Algorithmic Geometry of Triangulations
January 26-30, 2015 – INRIA Sophia Antipolis

EXERCISES

1. Power diagrams

Exercice 1. *Delaunay predicate*

Let $\mathcal{S}$ be a hypersphere of $\mathbb{R}^d$ passing through $d+1$ points $p_0, \ldots, p_d$. Show that a point p_{d+1} of $\mathbb{R}^d$ lies on $\mathcal{S}$, in the interior of the ball $B_{\mathcal{S}}$ bounded by $\mathcal{S}$ or outside of $B_{\mathcal{S}}$ depending whether the determinant of the $(d+2)\times(d+2)$ matrix

$$\texttt{in\_sphere}(p_0, \ldots, p_{d+1}) = \begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}$$

is 0, negative or positive.

Exercice 2. *k-order diagrams*

1. Recall the definitions of k-order Voronoi and power diagrams.
2. Show that a k-order Voronoi diagram is a power diagram and recall (or propose) an algorithm to construct a $(k+1)$-order Voronoi diagram from a k-order Voronoi diagram.

   We now focus on the $(n-1)$-order Voronoi diagram of a set P of n points.
3. This diagram is also called *farthest* Voronoi diagram. Justify this name.
4. Prove the following properties of the farthest Voronoi diagram :
   - p_i is a vertex of the convex hull of P if and only if its farthest Voronoi region is non-empty
   - the farthest Voronoi diagram is a tree
5. Show that the center of the smallest sphere enclosing P is either a vertex of the farthest Voronoi diagram or the intersection of an edge (bisector of two sites A and B) of the farthest Voronoi diagram and $[AB]$.

Exercice 3. *Möbius diagrams*

1. Recall the definitions of affine and Möbius diagrams.
2. Show that the intersection of a power diagram with a hyperplane is a power diagram.
3. Prove the following lemma (also called *linearization lemma*) : Given a set of weighted points $\{p_i\}_i$ in $\mathbb{R}^d$, we can associate to each p_i a hypersphere Σ_i of R^{d+1} so that the faces of the Möbius diagram of $\{p_i\}_i$ are obtained by projecting vertically the faces of the restriction of the power diagram of the paraboloid $\mathcal{P} : x_{d+1} = x^2$.

2. α-Shapes, Union of balls

Exercice 4. We consider a set $B = \{b_i, i = 1, \dots, n\}$ of n balls of $\mathbb{R}^d$ and use the following notations :

— ∂b_i denotes the sphere bounding b_i,

— $U(B)$ denotes the union of balls in B,

— $\partial U(B)$ denotes the boundary of $U(B)$,

— $\mathrm{Vor}(B)$ denotes the power diagram of B,

— $V(b_i)$ denotes the cell of b_i in $\mathrm{Vor}(B)$

1. Prove the following equalities

$$\forall b_i \in B, V(b_i) \cap b_i = V(b_i) \cap U(B), \quad (1)$$

$$\forall b_i \in B, V(b_i) \cap \partial b_i = V(b_i) \cap \partial U(B), \quad (2)$$

$$U(B) = \cup_i (V(b_i) \cap b_i), \quad (3)$$

$$\partial U(B) = \cup_i (V(b_i) \cap \partial b_i). \quad (4)$$

2. Using the facts proved in the previous question, show that, in the space of dimension 2, the union of n balls has a linear complexity, i.e. that that the number of vertices and arcs on $\partial U(B)$ is $O(n)$. Propose an algorithm to compute the union of n-balls in $\mathbb{R}^2$ in $O(n \log n)$.

   Hereafter, we denote by :

   — $\mathrm{Reg}(B)$, the weighted Delaunay triangulation of B

   — $\mathcal{W}_\alpha(B)$, the α-shape of B for a given value of the parameter α

   — $\mathcal{W}_0(B)$, the α-shape of B for the value $\alpha = 0$

   For any subset $T \subset B$ with cardinality less than $(d+1)$, we note $\sigma(T)$ the simplex whose vertices are the centers of ball in T, and by $f(T)$ the intersection (which may be empty) of the spheres bounding the balls in T : $f(T) = \cap_{b_i \in T} \partial b_i$.

3. Recall the definition of the α-shape $\mathcal{W}_\alpha(B)$.
4. Show that a $(d-1)$-simplex $\sigma(T)$ of $\mathrm{Reg}(B)$ belongs to the boundary $\partial \mathcal{W}_\alpha(B)$ of $\mathcal{W}_\alpha(B)$ if and only if there is a ball with squared radius α orthogonal to any ball in T and further than orthogonal to any ball of $B \backslash T$.

   hint : Consider the pencil $L(T)$ of balls that are orthogonal to all the balls in T.
5. Equation 3 defines a cover of the union of balls $U(B)$. Show that $\mathcal{W}_0(B)$ is a realization of the nerve of this cover. i.e. that a simplex $\sigma(T)$ of $\mathrm{Reg}(B)$ belongs to $\mathcal{W}_0(B)$ if and only if the intersection $\cap_{b \in T} V(b) \cap b$ is non empty.
6. Show that a simplex $\sigma(T)$ of $\mathrm{Reg}(B)$ belongs to the boundary $\partial \mathcal{W}_0(B)$ if and only if $f(T) \cup \partial U(B)$ is non-empty.