Mesh Generation

Jean-Daniel Boissonnat
Geometrica, INRIA
http://www-sop.inria.fr/geometrica

Winter School, University of Nice Sophia Antipolis
January 26-30, 2015
Meshing surfaces and 3D domains

- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medicine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing: meshes for FEM
### Grid methods
- Lorensen & Cline [87]: marching cube
- Lopez & Brodlie [03]: topological consistency
- Plantiga & Vegter [04]: certified topology using interval arithmetic

### Morse theory
- Stander & Hart [97]
- B., Cohen-Steiner & Vegter [04]: certified topology

### Delaunay refinement
- Hermeline [84]
- Ruppert [95]
- Shewchuk [02]
- Chew [93]
- B. & Oudot [03, 04]
- Cheng et al. [04]
Main issues

Sampling
- How do we choose points in the domain?
- What information do we need to know/measure about the domain?

Meshing
- How do we connect the points?
- Under what sampling conditions can we compute a good approximation of the domain?
Restricted Delaunay triangulation

**Definition**

The restricted Delaunay triangulation $\text{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\text{Vor}(\mathcal{P}) \cap X$.

If $\mathcal{P}$ is an $\varepsilon$-sample, any ball centered on $X$ that circumscribes a facet of $\text{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon \text{rch}(\mathcal{M})$. 
Definition

The restricted Delaunay triangulation $\text{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\text{Vor}(\mathcal{P}) \cap X$.

If $P$ is an $\varepsilon$-sample, any ball centered on $X$ that circumscribes a facet of $\text{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon \text{rch}(M)$.
**Definition**

The restricted Delaunay triangulation $\text{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\text{Vor}(\mathcal{P}) \cap X$

If $\mathcal{P}$ is an $\varepsilon$-sample, any ball centered on $X$ that circumscribes a facet of $\text{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon \text{rch}(\mathcal{M})$
**Definition**

The restricted Delaunay triangulation $\text{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\text{Vor}(\mathcal{P}) \cap X$.
A variant of the nerve theorem

Let $\mathbb{M}$ be a compact manifold without boundary. If, for any face $f \in \text{Vor}(P)$ s.t. $f \cap \mathbb{M} \neq \emptyset$,

1. $f$ intersects $\mathbb{M}$ transversally
2. $f \cap \mathbb{M} = \emptyset$ or is a topological ball

then $\text{Del}_\mathbb{M}(P) \approx \mathbb{M}$
A variant of the nerve theorem [Edelsbrunner & Shah 1997]

Let $\mathbb{M}$ be a compact manifold without boundary. If, for any face $f \in \text{Vor}(P)$ s.t. $f \cap \mathbb{M} \neq \emptyset$,

1. $f$ intersects $\mathbb{M}$ transversally
2. $f \cap \mathbb{M} = \emptyset$ or is a topological ball

then $\text{Del}_\mathbb{M}(P) \approx \mathbb{M}$
Proof of the closed ball property

Barycentric subdivision

of $\text{Vor}(P) \cap M$

of $\text{Del}_M(P)$
Good sampling, scale and dimension
Sampling conditions

[Federmeyer 1958], [Amenta & Bern 1998]

**Medial axis of** $\mathcal{M}$: $axis(\mathcal{M})$

set of points with at least two closest points on $\mathcal{M}$

**Reach**

$\forall x \in \mathcal{M}, \ rch(x) = \infimum$ of the radii of the medial balls tangent to $\mathcal{M}$ at $x$

$rch(\mathcal{M}) = \inf_{x \in \mathcal{M}} rch(x)$

$(\epsilon, \eta)$-net of $\mathcal{M}$

1. $\mathcal{P} \subset \mathcal{M}, \ \forall x \in \mathcal{M} : \ d(x, \mathcal{P}) \leq \epsilon \ rch(x)$
2. $\forall p, q \in \mathcal{P}, \ \|p - q\| \geq \eta \ \min(rch(p), rch(q))$
Sampling conditions

[Federer 1958], [Amenta & Bern 1998]

**Medial axis of** $\mathbb{M}$: axis($\mathbb{M}$)

set of points with at least two closest points on $\mathbb{M}$

**Reach**

$\forall x \in \mathbb{M}, \ rch(x) = \infimum \ of \ the \ radii \ of \ the \ medial \ balls \ tangent \ to \ \mathbb{M} \ at \ x$

$rch(\mathbb{M}) = \inf_{x \in \mathbb{M}} rch(x)$

*(\epsilon, \eta)*-net of $\mathbb{M}$

1. $\mathcal{P} \subset \mathbb{M}, \ \forall x \in \mathbb{M}: \ d(x, \mathcal{P}) \leq \epsilon \ rch(x)$
2. $\forall p, q \in \mathcal{P}, \ \|p - q\| \geq \eta \ \min(rch(p), rch(q))$
Restricted Delaunay triang. of $(\varepsilon, \eta)$-nets

[Amenta et al. 1998-], [B. & Oudot 2005]

If

- $S \subset \mathbb{R}^3$ is a compact surface of positive reach without boundary
- $\mathcal{P}$ is an $(\varepsilon, \eta)$-net with $\varepsilon/\eta \leq \xi_0$ and $\varepsilon$ small enough

then

- $\text{Del}_{|S}(\mathcal{S})$ provides good estimates of normals
- There exists a homeomorphism $\phi : \text{Del}_{|S}(\mathcal{P}) \rightarrow S$
- $\sup_x (\|\phi(x) - x\|) = O(\varepsilon^2)$
Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

\[ \phi : S \rightarrow \mathbb{R} = \text{Lipschitz function} \]
\[ \forall x \in S, \ 0 < \phi_{\text{min}} \leq \phi(x) < \epsilon \text{rch}(x) \]

**ORACLE**: For a facet \( f \) of \( \text{Del}|_S(P) \),
return \( c_f, r_f \) and \( \phi(c_f) \)

A facet \( f \) is **bad** if \( r_f > \phi(c_f) \)

**Algorithm**

INIT compute an initial (small) sample \( P_0 \subset S \)

REPEAT
IF \( f \) is a bad facet
    insert_in_Del3D(c_f),
    update \( P \) and \( \text{Del}|_S(P) \)
UNTIL all facets are good
Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

\( \phi : S \rightarrow \mathbb{R} = \) Lipschitz function
\[
\forall x \in S, \ 0 < \phi_{\min} \leq \phi(x) < \varepsilon \text{rch}(x)
\]

**ORACLE** : For a facet \( f \) of \( \text{Del}_{|S}(\mathcal{P}) \), return \( c_f, r_f \) and \( \phi(c_f) \)

A facet \( f \) is **bad** if \( r_f > \phi(c_f) \)

**Algorithm**

**INIT** compute an initial (small) sample \( \mathcal{P}_0 \subset S \)

**REPEAT**  
\[ \text{IF } f \text{ is a bad facet} \]
\[ \text{insert_in_Del3D}(c_f) , \]
\[ \text{update } \mathcal{P} \text{ and } \text{Del}_{|S}(\mathcal{P}) \]

**UNTIL** all facets are good
The meshing algorithm in action
The algorithm outputs a good sample

The output sample $\mathcal{P}$ is sparse

$$\forall p \in \mathcal{P}, d(p, \mathcal{P} \setminus \{p\}) = \|p - q\| \geq \min(\phi(p), \phi(q)) \geq \phi(p) - \|p - q\|$$

$$\Rightarrow \|p - q\| \geq \frac{1}{2} \phi(p) \geq \frac{1}{2} \phi_0 > 0$$

the algorithm terminates

$\mathcal{P}$ is a loose $\varepsilon$-sample of $S$

- Each facet has a $S$-radius $r_f \leq \phi(c_f) < \varepsilon \text{rch}(S)$

- $\text{Del}_S(\mathcal{P})$ has a vertex on each cc of $S$  \hspace{1cm} (\text{INIT})
Size of the sample $= \Theta \left( \int_S \frac{dx}{\phi^2(x)} \right)$

Upper bound

$B_p = B(p, \frac{\phi(p)}{2})$, $p \in \mathcal{P}$

\[
\int_S \frac{dx}{\phi^2(x)} \geq \sum_p \int_{(B_p \cap S)} \frac{dx}{\phi^2(x)}
\]

\[
\geq \frac{4}{9} \sum_p \frac{\text{area}(B_p \cap S)}{\phi^2(p)}
\]

\[
\geq \sum_p C = C |\mathcal{P}|
\]

(the $B_p$ are disjoint)

\[
\phi(x) \leq \phi(p) + ||p - x||
\]

\[
\leq \frac{3}{2} \phi(p)
\]

Lower bound

Use a covering instead of a packing
Size of the sample \( = \Theta \left( \int_S \frac{dx}{\phi^2(x)} \right) \)

**Upper bound**

\( B_p = B(p, \frac{\phi(p)}{2}) \), \( p \in \mathcal{P} \)

\[ \int_S \frac{dx}{\phi^2(x)} \geq \sum_p \int_{(B_p \cap S)} \frac{dx}{\phi^2(x)} \]

\[ \geq \frac{4}{9} \sum_p \frac{\text{area}(B_p \cap S)}{\phi^2(p)} \]

\[ \geq \sum_p C = C |\mathcal{P}| \]

(the \( B_p \) are disjoint)

\[ \phi(x) \leq \phi(p) + \|p - x\| \leq \frac{3}{2} \phi(p) \]

**Lower bound**

Use a covering instead of a packing
The full result

The Delaunay refinement algorithm produces

- a good (dense and sparse) sample
- a triangulated surface $\hat{S}$
  - homeomorphic to $S$
  - close to $S$ (Hausdorff/Fréchet distance, approximation of normals)
Applications

- Implicit surfaces $f(x, y, z) = 0$
- Isosurfaces in a 3d image (Medical images)
- Triangulated surfaces (Remeshing)
- Point sets (Surface reconstruction)

see cgal.org, CGALmesh project
Results on smooth implicit surfaces
Meshing 3D domains
Input from segmented 3D medical images

[INSERM]  [SIEMENS]
Comparison with the Marching Cube algorithm

Delaunay refinement

Marching cube
Meshing with sharp features
A polyhedral example
Meshing 3D multi-domains
Input from segmented 3D medical images [IRCAD]

<table>
<thead>
<tr>
<th>Size bound (mm)</th>
<th>vertices nb</th>
<th>facets nb</th>
<th>tetrahedra nb</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>3,743</td>
<td>3,735</td>
<td>19,886</td>
<td>0.880</td>
</tr>
<tr>
<td>8</td>
<td>27,459</td>
<td>19,109</td>
<td>159,120</td>
<td>6.97</td>
</tr>
<tr>
<td>4</td>
<td>199,328</td>
<td>76,341</td>
<td>1,209,720</td>
<td>54.1</td>
</tr>
<tr>
<td>2</td>
<td>1,533,660</td>
<td>311,420</td>
<td>9,542,295</td>
<td>431</td>
</tr>
</tbody>
</table>
Surface reconstruction from unorganized point sets

Courtesy of P. Alliez