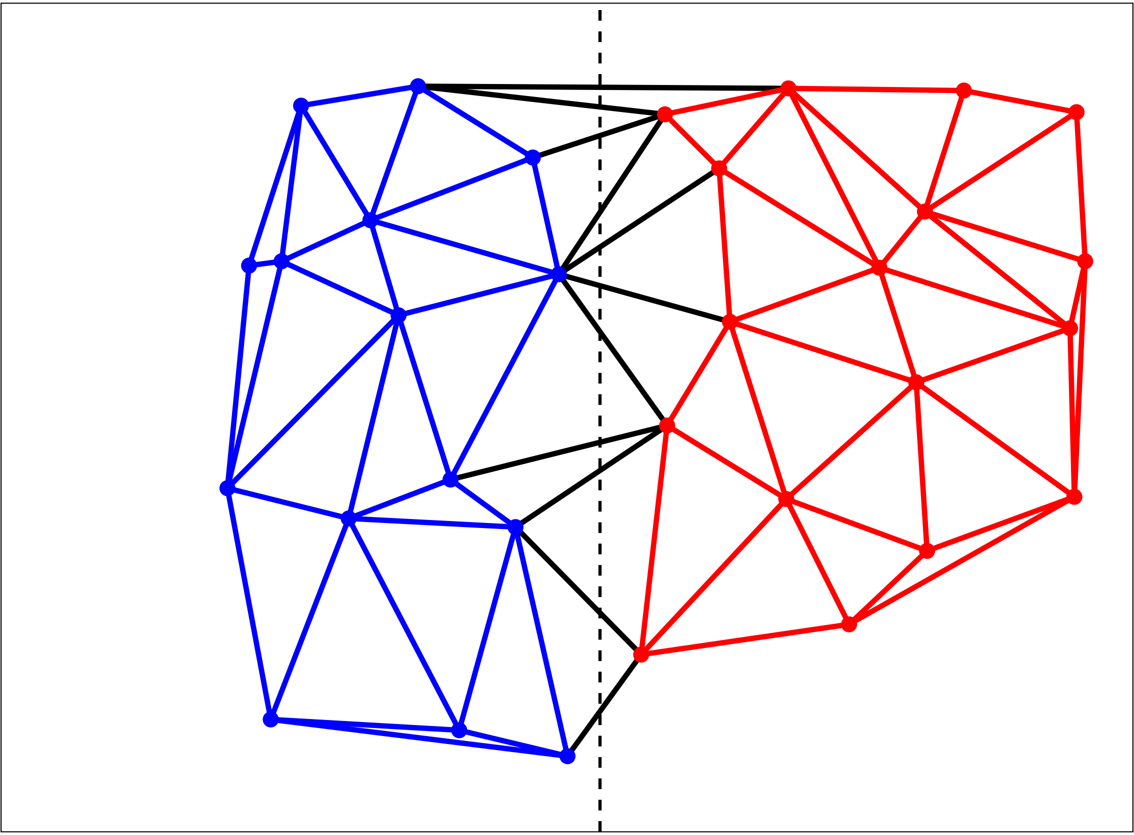


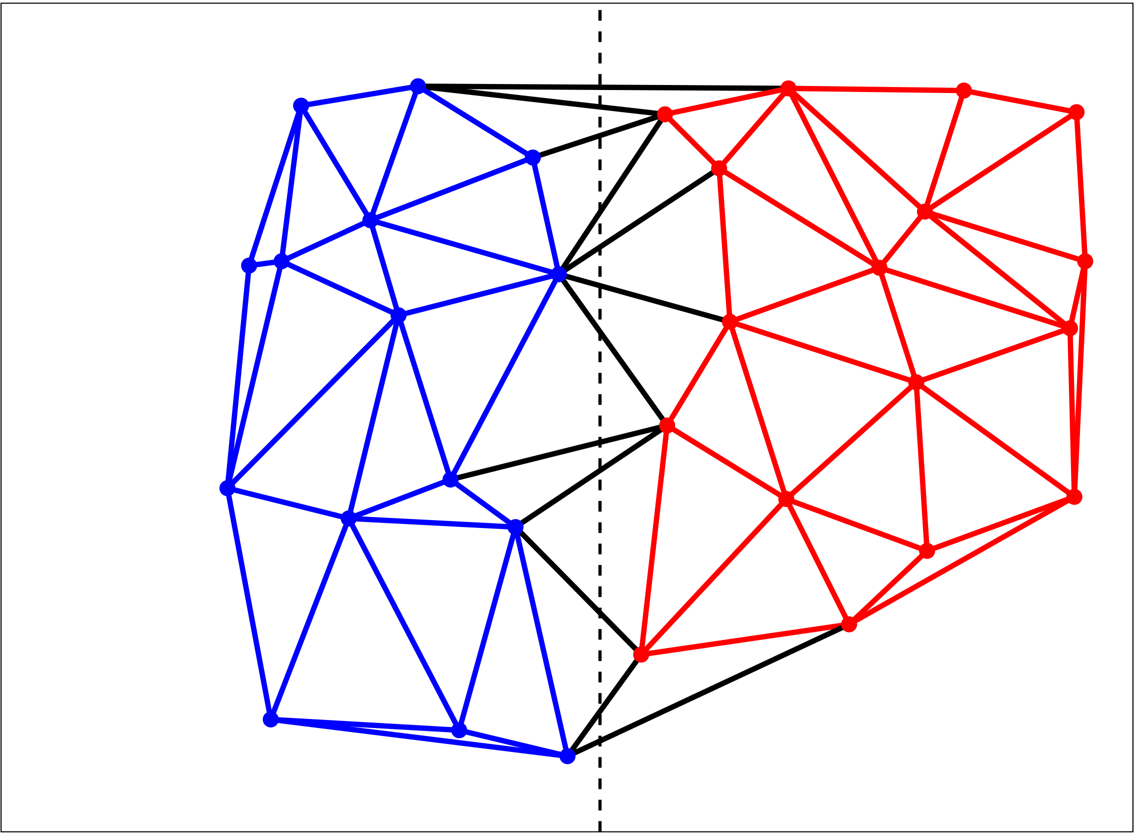
p
b
r
b_{next}


p
b
r









Complexité

Complexité

A chaque étape de la recherche de r_{next}

Complexité

A chaque étape de la recherche de r_{next}

On efface une arête rouge

Complexité

A chaque étape de la recherche de r_{next}

On efface une arête rouge

A chaque étape de la recherche de b_{next}

Complexité

A chaque étape de la recherche de r_{next}

On efface une arête rouge

A chaque étape de la recherche de b_{next}

On efface une arête bleue

Complexité

A chaque étape de la recherche de r_{next}

On efface une arête rouge

A chaque étape de la recherche de b_{next}

On efface une arête bleue

Choisir entre r_{next} et b_{next}

Complexité

A chaque étape de la recherche de r_{next}

On efface une arête rouge

A chaque étape de la recherche de b_{next}

On efface une arête bleue

Choisir entre r_{next} et b_{next}

On trace une arête noire

Complexité

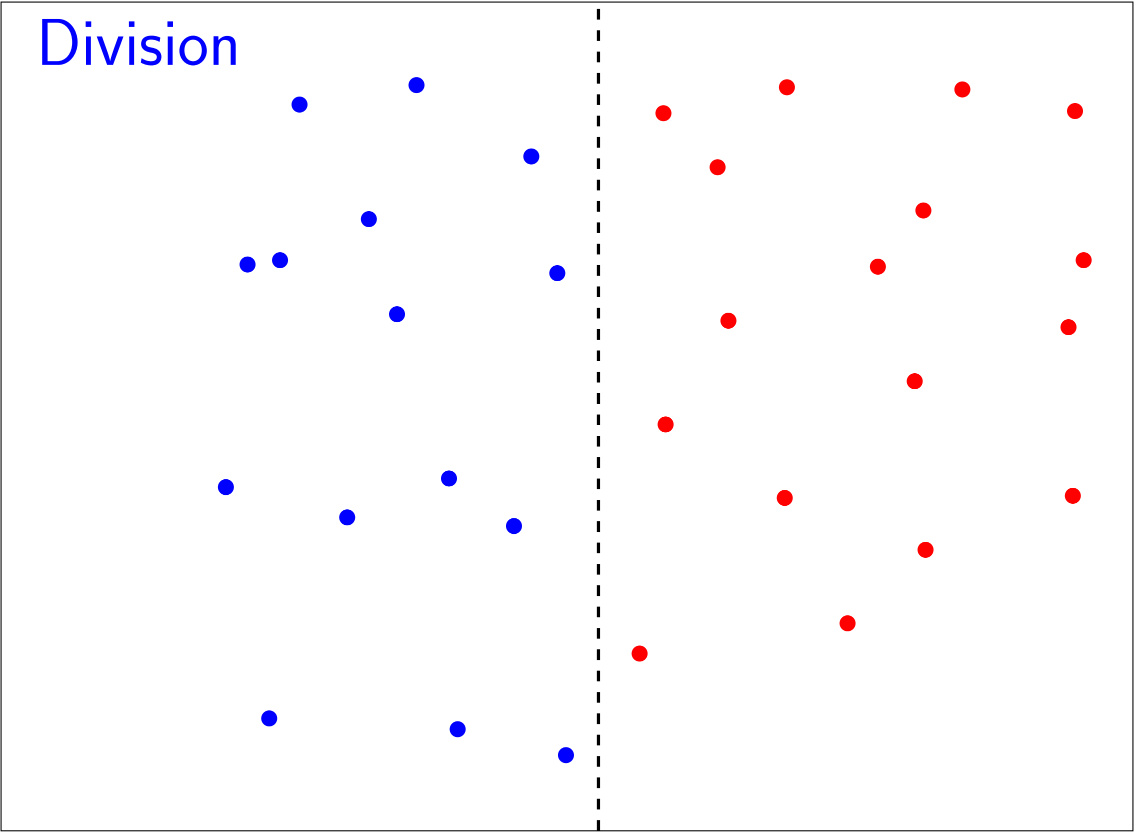
$$\text{Complexité} \leq \begin{array}{l} \sharp\ \text{arêtes rouges} \\ +\sharp\ \text{arêtes bleues} \\ +\sharp\ \text{arêtes noires} \end{array}$$

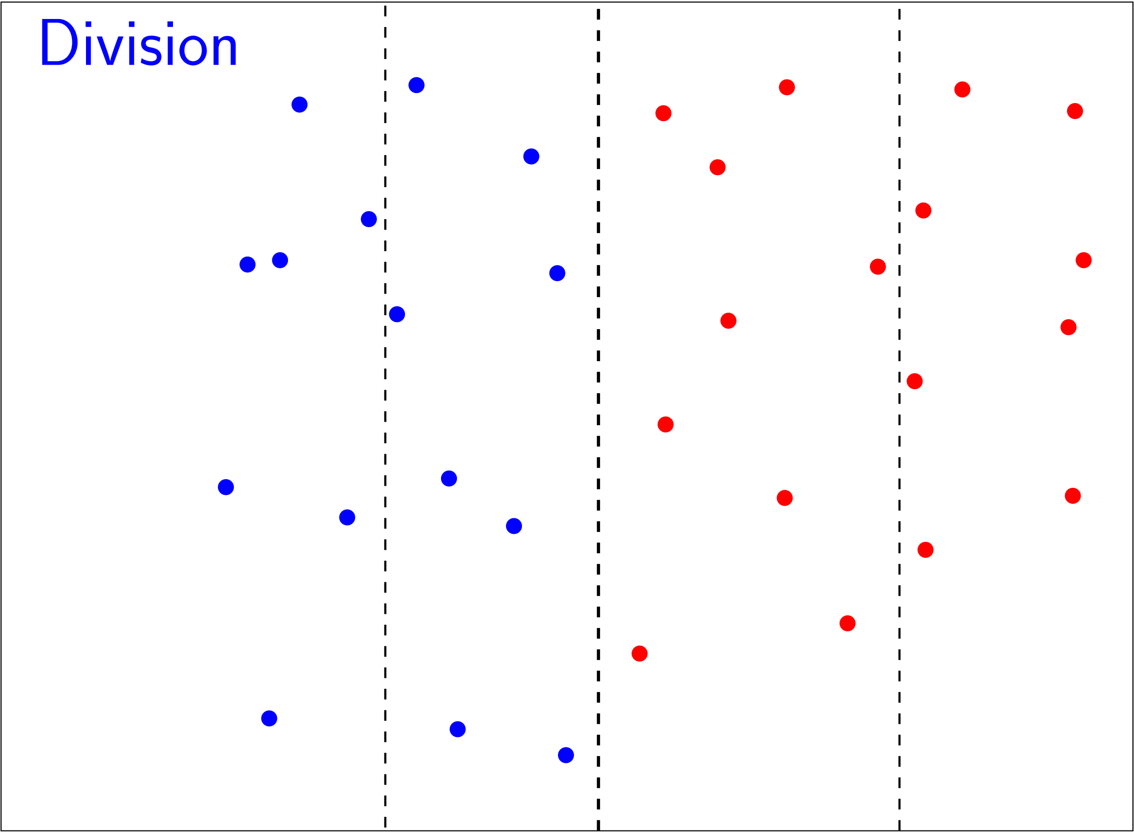
Complexité

Complexité $\leq$ $\sharp$ arêtes rouges
$+\sharp$ arêtes bleues
$+\sharp$ arêtes noires

$$\leq 3n + 3n = O(n)$$

Division-Fusion $\Longrightarrow O(n \log n)$


Division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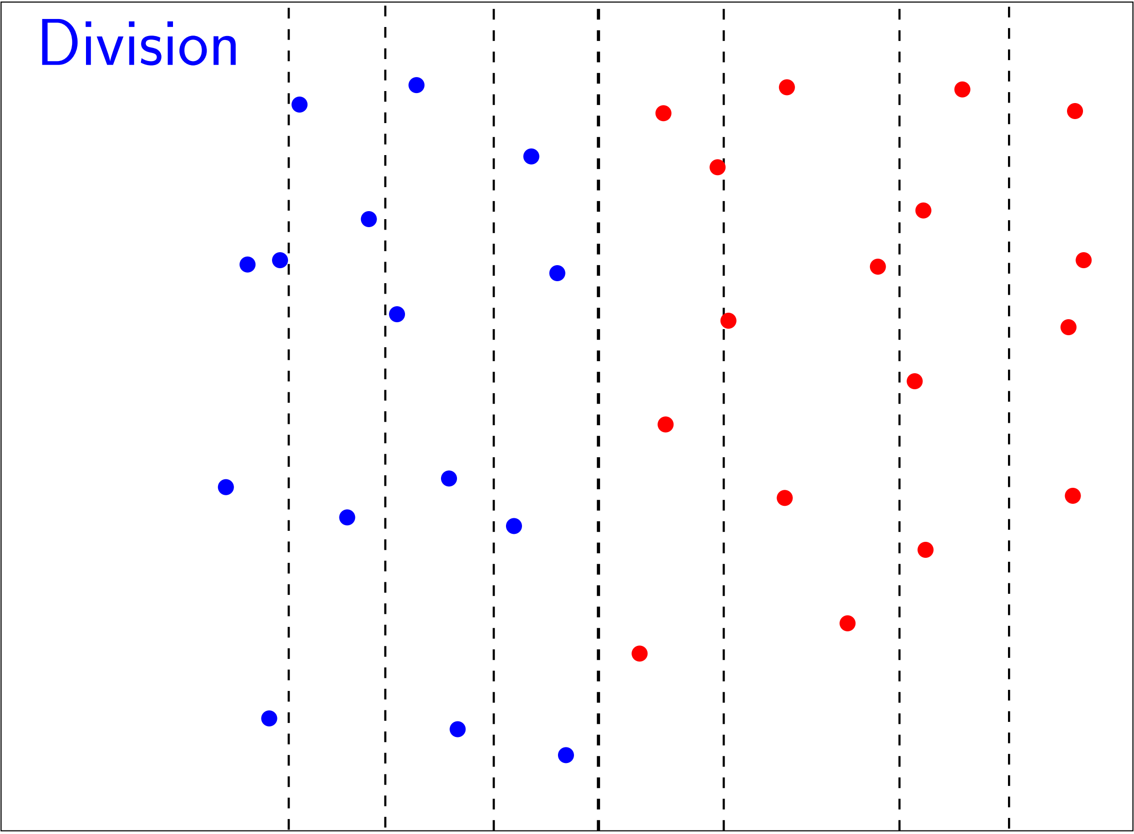
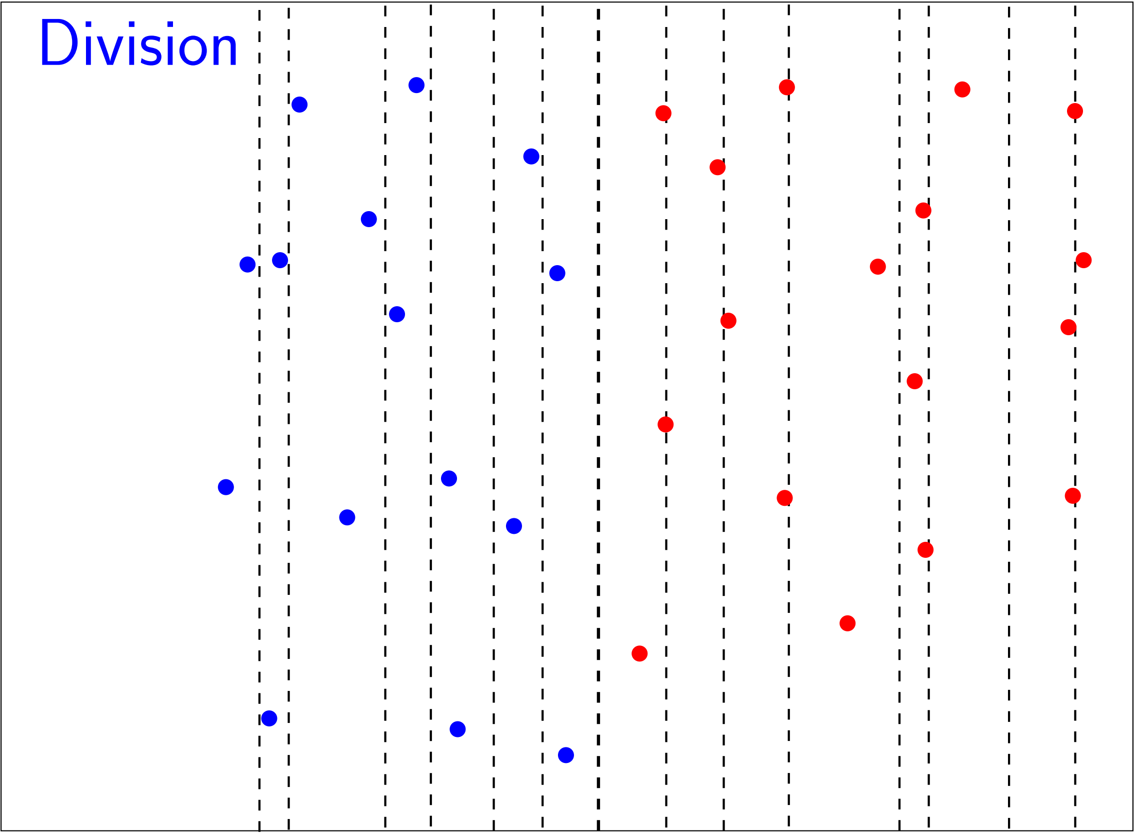

Division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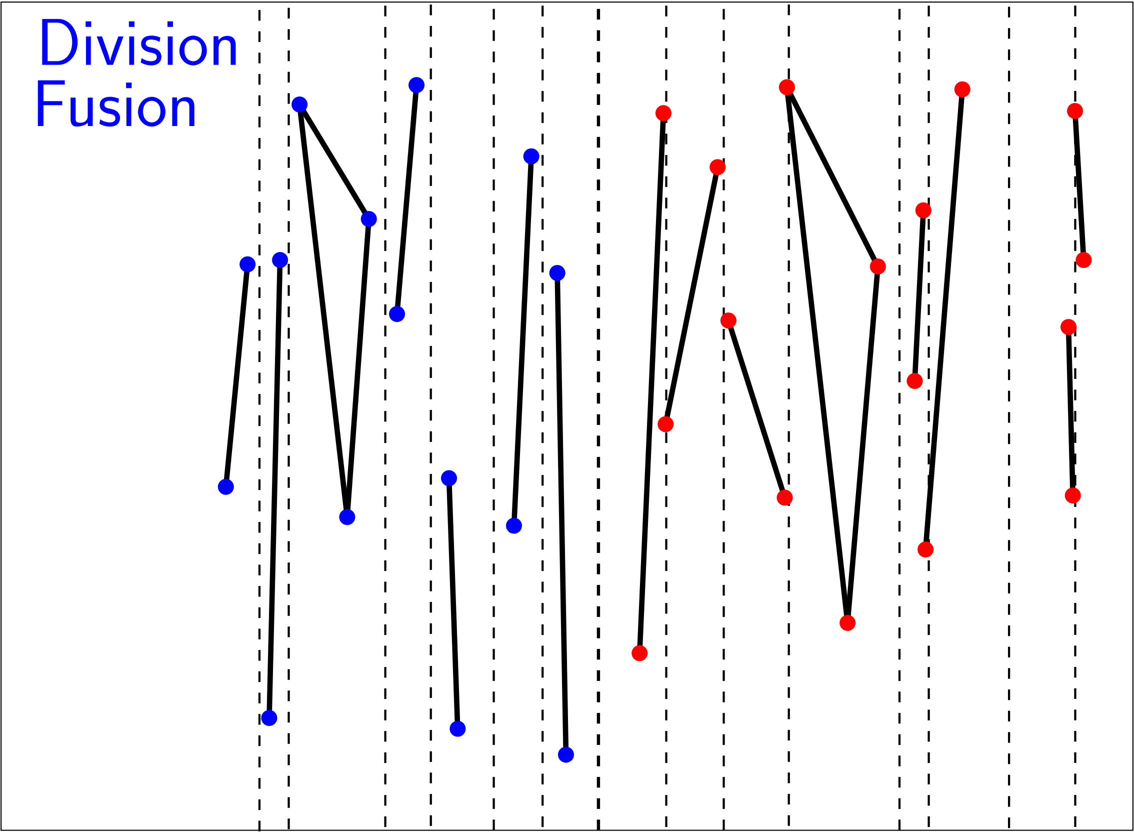
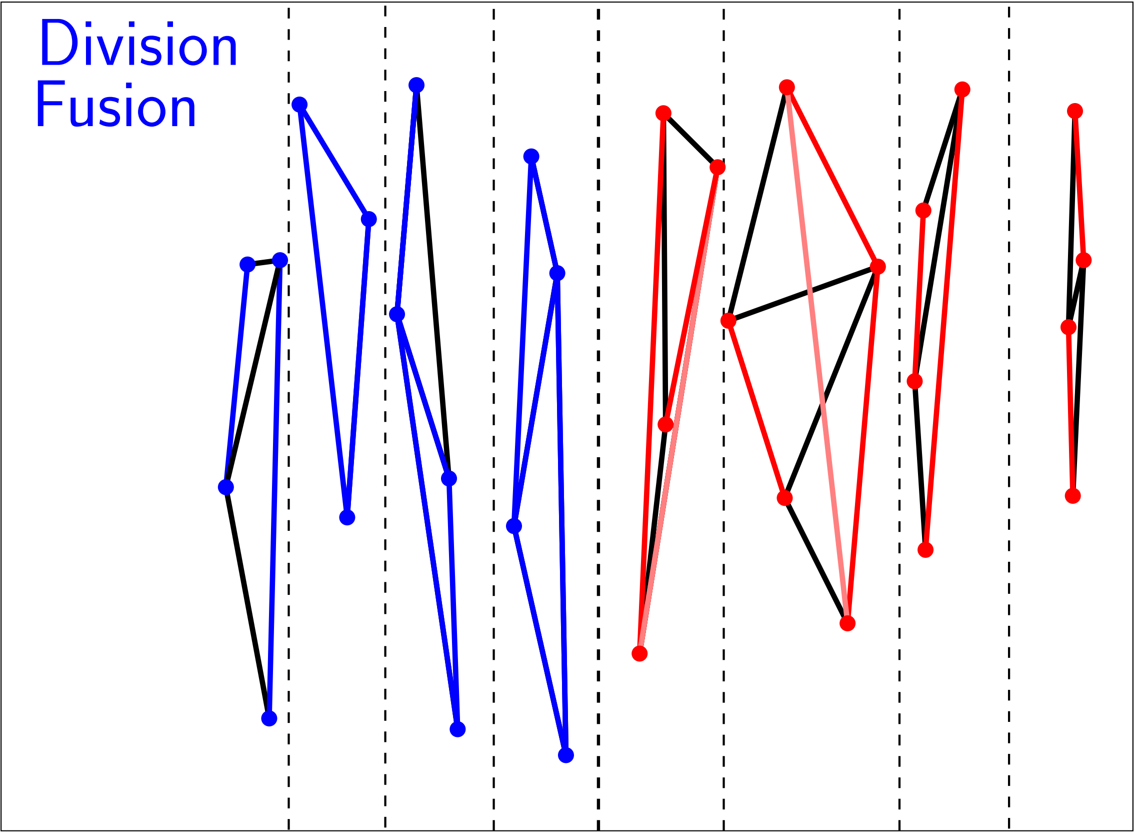
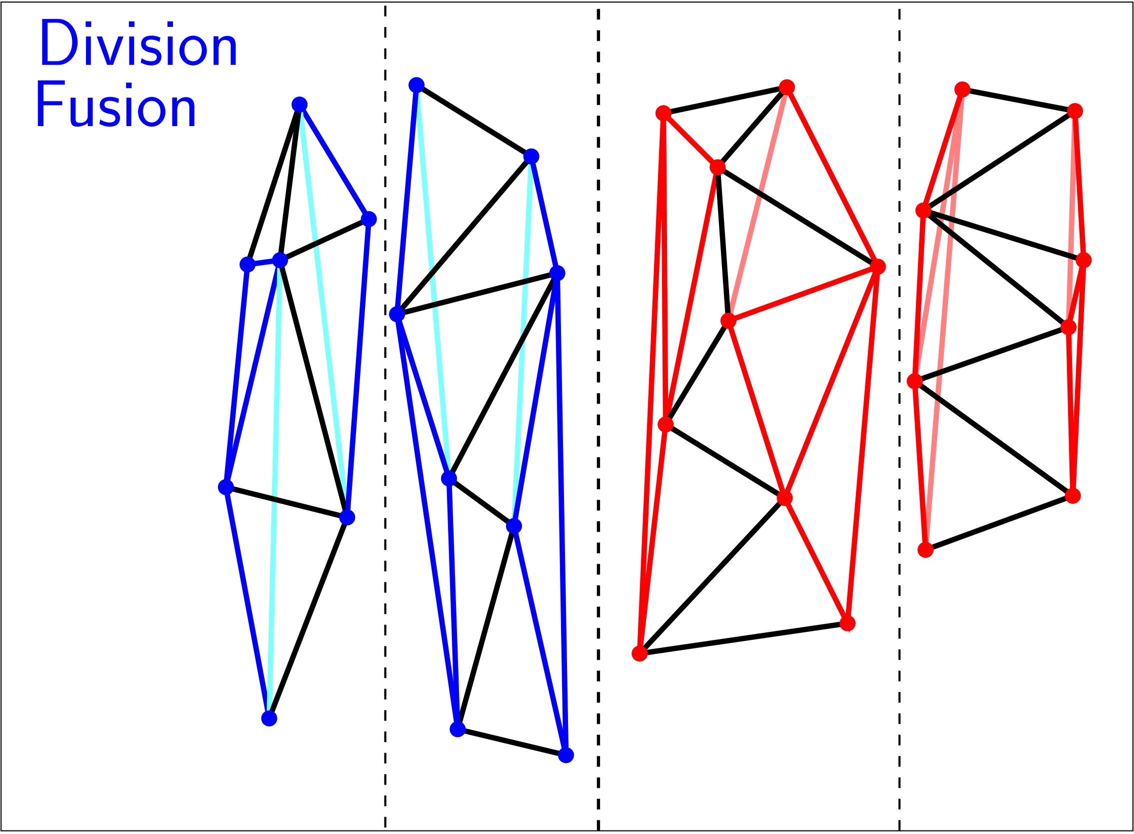
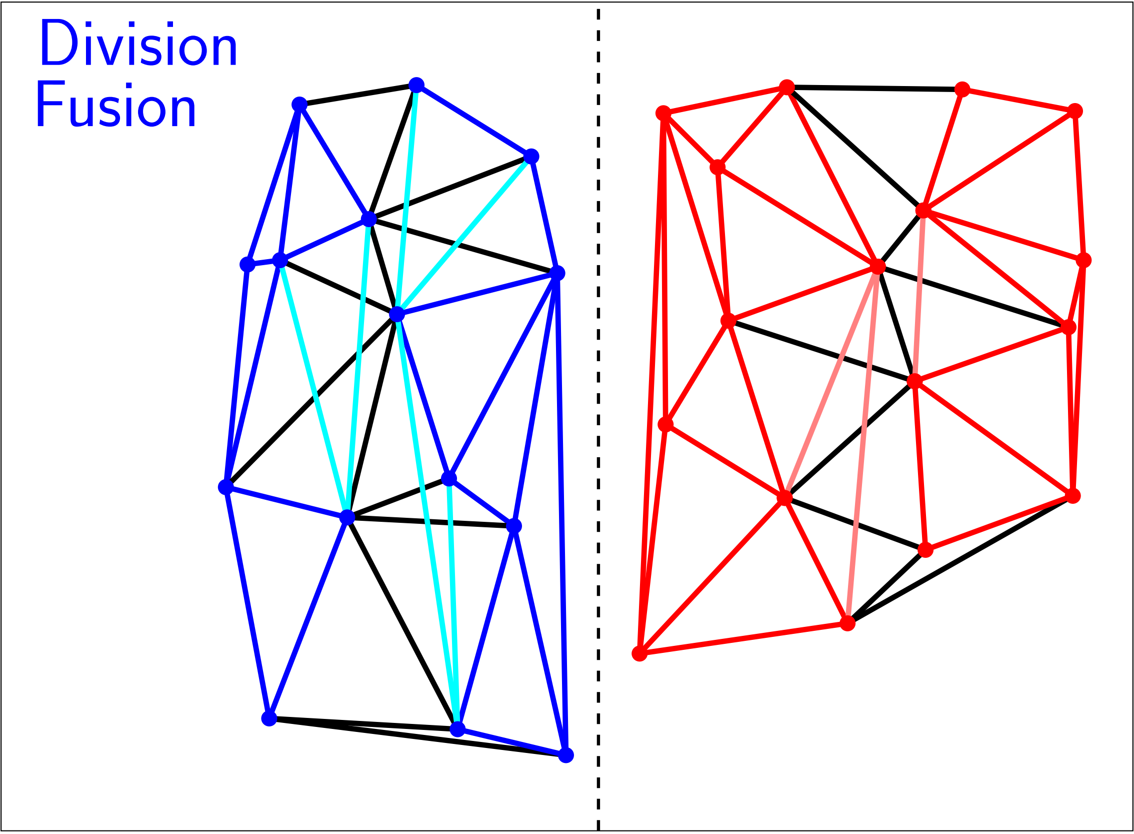
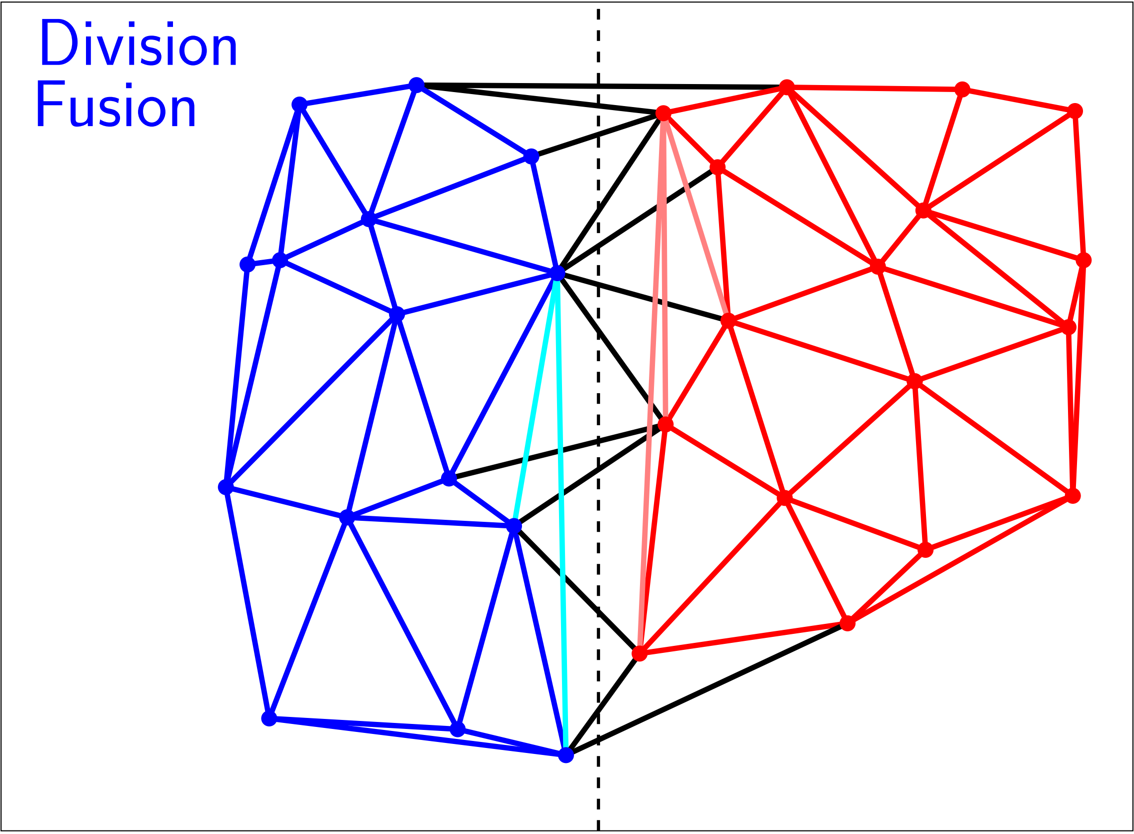
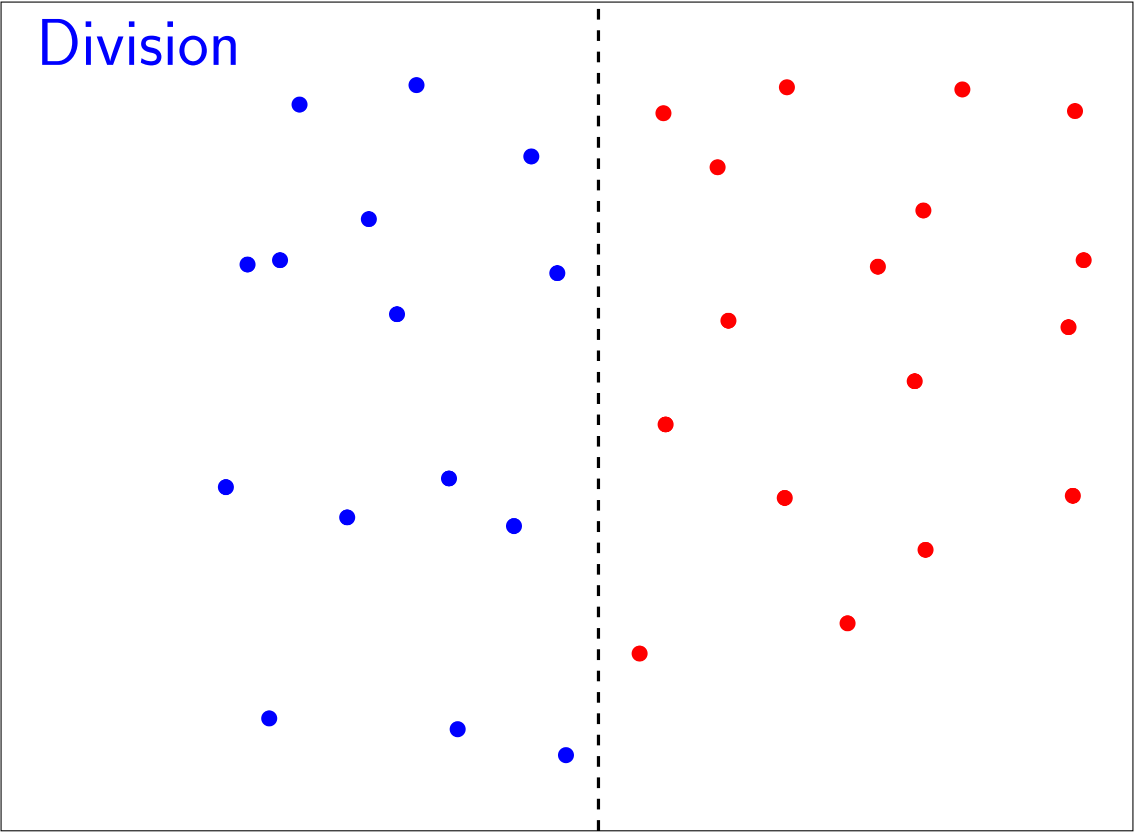

Division
Fusion


Division
Fusion


Division
Fusion


Division
Fusion


Division
Fusion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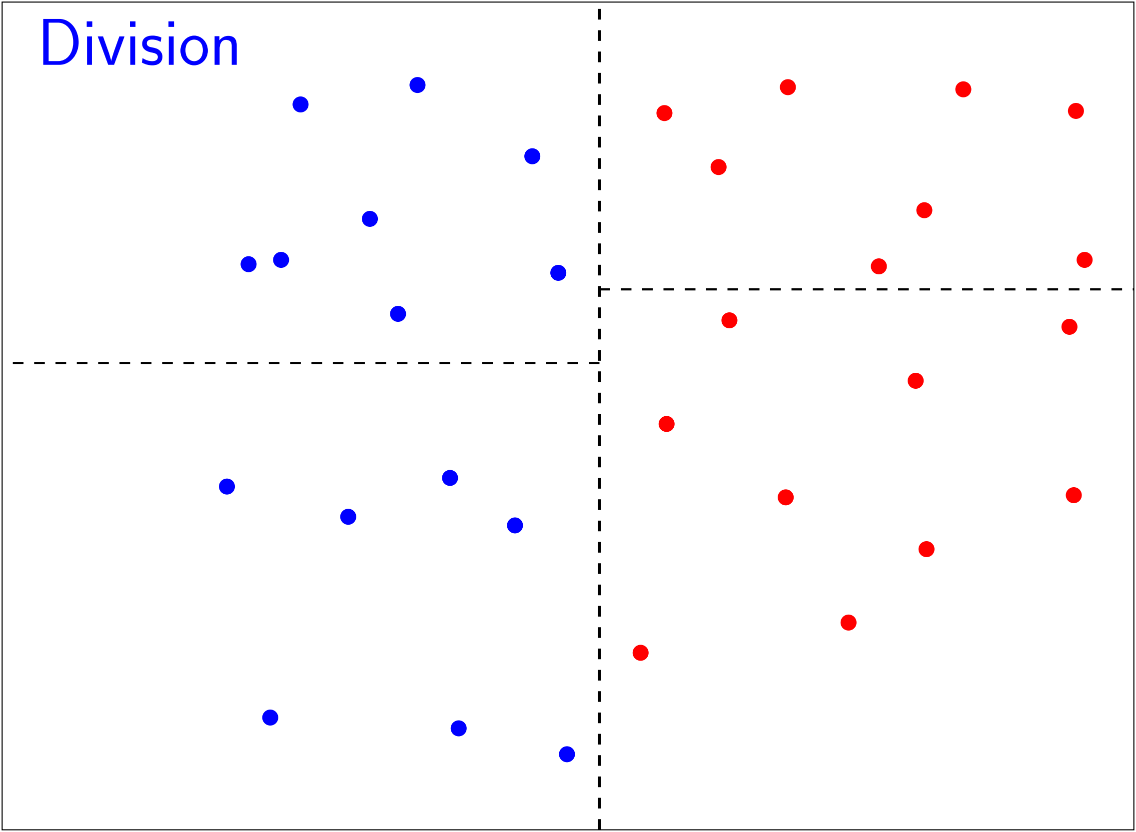
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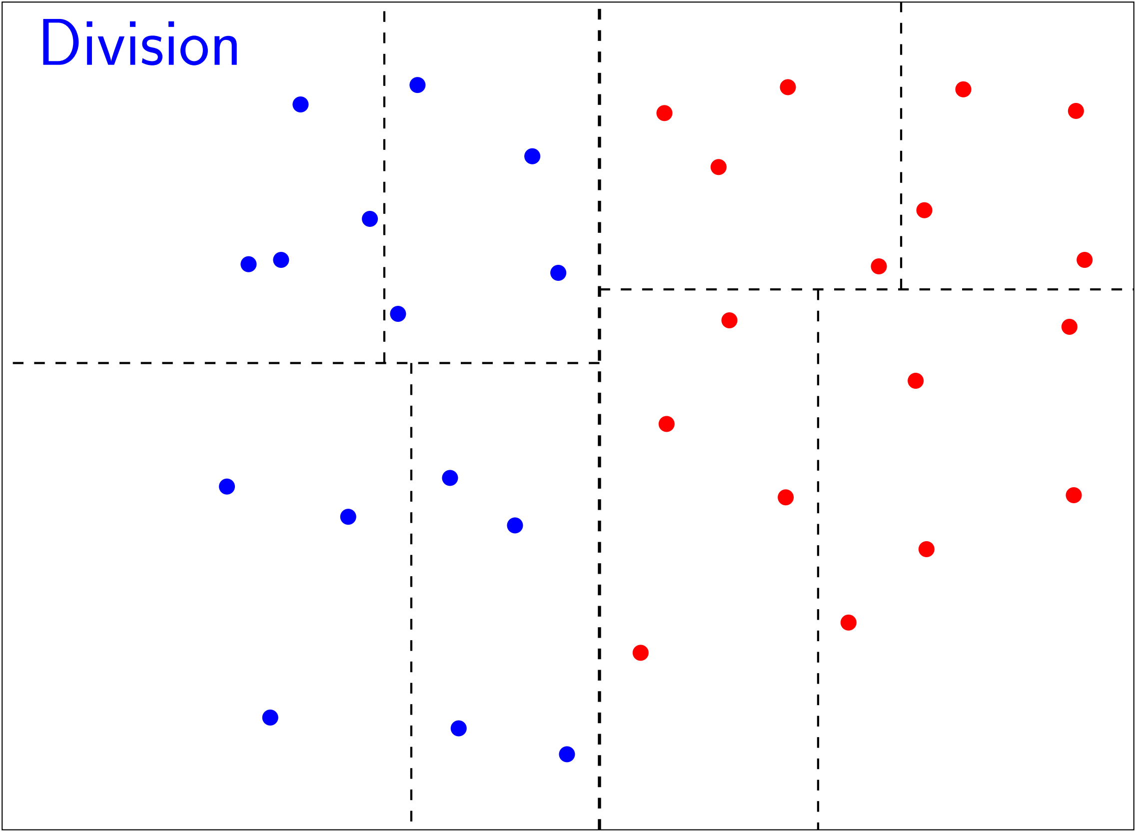
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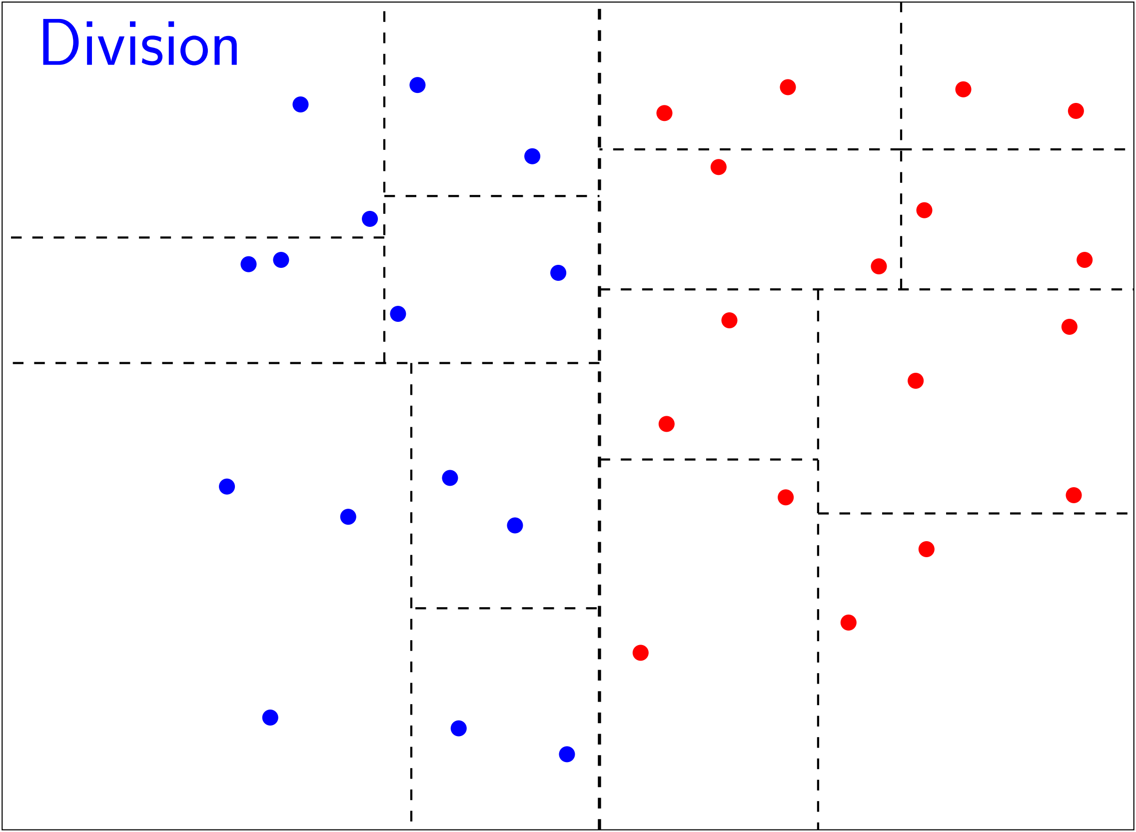
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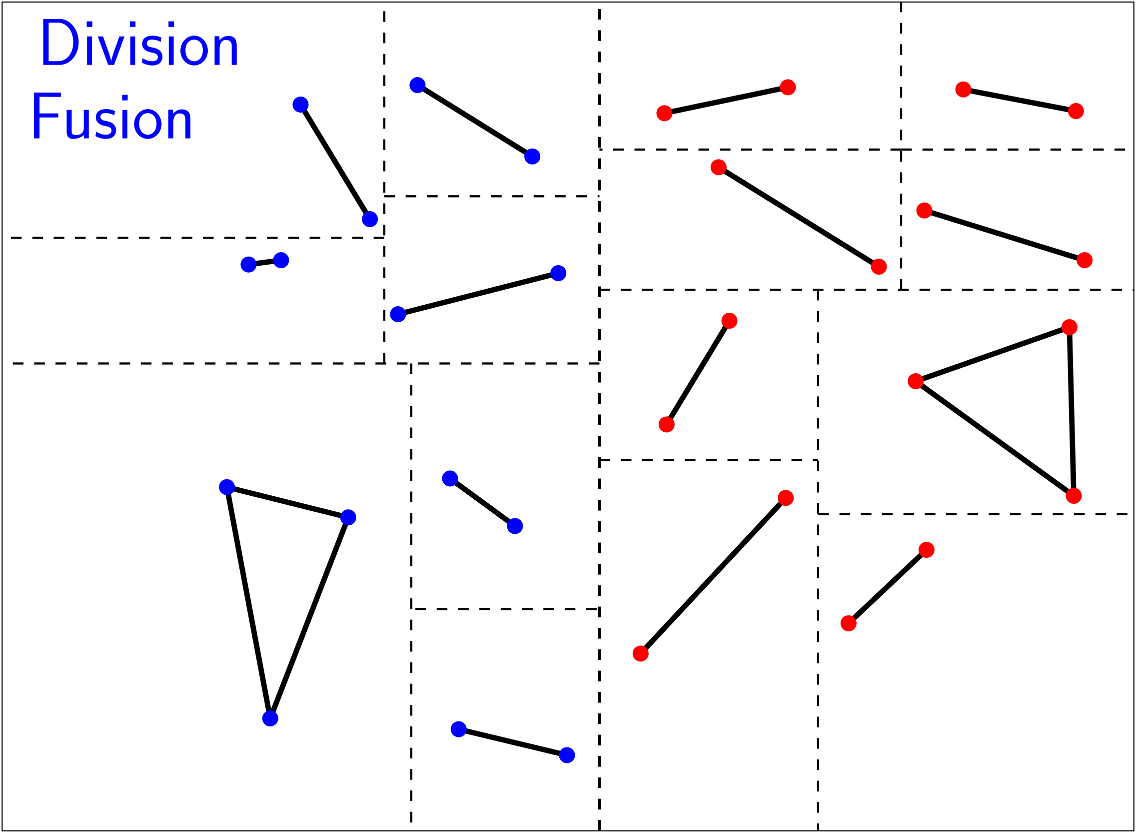
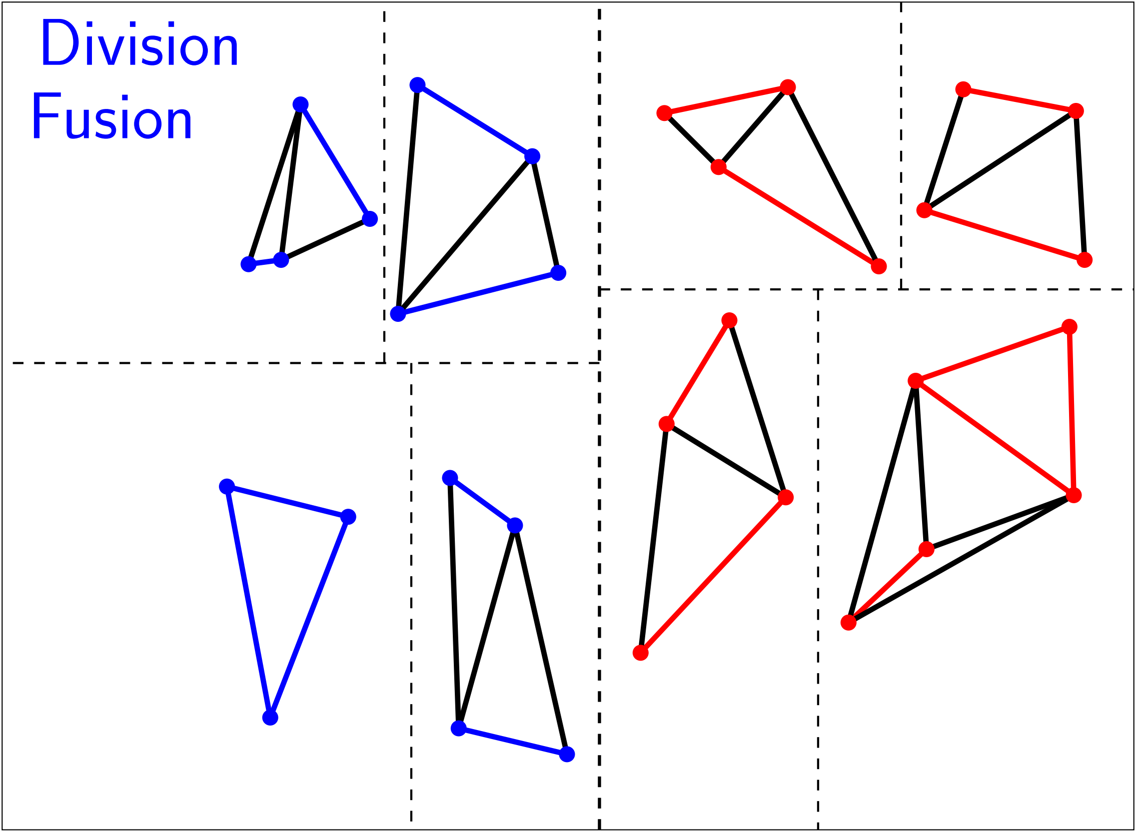

Division
Fusion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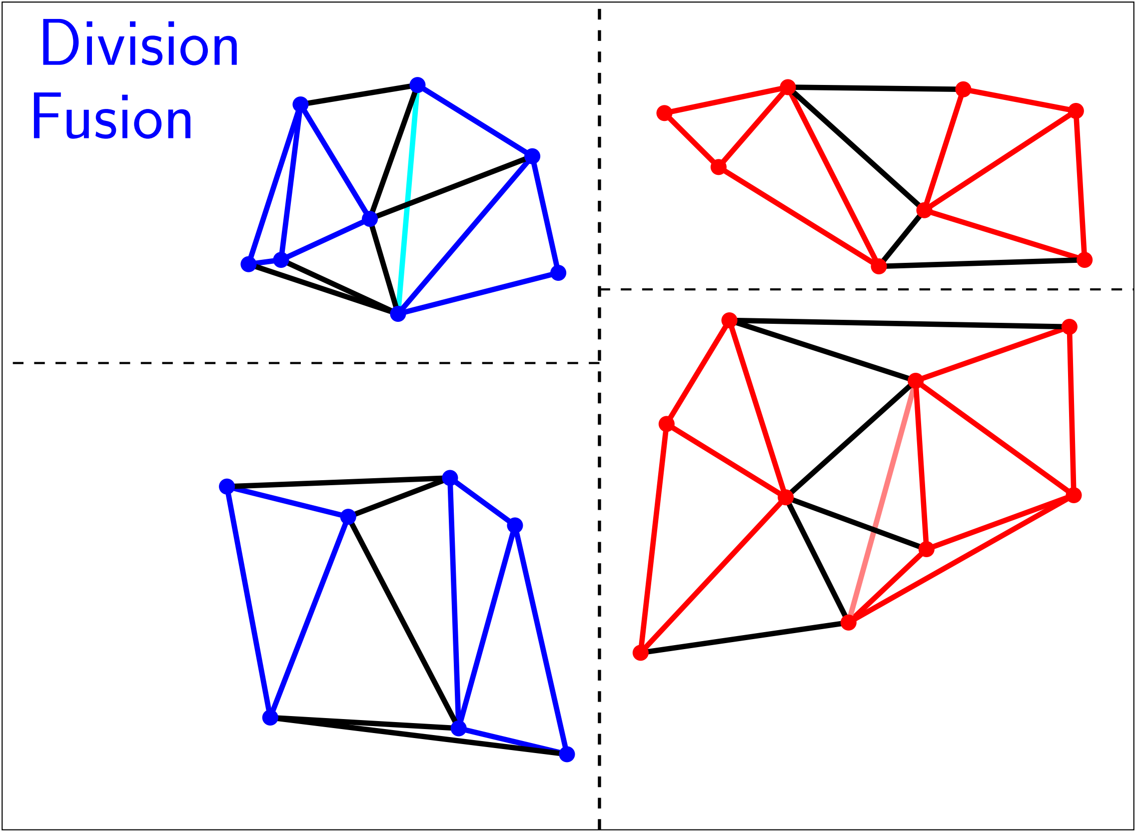
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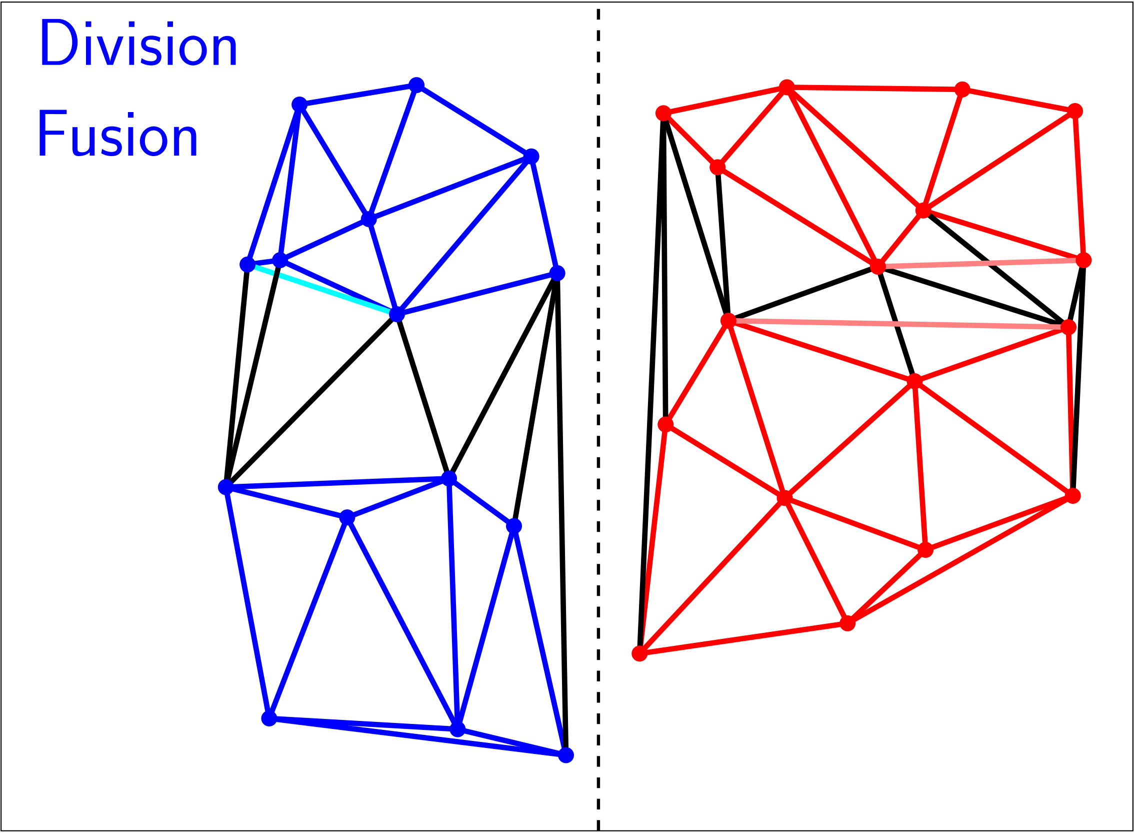
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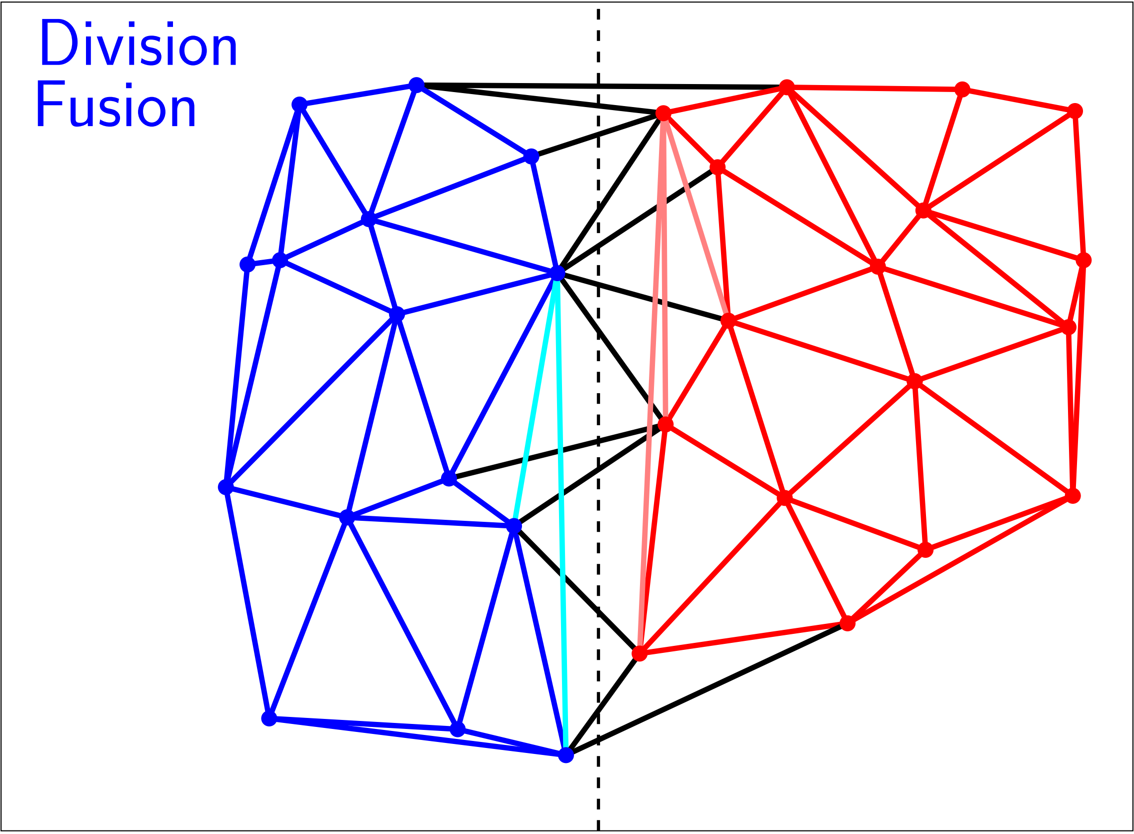
Fusion


Division
Fusion

Complexité

Complexité

Fusion

Complexité

Fusion $O(n)$

Complexité

Fusion $O(n)$

Division ?

Complexité

Fusion $O(n)$

Division ? médian ?

Complexité

Fusion $O(n)$

Division ? médian ?

plus difficile

Complexité

Fusion $O(n)$

Division ? médian ?

$O(n)$

Complexité

		Points aléatoires
Fusion	$O(n)$	$O(\sqrt{n})$
Division ?	médian ? $O(n)$	$O(1)$

$$f(n) = O(\sqrt{n}) + 2f(\tfrac{n}{2})$$

$$f(n) = \sqrt{n} + 2f(\tfrac{n}{2})$$

$$\sqrt{n} + 2\left(\sqrt{\tfrac{n}{2}} + f(\tfrac{n}{4})\right)$$

$$\sqrt{n} + 2\left(\sqrt{\tfrac{n}{2}} + 2\left(\sqrt{\tfrac{n}{4}} + f(\tfrac{n}{8})\right)\right)$$

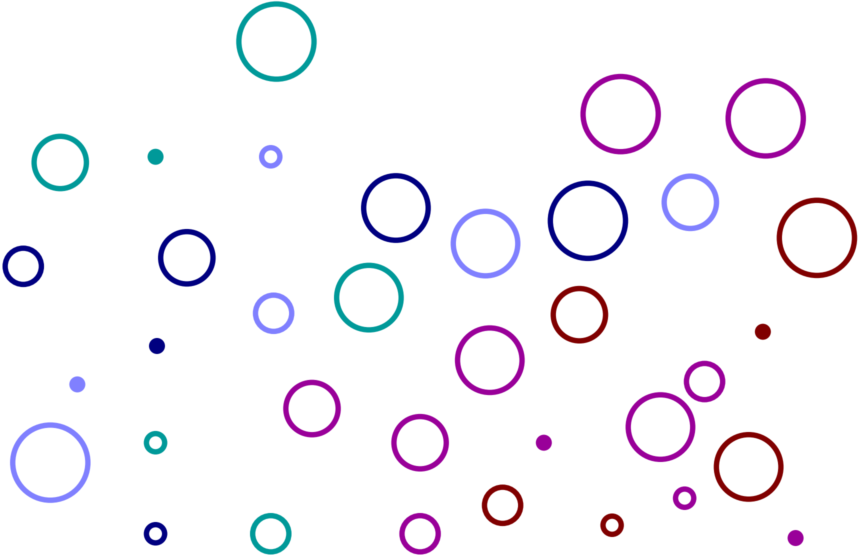
$$\underbrace{\sqrt{n} + 2\sqrt{\tfrac{n}{2}} + \ldots + \tfrac{n}{2}\sqrt{2} + n\sqrt{1}}_{\log_2 n}$$

$$f(n) \leq n + 2^{-\frac{1}{2}}n + \ldots + 2^{-\frac{i}{2}}n + \ldots$$

$$\boxed{f(n) = O(n)}$$

Médian linéaire

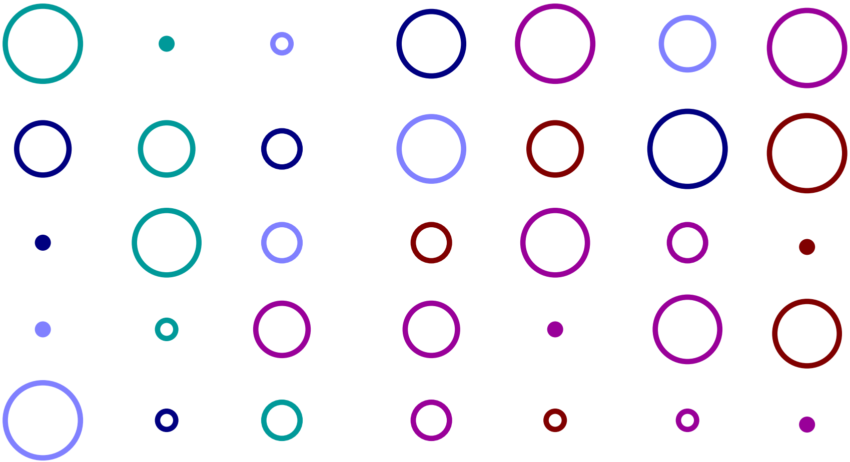
Médian linéaire



n nombres

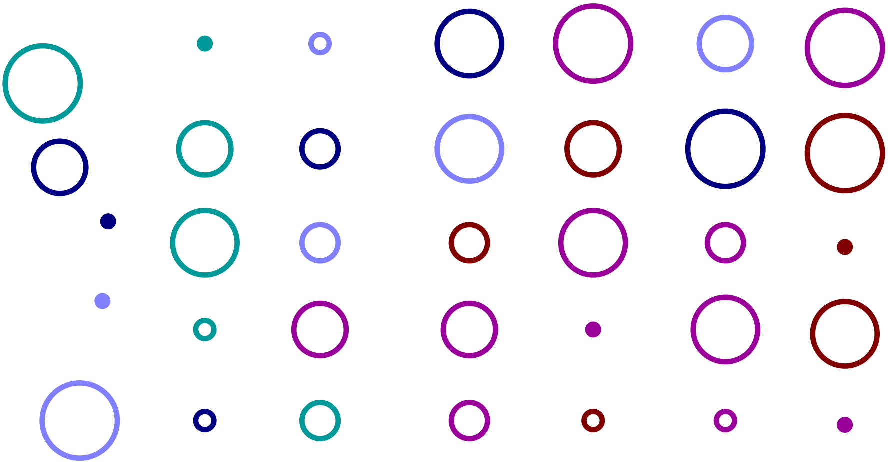
Médian linéaire

des paquets de 5



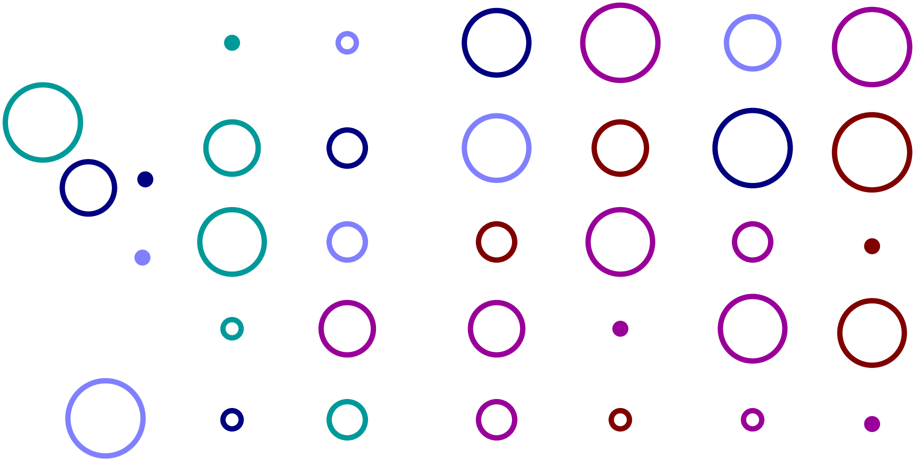
Médian linéaire

des paquets de 5



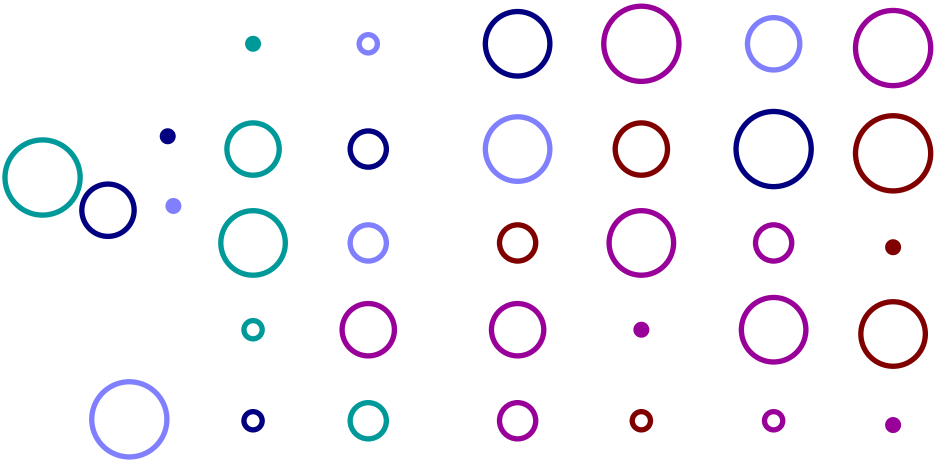
Médian linéaire

des paquets de 5



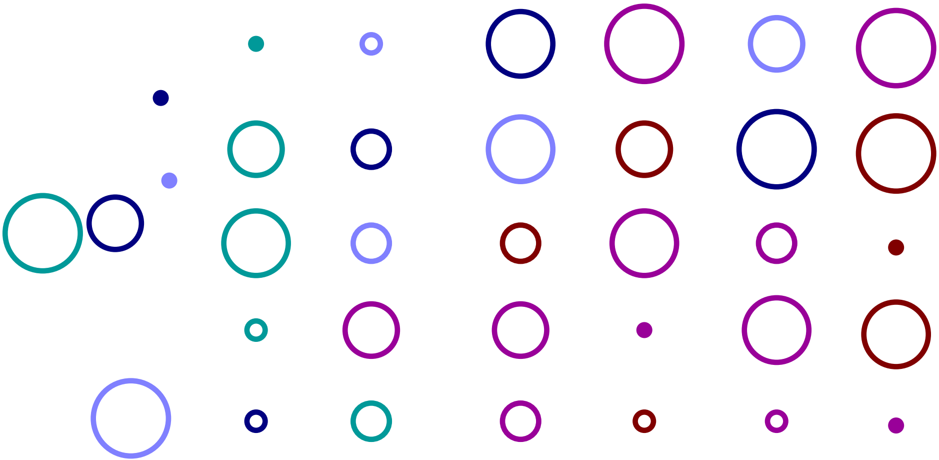
Médian linéaire

des paquets de 5



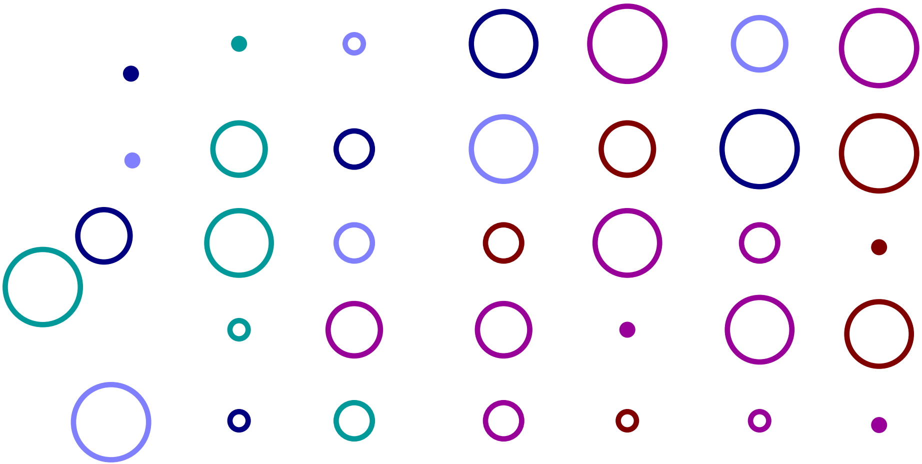
Médian linéaire

des paquets de 5



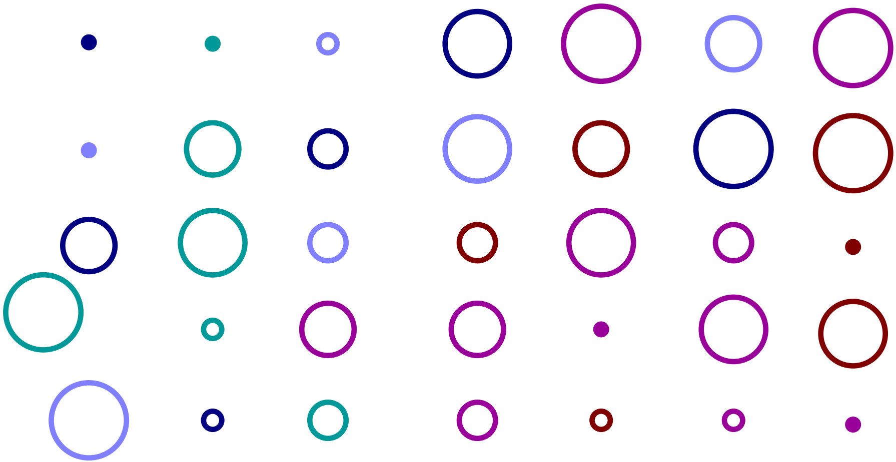
Médian linéaire

des paquets de 5



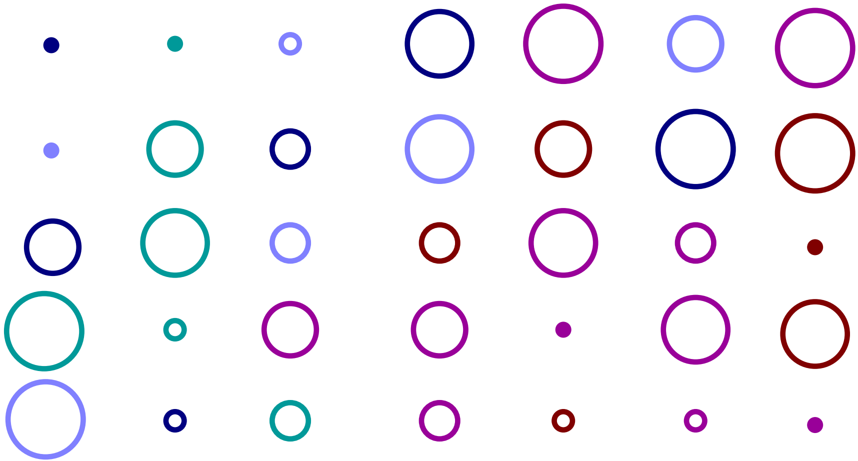
Médian linéaire

des paquets de 5



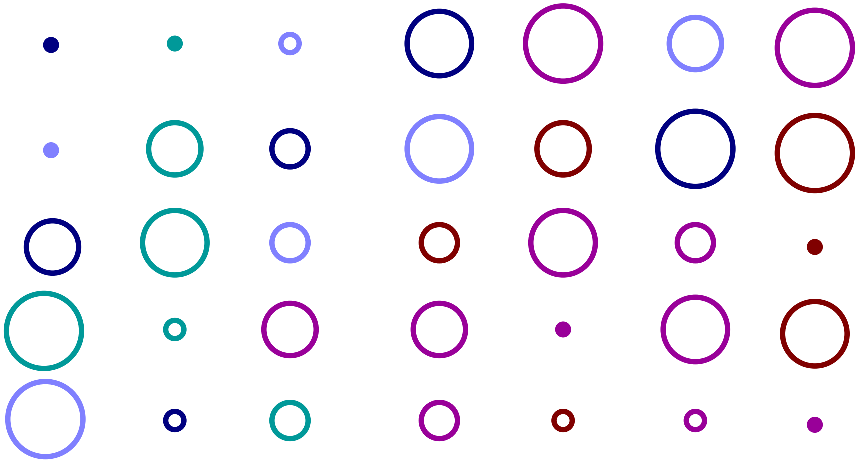
Médian linéaire

des paquets de 5



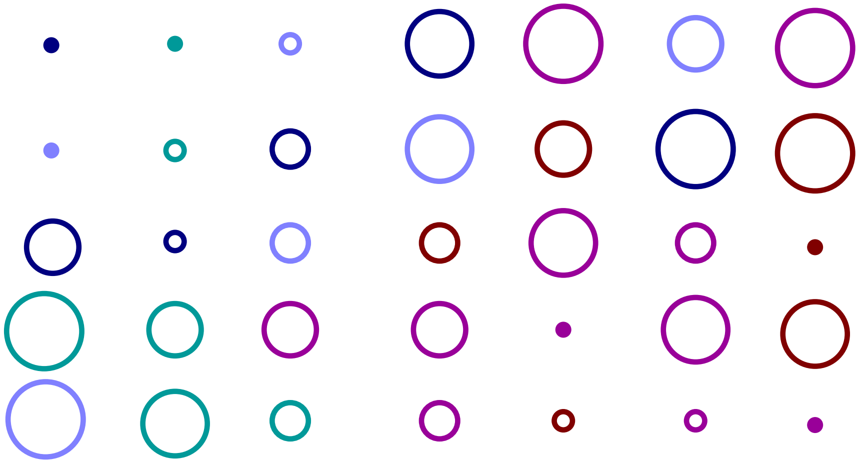
Médian linéaire

médian de 5



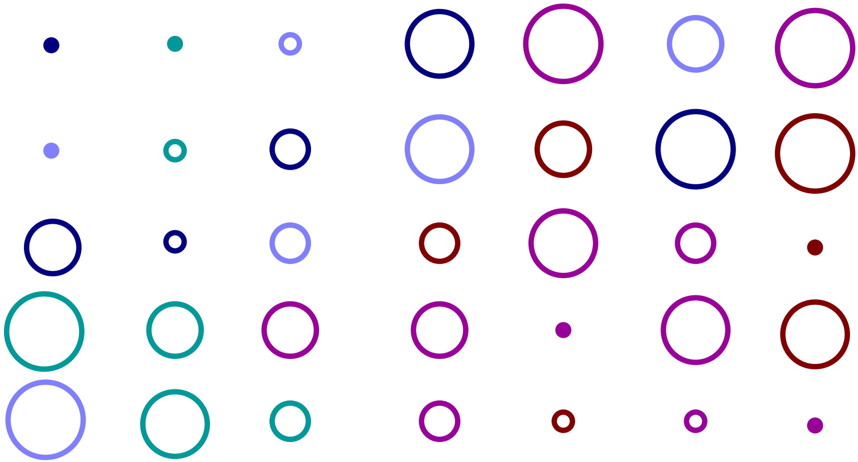
Médian linéaire

médian de 5



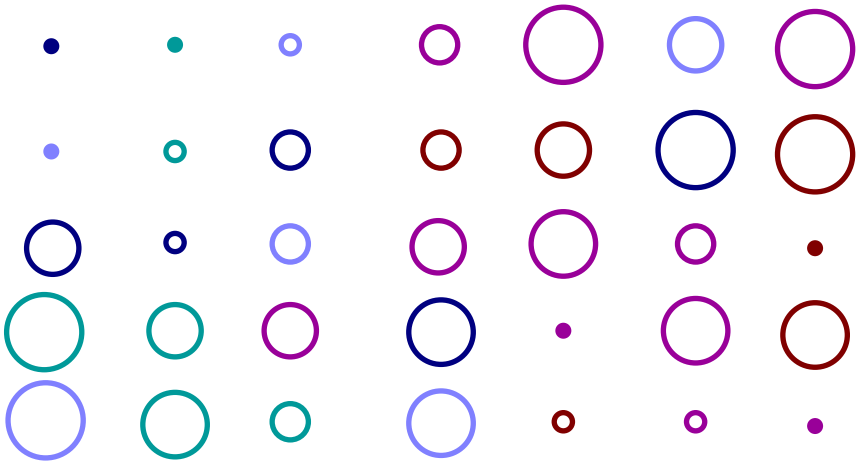
Médian linéaire

médian de 5



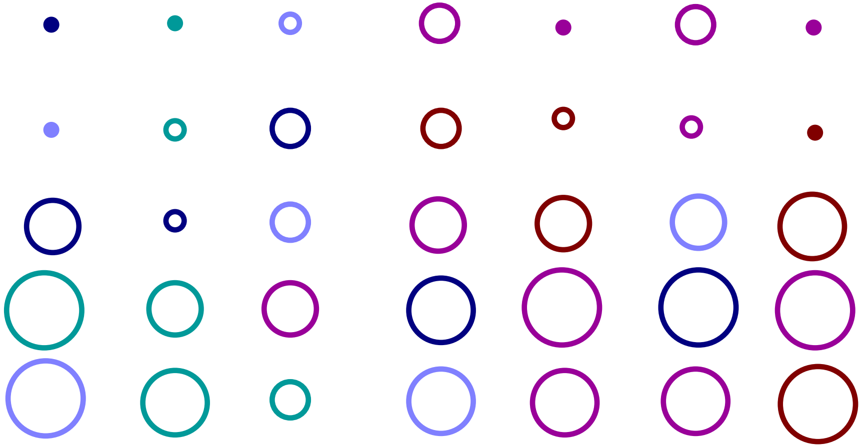
Médian linéaire

médian de 5



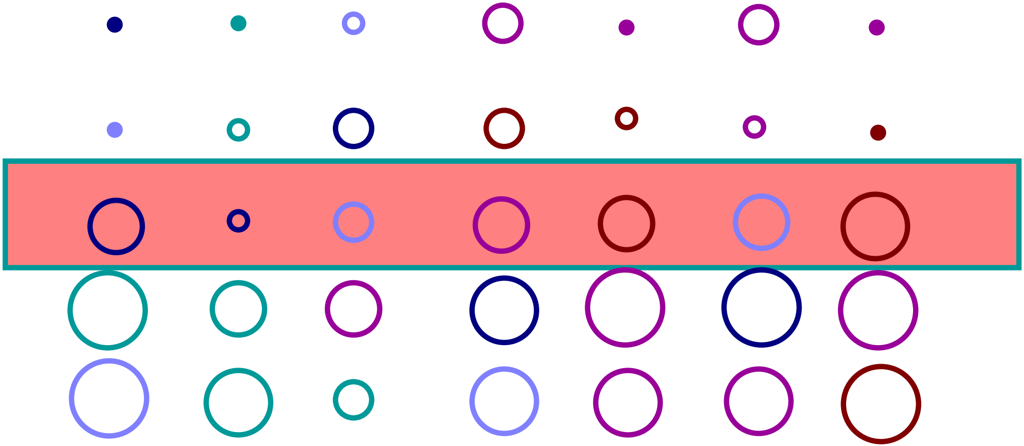
Médian linéaire

médian de 5



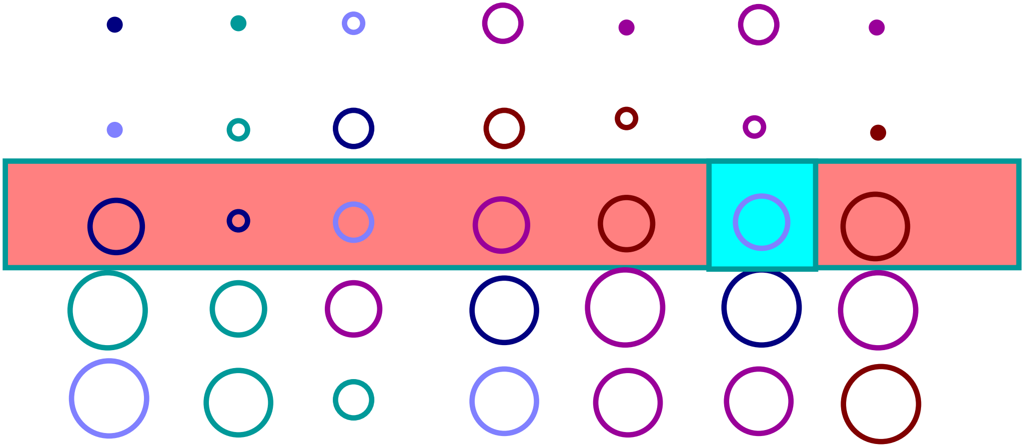
Médian linéaire

médian de $\frac{n}{5}$



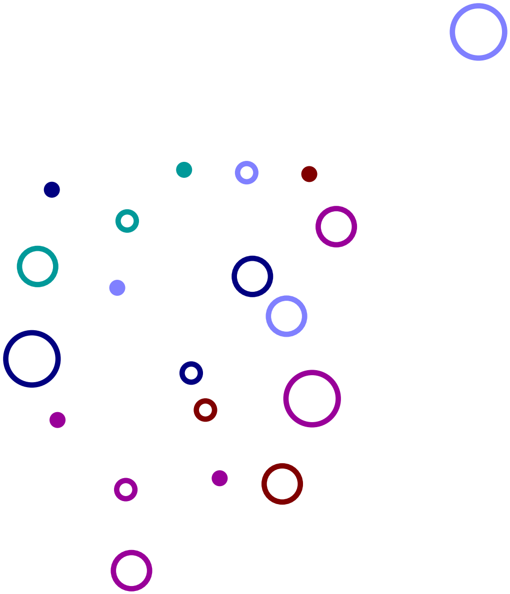
Médian linéaire

médian de $\frac{n}{5}$



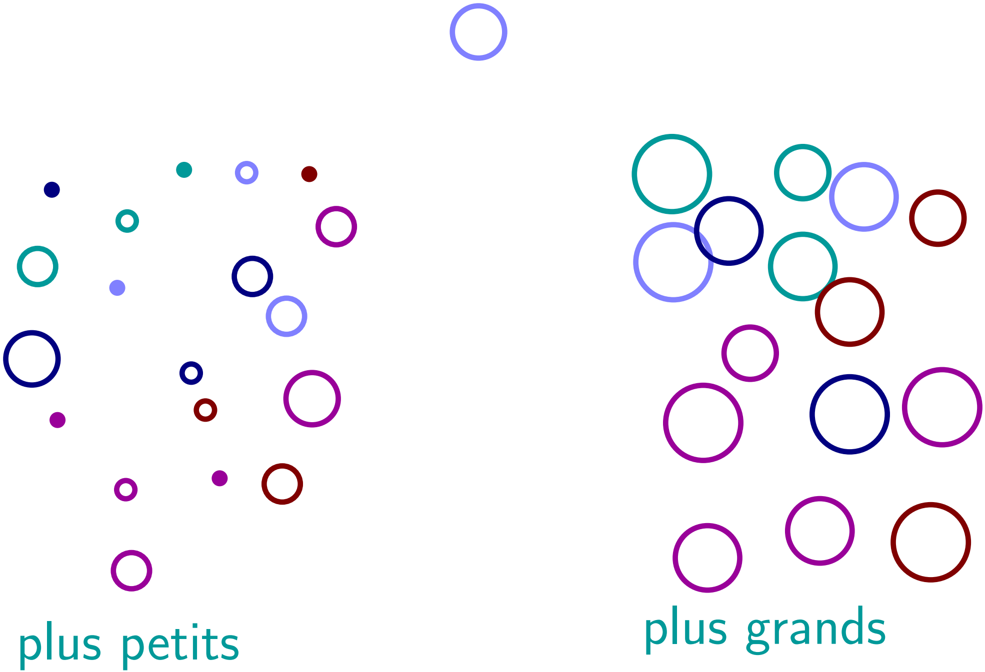
Médian linéaire

Médian linéaire

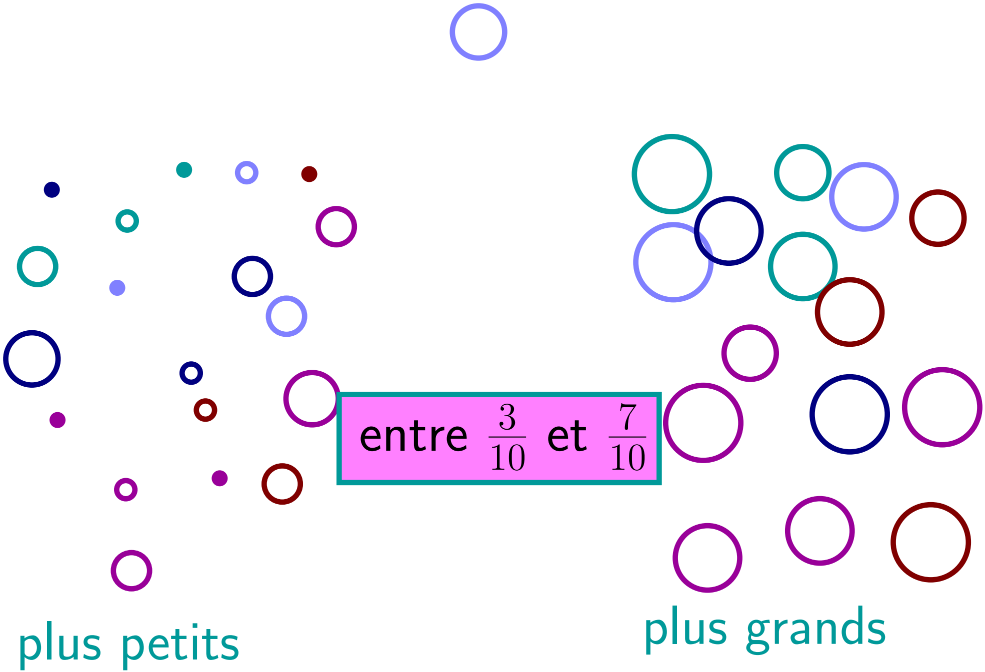


plus petits

Médian linéaire

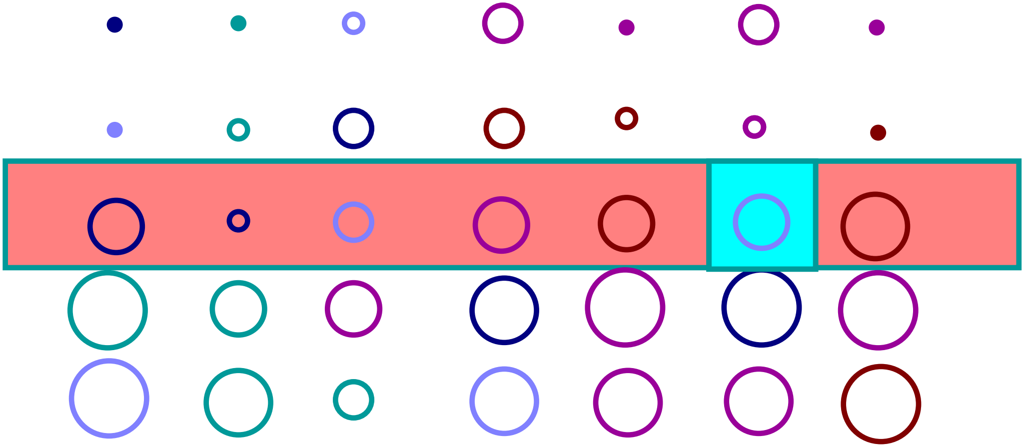


Médian linéaire

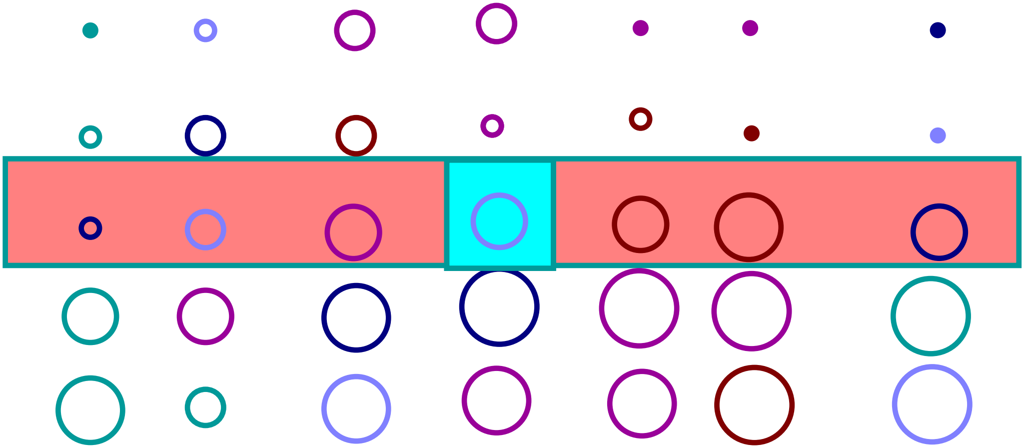


Médian linéaire

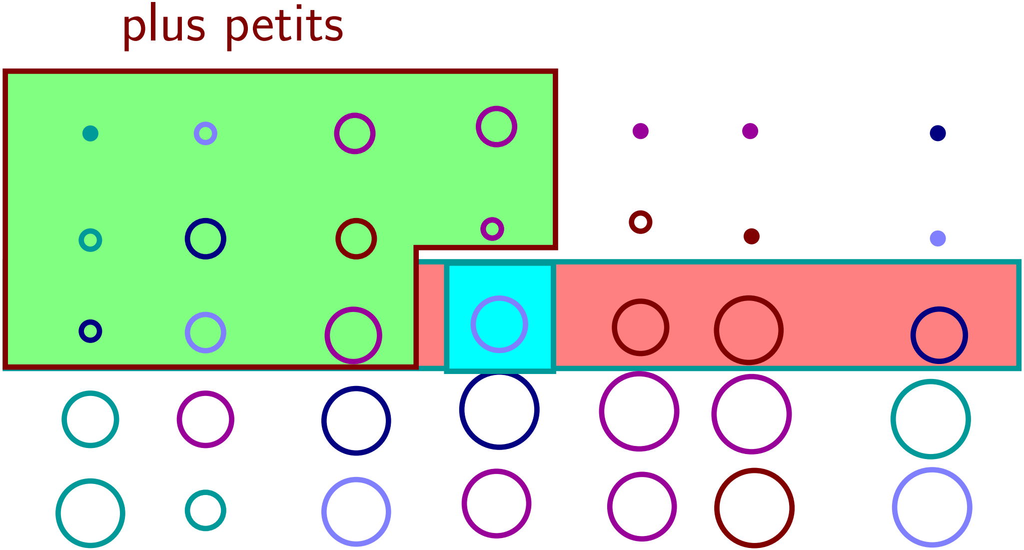
médian de $\frac{n}{5}$



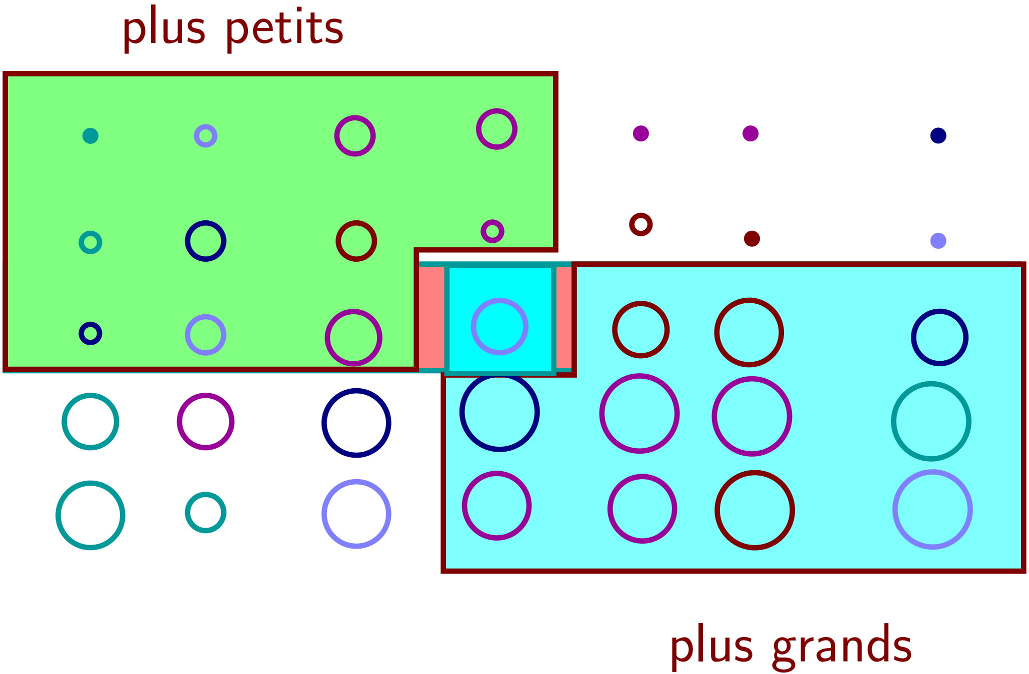
Médian linéaire



Médian linéaire



Médian linéaire



$$f(n) = O(n) + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$f(n) = \; n + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$n + \frac{n}{5} + \frac{7n}{10} + f(\frac{n}{25}) + f(\frac{7n}{50}) + f(\frac{7n}{50}) + f(\frac{49n}{100})$$

$$n + \frac{9}{10}n + \left(\frac{9}{10}\right)^2 n + f(\ldots) + f(\ldots) + \ldots$$

$$f(n) \leq \; 10n\alpha$$

$$f(n) = O(n)$$

$$f(n) = O(n) + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$f(n) = \ n + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$n + \frac{n}{5} + \frac{7n}{10} + f(\frac{n}{25}) + f(\frac{7n}{50}) + f(\frac{7n}{50}) + f(\frac{49n}{100})$$

$$n + \frac{9}{10}n + \left(\frac{9}{10}\right)^2 n + f(\ldots) + f(\ldots) + \ldots$$

$$f(n) \leq \ 10n\alpha \qquad f(n) = O(n)$$

α médian de 5

séparation

$$f(n) = O(n) + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$f(n) = n + f(\frac{n}{5}) + f(\frac{7n}{10})$$

$$n + \frac{n}{5} + \frac{7n}{10} + f(\frac{n}{25}) + f(\frac{7n}{50}) + f(\frac{7n}{50}) + f(\frac{49n}{100})$$

$$n + \frac{9}{10}n + \left(\frac{9}{10}\right)^2 n + f(\ldots) + f(\ldots) + \ldots$$

$$f(n) \leq 10n\alpha$$

$$f(n) = O(n)$$

α

médian de 5

séparation

$$log_2(5!) = 7$$

C'est tout pour aujourd'hui