# Thick Triangulations

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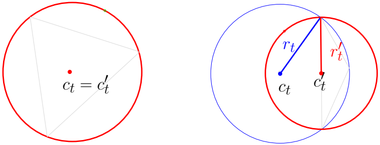
# Some optimality properties of Delaunay triangulations

Among all possible triangulations of $\mathcal{P}$, $\mathrm{Del}(\mathcal{P})$

1. (2d) maximizes the smallest angle [Lawson]
2. (2d) Linear interpolation of $\{(p_i, f(p_i))\}$ that minimizes [Rippa]

   $R(T) = \sum_i \int_{T_i} \left( \left(\frac{\partial \phi_i}{\partial x}\right)^2 + \left(\frac{\partial \phi_i}{\partial y}\right)^2 \right) dx\, dy$ (Dirichlet energy)

   $\phi_i =$ linear interpolation of the $f(p_j)$ over triangle $T_i \in T$
3. minimizes the radius of the maximal smallest ball enclosing a simplex ) [Rajan]



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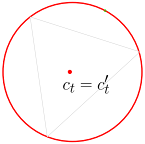


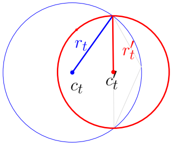

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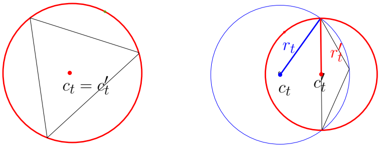


# Optimizing the angular vector ($d = 2$)

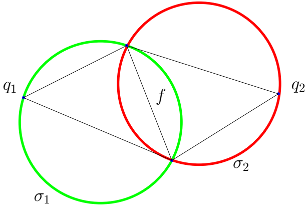
Angular vector of a triangulation $T(\mathcal{P})$

$$\text{ang}\,(T(\mathcal{P})) = (\alpha_1, \ldots, \alpha_{3t}), \qquad \alpha_1 \leq \ldots \leq \alpha_{3t}$$

Optimality Any triangulation of a given point set $\mathcal{P}$ whose angular vector is maximal (for the lexicographic order) is a Delaunay triangulation of $\mathcal{P}$

Good for matrix conditioning in FE methods

# Local characterization of Delaunay complexes




Pair of regular simplices

$\sigma_2(q_1) \geq 0 \quad \text{and} \quad \sigma_1(q_2) \geq 0$

$\Leftrightarrow \hat{c}_1 \in h^+_{\sigma_2}$ and $\hat{c}_2 \in h^+_{\sigma_1}$

Theorem A triangulation $T(P)$ such that all pairs of simplexes are regular is a Delaunay triangulation $\mathrm{Del}(P)$

Proof The PL function whose graph $G$ is obtained by lifting the triangles is locally convex and has a convex support

$$\Rightarrow \quad G = \mathrm{conv}^-(\hat{Q}) \quad \Rightarrow \quad T(Q) = \mathrm{Del}(Q)$$

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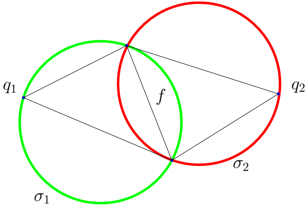


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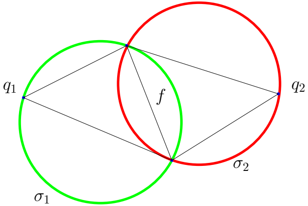


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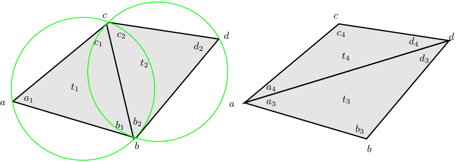
# Constructive proof using flips




While $\exists$ a non regular pair $(t_3, t_4)$

/* $t_3 \cup t_4$ is convex */

replace $(t_3, t_4)$ by $(t_1, t_2)$

Regularize $\Leftrightarrow$ improve $\text{ang}\,(T(\mathcal{P}))$

$\text{ang}\,(t_1, t_2) \geq \text{ang}\,(t_3, t_4)$

$$a_1 = a_3 + a_4,\ d_2 = d_3 + d_4,$$
$$c_1 \geq d_3,\ \ b_1 \geq d_4,\ \ b_2 \geq a_4,\ \ c_2 \geq a_3$$

- The algorithm terminates since the number of triangulations of $\mathcal{P}$ is finite and $\text{ang}(T(\mathcal{P}))$ cannot decrease

- The obtained triangulation is a Delaunay triangulation of $\mathcal{P}$ since all its edges are regular

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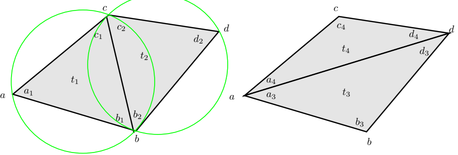


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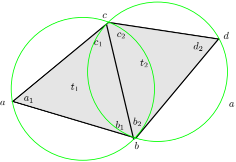


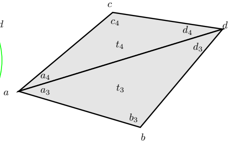


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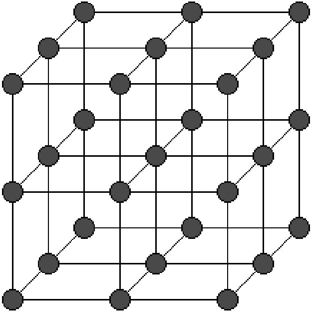
# Flat simplices may exist in higher dimensional DT



- Each square face can be circumscribed by an empty sphere
- This remains true if the grid points are slightly perturbed therefore creating flat tetrahedra

# The long quest for thick triangulations

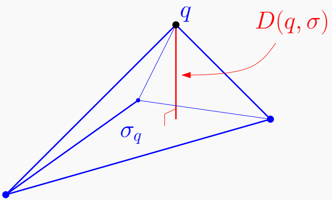
Differential Topology [Cairns], [Whitehead], [Whitney], [Munkres]

Differential Geometry [Cheeger et al.]

Geometric Function Theory [Peltonen], [Saucan]

# Simplex quality

## Altitudes




If $\sigma_q$, the face opposite $q$ in $\sigma$ is protected, The *altitude* of $q$ in $\sigma$ is

$$D(q,\sigma) = d(q, \mathrm{aff}(\sigma_q)),$$

where $\sigma_q$ is the face opposite $q$.

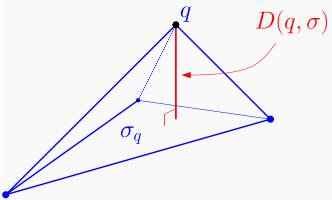
## Definition (Thickness [Cairns, Whitney, Whitehead et al.] )

The *thickness* of a $j$-simplex $\sigma$ with diameter $\Delta(\sigma)$ is

$$\Theta(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p\in\sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

# Simplex quality

## Altitudes

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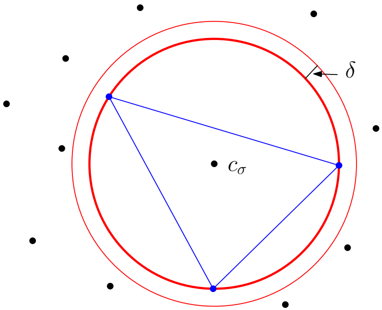
# Protection




$\delta$-protection We say that a Delaunay simplex $\sigma \subset L$ is $\delta$-protected if

$$\|c_\sigma - q\| > \|c_\sigma - p\| + \delta \qquad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma.$$

# Protection implies thickness

Let P be a $(\varepsilon, \bar{\eta})$-net, i.e.

- $\forall x \in \Omega, \quad d(x, P) \leq \varepsilon$
- $\forall p, q \in P, \quad \|p - q\| \geq \bar{\eta}\varepsilon$

if all $d$-simplices of $\mathrm{Del}(P)$ are $\bar{\delta}\varepsilon$-protected, then the thickness of all Delaunay simplices (of all dimensions) is lower bounded

$$\Theta(\sigma) > \Theta_0 = \frac{\bar{\delta}\,(\bar{\eta}+\bar{\delta})}{8d}$$

(except possibly near the boundary of $\mathrm{conv}(\mathrm{P})$)

# Protection implies thickness

## Proof : Case 1

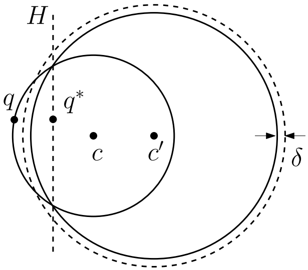



Fig in plane $cc'q$

$c$ and $c'$ are on the same side of $H$

$\sigma$ $d$-simplex of $\mathrm{Del}(P)$

$\sigma = q * \tau \qquad \sigma' \supset \tau \ \wedge \ q \notin \sigma'$

$H = \mathrm{aff}(\tau)$

$q^* \in B' \quad \Rightarrow \quad \|qq^*\| > \delta$

# Protection implies thickness

## Proof : Case 2

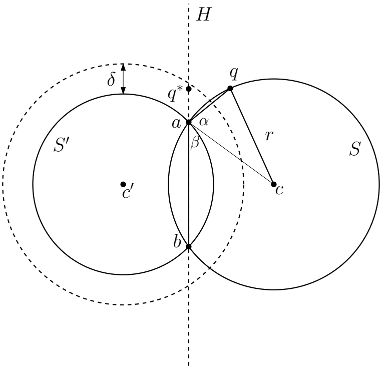



Fig in plane $cc'q$

$H$ separates $c$ and $c'$

$\sigma$ $d$-simplex of $\mathrm{Del}(P)$

$\sigma = q * \tau \qquad \sigma' \supset \tau \ \wedge \ q \notin \sigma'$

$H = \mathrm{aff}(\tau)$

$\gamma = \angle qab, \quad \alpha = \angle qac, \quad \beta = \angle cab$

wlog $\gamma \geq \angle qba$

$\gamma = \alpha + \beta \geq \frac{\pi}{2}$ (otherwise easy)

$\cos\alpha = \frac{\|a-q\|}{2r} \geq \frac{\delta}{2\varepsilon}$

$\cos\beta = \frac{\|a-b\|}{2r} \geq \frac{\eta}{2\varepsilon}$

$$\|qq^*\| = \|aq\| \sin\gamma > \delta \sin\gamma > \frac{\delta}{4\varepsilon}(\eta + \delta)$$

$\Delta(\sigma) \leq 2\varepsilon$

Given: A set of (bad) events $A_1, ..., A_N$,
each happens with $\text{proba}(A_i) \leq p < 1$

Question : what is the probability that none of the events occur?

The case of independent events

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) \geq (1-p)^N > 0$$

What if we allow a limited amount of dependency among the events?

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# LLL : symmetric version [Lovász & Erdös 1975]

Under the assumptions

1. $\text{proba}(A_i) \leq p < 1$
2. $A_i$ depends of $\leq \Gamma$ other events $A_j$
3. $\text{proba}(A_i) \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) > 0$$

# Moser and Tardos' constructive proof of the LLL [2010]

$\mathcal{P}$ a finite set of mutually independent random variables

$\mathcal{A}$ a finite set of events that are determined by the values of $S \subseteq \mathcal{P}$

Two events are independent iff they share no variable

**Algorithm**

**for all** $P \in \mathcal{P}$ **do**
  $v_P \leftarrow$ a random evaluation of $P$;

**while** $\exists A \in \mathcal{A}$ : $A$ happens when $(P = v_P, P \in \mathcal{P})$ **do**

  pick an arbitrary happening event $A \in \mathcal{A}$;

  **for all** $P \in \mathrm{variables}(A)$ **do**
    $v_P \leftarrow$ a new random evaluation of $P$;

**return** $(v_P)_{P \in \mathcal{P}}$;

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# Moser and Tardos' theorem

**if**

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2. $A_i$ depends of $\leq \Gamma$ other events $A_j$
3. $\text{proba}(A_i) \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

**then** $\exists$ an assignment of values to the variables $\mathcal{P}$ such that no event in $\mathcal{A}$ happens

The randomized algorithm resamples an event $A \in \mathcal{A}$ at most $\boxed{\frac{1}{\Gamma}}$ expected times before it finds such an evaluation

The expected total number of resampling steps is at most $\boxed{\frac{N}{\Gamma}}$

# Home work

- Read the beautiful (rather simple) proof of Moser & Tardos
- Learn about the parallel and the derandomized versions
- Listen to a talk by Aravind Srinivasan on further extensions

  `https://video.ias.edu/csdm/2014/0407-AravindSrinivasan`

# Protecting Delaunay simplices via perturbation

Picking regions : pick $p'$ in $B(p, \rho)$ Hyp. $\rho \leq \frac{\eta}{4} \ (\leq \frac{1}{2})$

Sampling parameters of a perturbed point set

If P is an $(\varepsilon, \bar{\eta})$-net, P′ is an $(\varepsilon', \bar{\eta}')$-net, where

$$\varepsilon' = \varepsilon(1 + \bar{\rho}) \text{ and } \bar{\eta}' = \frac{\bar{\eta} - 2\bar{\rho}}{1 + \bar{\rho}} \geq \frac{\bar{\eta}}{3}$$

Notation : $\bar{x} = \frac{x}{\varepsilon}$

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# The LLL framework

Random variables : $P'$ a set of random points $\{p',\ p' \in B(p,\rho), p \in \mathsf{P}\}$

Event: $\exists \phi' = (\sigma', p')$ (Bad configuration)
$\sigma'$ is a $d$ simplex with $R_{\sigma'} \leq \varepsilon + \rho$
$p' \in Z_\delta(\sigma')$ $\quad Z_\delta(\sigma') = B(c_{\sigma'}, R_{\sigma'} + \delta) \setminus B(c_{\sigma'}, R_{\sigma'})$

Algorithm

**Input:** P, $\rho$, $\delta$

**while** an event $\phi' = (\sigma', p')$ occurs **do**

resample the points of $\phi'$

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# Bounding $\Gamma$

Lemma : An event is independent of all but at most $\Gamma$ other bad events where $\Gamma$ depends on $\bar{\eta}$, $\bar{\rho}$, $\bar{\delta}$ and $d$

Proof :

- Let $\phi' = (\sigma', p')$ be a bad configuration.

$$\forall p' \in \phi', \quad \|p' - c_{\sigma'}\| \leq R_{\sigma'} + \delta = R = \varepsilon + \rho + \delta = \varepsilon\,(1 + \bar{\rho} + \bar{\delta})$$

- the number of events that may not be independent from an event $(\sigma', p')$ is at most the number of subsets of $(d+1)$ points in $B(c_{\sigma'}, 3R)$.

- Since P′ is $\eta'$-sparse,

$$\Gamma = \left(\frac{3R + \frac{\eta'}{2}}{\frac{\eta'}{2}}\right)^{d(d+2)}$$

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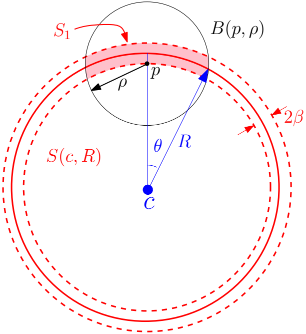
# Bounding $\text{proba}(\sigma, p)$ be a bad configuration




$S(c, R)$ a hypersphere of $\mathbb{R}^d$

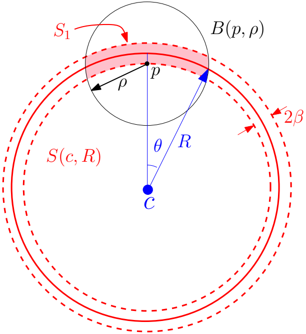
$T_\delta = B(c, R + \delta) \setminus B(c, R)$

$B_\rho$ any $d$-ball of radius $\rho < R$

$\text{vol}_d(T_\delta \cap B_\rho) \leq U_{d-1} \left(\frac{\pi}{2}\rho\right)^{d-1} \delta,$

$\text{proba}(p' \in Z_\delta(\sigma')) \leq \varpi = \frac{U_{d-1}}{U_d} \frac{2}{\pi} \frac{\delta}{\rho} < \frac{2^{d+1}\delta}{\pi\rho}$

# Bounding $\mathrm{proba}(\sigma, p)$ be a bad configuration




$S(c,R)$ a hypersphere of $\mathbb{R}^d$

$T_\delta = B(c, R+\delta) \setminus B(c,R)$

$B_\rho$ any $d$-ball of radius $\rho < R$

$$\mathrm{vol}_d(T_\delta \cap B_\rho) \leq U_{d-1} \left(\frac{\pi}{2}\rho\right)^{d-1} \delta,$$

$$\mathrm{proba}(p' \in Z_\delta(\sigma')) \leq \varpi = \frac{U_{d-1}}{U_d}\,\frac{2}{\pi}\,\frac{\delta}{\rho} < \frac{2^{d+1}\,\delta}{\pi\,\rho}$$

# Bound on the number of events

$\Sigma(p')$ : number of $d$-simplices that can possibly make a bad configuration with $p' \in P'$ for some perturbed set $P'$

$R = \varepsilon + \rho + \delta$

$$\sum_{p' \in P'} \Sigma(p') \leq n \times |P' \cap B(p', 2R)| \leq N = n \left( \frac{2R + \frac{\eta'}{2}}{\frac{\eta'}{2}} \right)^{d(d+1)}$$

# Main result

Under condition

$$\frac{2^{d+1}e}{\pi}(\Gamma+1)\,\delta \leq \rho \leq \frac{\eta}{4}$$

the algorithm terminates.

Guarantees on the output

- $d_H(P, P') \leq \rho$
- the $d$-simplices of $\mathrm{Del}(P')$ are $\delta$-protected
- and therefore have a positive thickness

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# Complexity of the algorithm

- The number of resamplings executed by the algorithm is at most

$$N/\Gamma \leq C\,n$$

where $C$ depends on $\bar{\eta}$, $\bar{\rho}$, $\bar{\delta}$ and $d$

- Each resampling consists in perturbing $O(1)$ points
- Updating the Delaunay triangulation after each resampling takes $O(1)$ time
- The expected complexity is linear in the number of points