Mesh Generation

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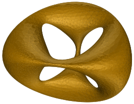
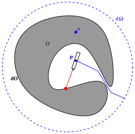
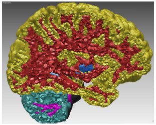
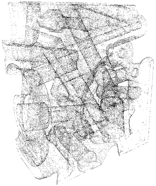
Meshing surfaces and 3D domains

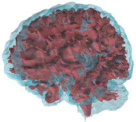
- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medecine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing : meshes for FEM











Mesh generation : from art to science

Grid methods

Lorensen & Cline [87] : marching cube
Lopez & Brodlie [03] : topological consistency
Plantiga & Vegter [04] : certified topology using interval arithmetic

Morse theory

Stander & Hart [97]
B., Cohen-Steiner & Vegter [04] : certified topology

Delaunay refinement

Hermeline [84]
Ruppert [95]
Shewchuk [02]
Chew [93]
B. & Oudot [03,04]
Cheng et al. [04]

Main issues

Sampling

- How do we choose points in the domain ?
- What information do we need to know/measure about the domain ?

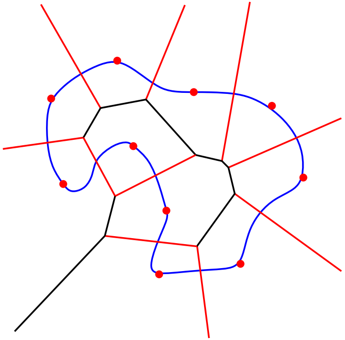
Meshing

- How do we connect the points ?
- Under what sampling conditions can we compute a good approximation of the domain ?

Restricted Delaunay triangulation [Chew 93]

Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$



If P is an ε-sample, any ball centered on X that circumscribes a facet of $\mathrm{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon\,\mathrm{rch}(\mathbb{M})$

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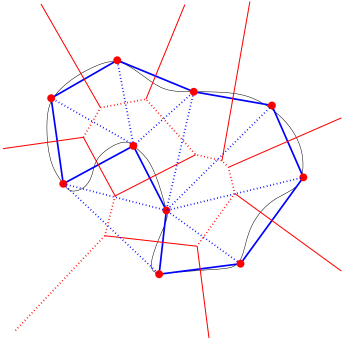

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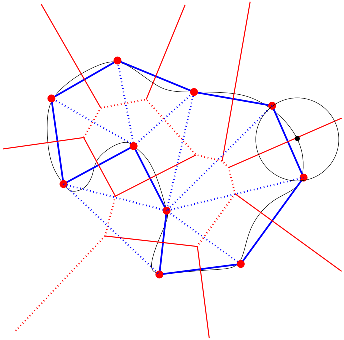

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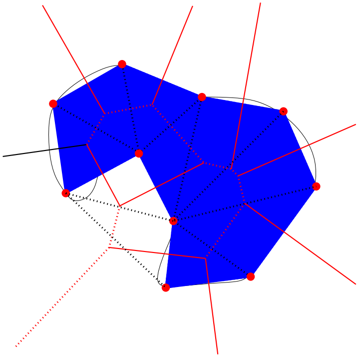
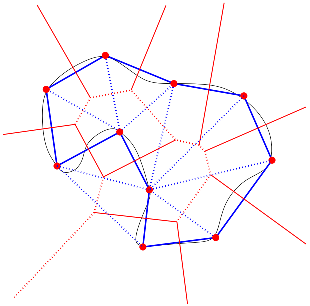


Let $\mathbb{M}$ be a compact manifold without boundary. If, for any face $f \in \mathrm{Vor}(\mathsf{P})$ s.t. $f \cap \mathbb{M} \neq \emptyset$,

1. f intersects $\mathbb{M}$ transversally
2. $f \cap \mathbb{M} = \emptyset$ or is a topological ball

then $\mathrm{Del}_{\mathbb{M}}(\mathsf{P}) \approx \mathbb{M}$



A variant of the nerve theorem [Edelsbrunner & Shah 1997]

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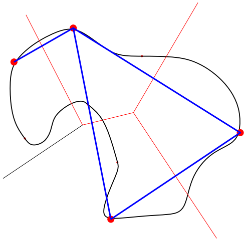
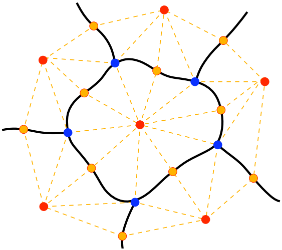


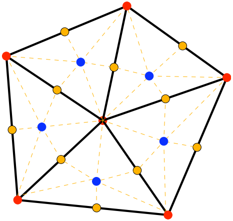
Proof of the closed ball property

Barycentric subdivision

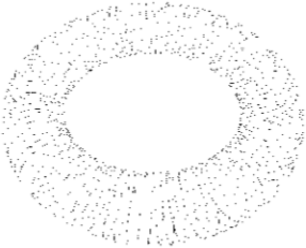
of $\mathrm{Vor}(\mathsf{P}) \cap \mathbb{M}$

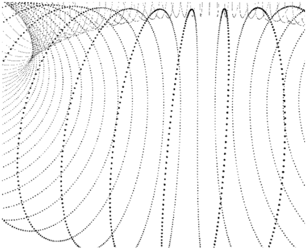


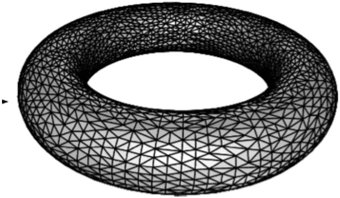
of $\mathrm{Del}_{\mathbb{M}}(\mathsf{P})$

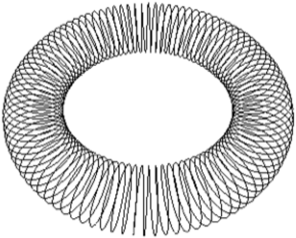


Good sampling, scale and dimension



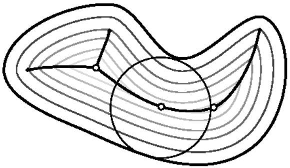






Medial axis of $\mathbb{M}$: $\mathrm{axis}(\mathbb{M})$

set of points with at least two closest points on $\mathbb{M}$



Reach

$\forall x \in \mathbb{M}, \quad \mathrm{rch}(x) =$ infimum of the radii of the medial balls tangent to $\mathbb{M}$ at x

$\mathrm{rch}(\mathbb{M}) = \inf_{x\in\mathbb{M}} \mathrm{rch}(x)$

(ϵ, η)-net of $\mathbb{M}$

1. $\mathcal{P} \subset \mathbb{M}, \forall x \in \mathbb{M}: \quad d(x, \mathcal{P}) \leq \epsilon\, \mathrm{rch}(x)$
2. $\forall p, q \in \mathcal{P}, \quad \|p - q\| \geq \eta\, \min(\mathrm{rch}(p), \mathrm{rch}(q))$

Sampling conditions [Federer 1958], [Amenta & Bern 1998]

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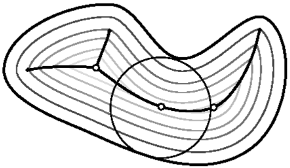

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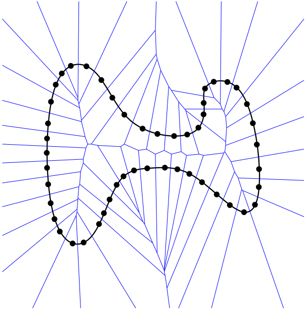
Restricted Delaunay triang. of (ε, η)-nets

[Amenta et al. 1998-], [B. & Oudot 2005]



If

- $\mathcal{S} \subset \mathbb{R}^3$ is a compact surface of positive reach without boundary
- $\mathcal{P}$ is an (ε, η)-net with $\varepsilon/\eta \leq \xi_0$ and ε small enough

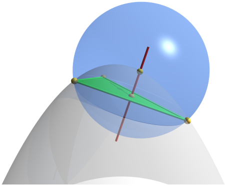
then

- $\mathrm{Del}_{|S}(\mathcal{S})$ provides good estimates of normals
- There exists a homeomorphism $\phi : \mathrm{Del}_{|S}(\mathcal{P}) \rightarrow \mathcal{S}$
- $\sup_x(\|\phi(x) - x\|) = O(\varepsilon^2)$

Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\ 0 < \phi_{\min} \leq \phi(x) < \varepsilon\, \mathrm{rch}(x)$



ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$

Algorithm

INIT compute an initial (small) sample $\mathcal{P}_0 \subset S$

REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ and $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL all facets are good

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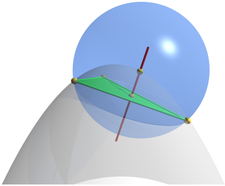

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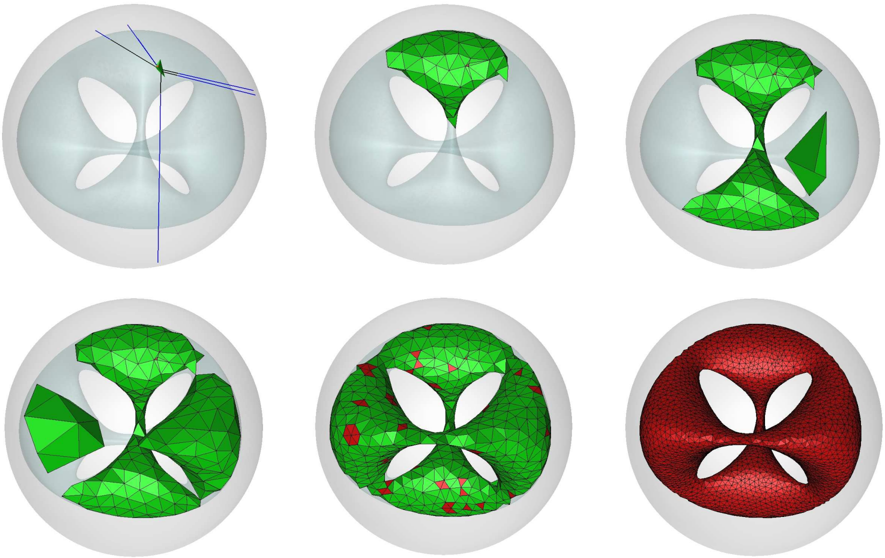
The meshing algorithm in action



The algorithm outputs a good sample

The output sample $\mathcal{P}$ is sparse

$$\begin{aligned}\forall p \in \mathcal{P}, d(p, \mathcal{P} \setminus \{p\}) &= \|p - q\| \\ &\geq \min(\phi(p), \phi(q)) \\ &\geq \phi(p) - \|p - q\|\end{aligned}$$

$$\Rightarrow \quad \|p - q\| \geq \frac{1}{2}\,\phi(p) \geq \frac{1}{2}\,\phi_0 > 0$$

the algorithm terminates

$\mathcal{P}$ is a loose ε-sample of $\mathcal{S}$

- Each facet has a $\mathcal{S}$-radius $r_f \leq \phi(c_f) < \varepsilon\,\mathrm{rch}(\mathcal{S})$
- $\mathrm{Del}_{|S}(\mathcal{P})$ has a vertex on each cc of $\mathcal{S}$ (INIT)

Size of the sample $= \Theta\left(\int_S \frac{dx}{\phi^2(x)}\right)$

Upper bound

$B_p = B(p, \frac{\phi(p)}{2}),\, p \in \mathcal{P}$

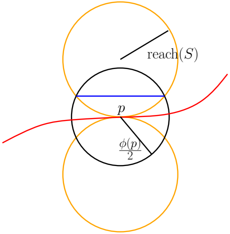
$$\int_S \frac{dx}{\phi^2(x)} \geq \sum_p \int_{(B_p \cap S)} \frac{dx}{\phi^2(x)}$$

(the B_p are disjoint)

$$\geq \frac{4}{9} \sum_p \frac{\text{area}(B_p \cap S)}{\phi^2(p)}$$

$$\phi(x) \leq \phi(p) + \|p - x\| \leq \frac{3}{2}\,\phi(p))$$

$$\geq \sum_p C \;=\; C\,|\mathcal{P}|$$



Lower bound

Use a covering instead of a packing

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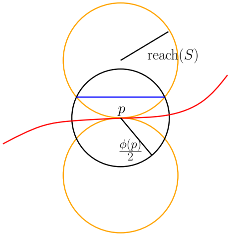


Lower bound

Use a covering instead of a packing

The full result

The Delaunay refinement algorithm produces

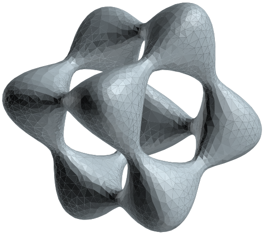
- a good (dense and sparse) sample
- a triangulated surface $\hat{S}$
  - homeomorphic to S
  - close to $\mathcal{S}$ (Hausdorff/Fréchet distance, approximation of normals)

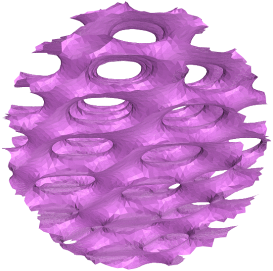
Applications

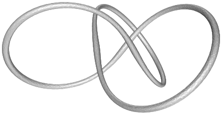
- Implicit surfaces $f(x, y, z) = 0$
- Isosurfaces in a 3d image (Medical images)
- Triangulated surfaces (Remeshing)
- Point sets (Surface reconstruction)

see cgal.org, CGALmesh project

Results on smooth implicit surfaces



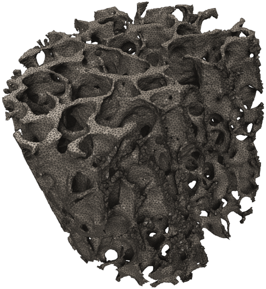




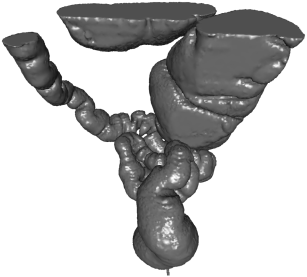
Meshing 3D domains

Input from segmented 3D medical images

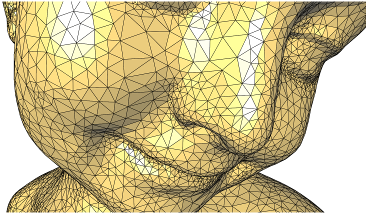
[INSERM]



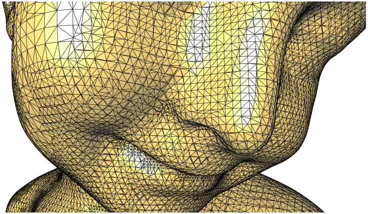
[SIEMENS]



Comparison with the Marching Cube algorithm



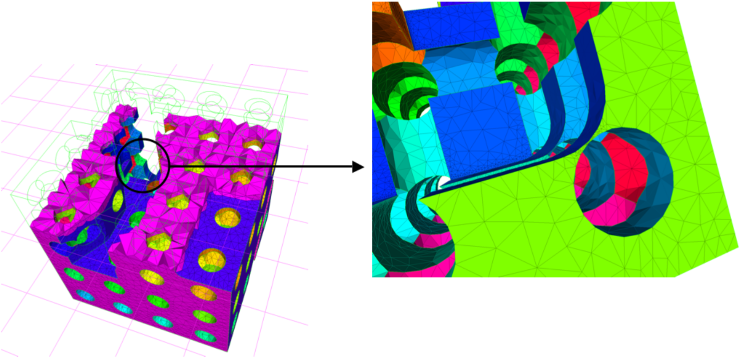
Delaunay refinement



Marching cube

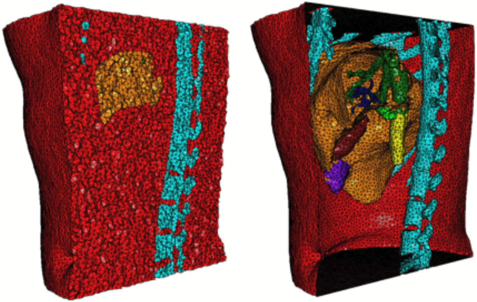
Meshing with sharp features

A polyhedral example



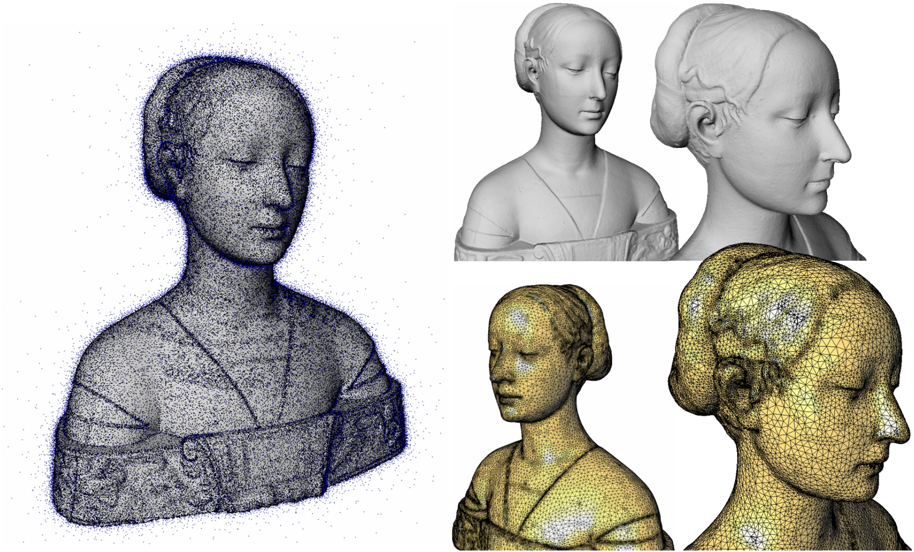
Meshing 3D multi-domains

Input from segmented 3D medical images [IRCAD]



Size bound (mm)	vertices nb	facets nb	tetrahedra nb	CPU Time (s)
16	3,743	3,735	19,886	0.880
8	27,459	19,109	159,120	6.97
4	199,328	76,341	1,209,720	54.1
2	1,533,660	311,420	9,542,295	431

Surface reconstruction from unorganized point sets



Courtesy of P. Alliez