Union of Balls and $\alpha$-Complexes

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Laguerre geometry

Power distance of two balls or of two weighted points.

ball $b_1(p_1, r_1)$, center $p_1$ radius $r_1$ $\leftrightarrow$ weighted point $(p_1, r_1^2) \in \mathbb{R}^d$

ball $b_2(p_2, r_2)$, center $p_2$ radius $r_2$ $\leftrightarrow$ weighted point $(p_2, r_2^2) \in \mathbb{R}^d$

\[
\pi(b_1, b_2) = (p_1 - p_2)^2 - r_1^2 - r_2^2
\]

Orthogonal balls

$b_1, b_2$ closer $\iff$ $\pi(b_1, b_2) < 0 \iff (p_1 - p_2)^2 \leq r_1^2 + r_2^2$

$b_1, b_2$ orthogonal $\iff$ $\pi(b_1, b_2) = 0 \iff (p_1 - p_2)^2 = r_1^2 + r_2^2$

$b_1, b_2$ further $\iff$ $\pi(b_1, b_2) > 0 \iff (p_1 - p_2)^2 \leq r_1^2 + r_2^2$
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Orthogonal balls

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$b_1, b_2$ further $\iff \pi(b_1, b_2) > 0 \iff (p_1 - p_2)^2 \leq r_1^2 + r_2^2$
Power distance of a point wrt a ball

If $b_1$ is reduced to a point $p$:
$$\pi(p, b_2) = (p - p_2)^2 - r_2^2$$

Normalized equation of bounding sphere:
\[
p \in \partial b_2 \iff \pi(p, b_2) = 0
\]
\[
p \in \text{int} b_2 \iff \pi(p, b) < 0
\]
\[
p \in \partial b_2 \iff \pi(p, b) = 0
\]
\[
p \notin b_2 \iff \pi(p, b) > 0
\]

Tangents and secants through $p$
$$\pi(p, b) = pt^2 = \overline{pm} \cdot \overline{pm'} = \overline{pn} \cdot \overline{pn'}$$
Radical Hyperplane

The locus of point $x \in \mathbb{R}^d$ with same power distance to balls $b_1(p_1, r_1)$ and $b_2(p_2, r_2)$ is a hyperplane of $\mathbb{R}^d$

$$\pi(x, b_1) = \pi(x, b_2) \iff (x - p_1)^2 - r_1^2 = (x - p_2)^2 - r_2^2 \iff -2p_1 x + p_1^2 - r_1^2 = -2p_2 x + p_2^2 - r_2^2 \iff 2(p_2 - p_1)x + (p_1^2 - r_1^2) - (p_2^2 - r_2^2) = 0$$

A point in $h_{12}$ is the center of a ball orthogonal to $b_1$ and $b_2$
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A point in $h_{12}$ is the center of a ball orthogonal to $b_1$ and $b_2$
Power Diagrams
also named Laguerre diagrams or weighted Voronoi diagrams

Sites: $n$ balls $B = \{b_i(p_i, r_i), i = 1, \ldots n\}$

Power distance: $\pi(x, b_i) = (x - p_i)^2 - r_i^2$

Power Diagram: $\text{Vor}(B)$
One cell $V(b_i)$ for each site
$V(b_i) = \{x : \pi(x, b_i) \leq \pi(x, b_j). \forall j \neq i\}$

- Each cell is a polytope
- $V(b_i)$ may be empty
- $p_i$ may not belong to $V(b_i)$
Weighted Delaunay triangulations

\[ B = \{ b_i(p_i, r_i) \} \] a set of balls

\[ \text{Del}(B) = \text{nerve of Vor}(B): \]

\[ B_\tau = \{ b_i(p_i, r_i), i = 0, \ldots k \} \subseteq B \]

\[ B_\tau \in \text{Del}(B) \iff \bigcap_{b_i \in B_\tau} \text{V}(b_i) \neq \emptyset \]

To be proved (next slides):

under a general position condition on \( B \),

\[ B_\tau \rightarrow \tau = \text{conv}\{ p_i, i = 0 \ldots k \} \]

embeds \( \text{Del}(B) \) as a triangulation in \( \mathbb{R}^d \),
called the weighted Delaunay triangulation
Characteristic property of weighted Delaunay complexes

\[ \tau \in \text{Del}(B) \iff \bigcap_{b_i \in B_\tau} V(b_i) \neq \emptyset \]

\[ \iff \exists x \in \mathbb{R}^d \text{ s.t. } \forall b_i, b_j \in B_\tau, b_l \in B \setminus B_\tau \]

\[ \pi(x, b_i) = \pi(x, b_j) < \pi(x, b_l) \]

\[ \iff \exists \text{ ball } b(x, \omega) \text{ s.t. } \forall b_i \in B_\tau, b_l \in B \setminus B_\tau \]

\[ 0 = \pi(b, b_i) < \pi(b, b_l) \]
The space of spheres

\( b(p, r) \) ball of \( \mathbb{R}^d \)

\[ \rightarrow \text{point } \phi(b) \in \mathbb{R}^{d+1} \]

\[ \phi(b) = (p, s = p^2 - r^2) \]

\[ \rightarrow \text{polar hyperplane } h_b = \phi(b)^* \in \mathbb{R}^{d+1} \]

\[ \mathcal{P} = \{ \hat{x} \in \mathbb{R}^{d+1} : x_{d+1} = x^2 \} \]

\[ h_b = \{ \hat{x} \in \mathbb{R}^{d+1} : x_{d+1} = 2p \cdot x - s \} \]

- Balls will null radius are mapped onto \( \mathcal{P} \)
  - \( h_p \) is tangent to \( \mathcal{P} \) at \( \phi(p) \).

- The vertical projection of \( h_b \cap \mathcal{P} \) onto \( x_{d+1} = 0 \) is \( \partial b \)
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$h_b = \{ \hat{x} \in \mathbb{R}^{d+1} : x_{d+1} = 2p \cdot x - s \}$

- The vertical distance between $\hat{x} = (x, x^2)$ and $h_b$ is equal to

$$x^2 - 2p \cdot x + s = \pi(x, b)$$

- The faces of the power diagram of $B$ are the vertical projections onto $x_{d+1} = 0$ of the faces of the polytope $\mathcal{V}(B) = \bigcap_i h_b^+$ of $\mathbb{R}^{d+1}$
The space of spheres

\( b(p, r) \) ball of \( \mathbb{R}^d \)

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Power diagrams, weighted Delaunay triangulations and polytopes

\[ \mathcal{V}(B) = \cap_i \phi(b_i)^* \]

\[ \mathcal{D}(B) = \text{conv}^{-}(\hat{P}) \]
Proof of the theorem

\[ B_\tau \subset B, |B_\tau| = d + 1, \tau = \text{conv}(\{p_i, b_i(p_i, r_i) \in B_\tau\}), \]
\[ \phi(\tau) = \text{conv}(\{\phi(b_i), b_i \in B_\tau\}) \]

\[ \exists b(p, r) \text{ s.t. } h_b = \phi(b)^* = \text{aff}(\{\phi(b_i), b_i \in B_\tau\}) \]

\[ \phi(\tau) \in \mathcal{D}(B) = \text{conv}^{-}(\{\phi(b_i)\}) \]
\[ \iff \forall b_i \in B_\tau, \phi(b_i) \in h_b \quad \forall b_j \not\in B_\tau, \phi(b_j) \in h_b^{*+} \]
\[ \iff \forall b_i \in B_\tau, \pi(b, b_i) = 0 \quad \forall b_j \not\in B_\tau, \pi(b, b_j) > 0 \]
\[ \iff p \in \bigcap_{b_i \in B_\tau} V(b_i) \]
\[ \iff \tau \in \text{Del}(B) \]
Delaunay’s theorem extended

$B = \{ b_1, b_2 \ldots b_n \}$ is said to be in general position wrt spheres if

$\forall x \in \mathbb{R}^d$ with equal power to $d + 2$ balls of $B$

$P = \{ p_1, \ldots, p_n \}$: set of centers of the balls of $B$

**Theorem**

If $B$ is in general position wrt spheres, the simplicial map

$$f : \text{vert}(\text{Del}(B)) \rightarrow P$$

provides a realization of $\text{Del}(B)$

$\text{Del}(B)$ is a triangulation of $P' \subseteq P$ called the Delaunay triangulation of $B$
Delaunay’s theorem extended

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**Theorem**

If \( B \) is in general position \text{wrt spheres}, the simplicial map

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provides a realization of \( \text{Del}(B) \)

\( \text{Del}(B) \) is a triangulation of \( P' \subseteq P \) called the \textbf{Delaunay triangulation of} \( B \)
If $B$ is a set of balls in general position wrt spheres:

$$\mathcal{V}(B) = h_{b_1}^+ \cap \ldots \cap h_{b_n}^+ \quad \xrightarrow{\text{duality}} \quad \mathcal{D}(B) = \text{conv}^- (\{\phi(b_1), \ldots, \phi(b_n)\})$$

\[\uparrow\quad \xrightarrow{\text{nerve}} \quad \downarrow\]

Voronoi Diagram of $B$ \quad \xrightarrow{\text{nerve}} \quad Delaunay Complex of $B$
Complexity and algorithm for weighted VD and DT

Number of faces $= \Theta \left( n^{\left\lfloor \frac{d+1}{2} \right\rfloor} \right)$ (Upper Bound Th.)

Construction can be done in time $\Theta \left( n \log n + n^{\left\lfloor \frac{d+1}{2} \right\rfloor} \right)$ (Convex hull)

Main predicate

$$\text{power\_test}(b_0, \ldots, b_{d+1}) = \text{sign} \left| \begin{array}{cccc} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 - r_0^2 & \ldots & p_{d+1}^2 - r_{d+1}^2 \end{array} \right|$$
Complexity and algorithm for weighted VD and DT

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Power diagrams are maximization diagrams

Cell of \( b_i \) in the power diagram \( \text{Vor}(B) \)

\[
V(b_i) = \{ x \in \mathbb{R}^d : \pi(x, b_i) \leq \pi(x, b_j). \forall j \neq i \}
\]

\[
= \{ x \in \mathbb{R}^d : 2p_i x - s_i = \max_{j \in [1, \ldots, n]} \{ 2p_j x - s_j \} \}
\]

\( \text{Vor}(B) \) is the maximization diagram of the set of affine functions

\[
\{ f_i(x) = 2p_i x - s_i, i = 1, \ldots, n \}
\]
Affine diagrams (regular subdivisions)

Affine diagrams are defined as the maximization diagrams of a finite set of affine functions.

They are equivalently defined as the vertical projections of polyhedra intersection of a finite number of upper half-spaces of $\mathbb{R}^{d+1}$.

- Voronoi diagrams and power diagrams are affine diagrams.
- Any affine diagram of $\mathbb{R}^d$ is the power diagram of a set of balls.
- Delaunay and weighted Delaunay triangulations are regular triangulations.
- Any regular triangulation is a weighted Delaunay triangulation.
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Examples of affine diagrams

1. The intersection of a power diagram with an affine subspace (Exercise)

2. A Voronoi diagram defined with a quadratic distance function

\[ \|x - a\|_Q = (x - a)^t Q (x - a) \quad Q = Q^t \]

3. \(k\) order Voronoi diagrams
Let $P$ be a set of sites.
Each cell in the $k$-order Voronoi diagram $\text{Vor}_k(P)$ is the locus of points in $\mathbb{R}^d$ that have the same subset of $P$ as $k$-nearest neighbors.
\( k \)-order Voronoi diagrams are power diagrams

Let \( S_1, S_2, \ldots \) denote the subsets of \( k \) points of \( P \).
The \( k \)-order Voronoi diagram is the minimization diagram of \( \delta(x, S_i) \):

\[
\delta(x, S_i) = \frac{1}{k} \sum_{p \in S_i} (x - p)^2
\]

\[
= x^2 - \frac{2}{k} \sum_{p \in S_i} p \cdot x + \frac{1}{k} \sum_{p \in S_i} p^2
\]

\[
= \pi(b_i, x)
\]

where \( b_i \) is the ball

1. centered at \( c_i = \frac{1}{k} \sum_{p \in S_i} p \)
2. with \( s_i = \pi(o, b_i) = c_i^2 - r_i^2 = \frac{1}{k} \sum_{p \in S_i} p^2 \)
3. and radius \( r_i^2 = c_i^2 - \frac{1}{k} \sum_{p \in S_i} p^2 \).
Combinatorial complexity of $k$-order Voronoi diagrams

**Theorem**
If $P$ be a set of $n$ points in $\mathbb{R}^d$, the number of vertices and faces in all the Voronoi diagrams $\text{Vor}_j(P)$ of orders $j \leq k$ is:

$$O\left(k^{\left\lceil \frac{d+1}{2} \right\rceil} n^{\left\lfloor \frac{d+1}{2} \right\rfloor}\right)$$

**Proof**
uses:
- bijection between $k$-sets and cells in $k$-order Voronoi diagrams
- the sampling theorem (from randomization theory)
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\( k \)-sets and \( k \)-order Voronoi diagrams

\( P \) a set of \( n \) points in \( \mathbb{R}^d \)

\( k \)-sets

A \( k \)-set of \( P \) is a subset \( P' \) of \( P \) with size \( k \) that can be separated from \( P \setminus P' \) by a hyperplane

\( k \)-order Voronoi diagrams

\( k \) points of \( P \) have a cell in \( \text{Vor}_k(P) \) iff there exists a ball that contains those points and only those

\( \Rightarrow \) each cell of \( \text{Vor}_k(P) \) corresponds to a \( k \)-set of \( \phi(P) \)
$k$-sets and $k$-order Voronoi diagrams

$P$ a set of $n$ points in $\mathbb{R}^d$

$k$-sets

A $k$-set of $P$ is a subset $P'$ of $P$ with size $k$ that can be separated from $P \setminus P'$ by a hyperplane

$k$-order Voronoi diagrams

$k$ points of $P$ have a cell in $\text{Vor}_k(P)$ iff there exists a ball that contains those points and only those

$\Rightarrow$ each cell of $\text{Vor}_k(P)$ corresponds to a $k$-set of $\phi(P)$
For a set of points $P \in \mathbb{R}^d$, we consider the arrangement of the dual hyperplanes $\mathcal{A}(P^*)$.

- $h$ defines a $k$ set $P'$ $\Rightarrow$ $h$ separates $P'$ (below $h$) from $P \setminus P'$ (above $h$).
  $\Rightarrow$ $h^*$ is below the $k$ hyperplanes of $P'^*$ and above those of $P^* \setminus P'^*$.

- $k$-sets of $P$ are in 1-1 correspondence with the cells of $\mathcal{A}(P^*)$ of level $k$, i.e. with $k$ hyperplanes of $P^*$ above it.
For a set of points $P \in \mathbb{R}^d$, we consider the arrangement of the dual hyperplanes $\mathcal{A}(P^*)$.

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Bounding the number of \( k \)-sets

\[
c_k(P) : \text{Number of } k\text{-sets of } P = \text{Number of cells of level } k \text{ in } A(P^*)
\]

\[
c_{\leq k}(P) = \sum_{l \leq k} c_l(P)
\]

\[
c'_{\leq k}(P) : \text{Number of vertices of } A(P^*) \text{ with level at most } k
\]

\[
c_{\leq k}(n) = \max_{|P|=n} c_{\leq k}(P) \quad c'_{\leq k}(n) = \max_{|P|=n} c'_{\leq k}(P)
\]

Hyp. in general position: each vertex \( \in d \) hyperplanes incident to \( 2^d \) cells

Vertices of level \( k \) are incident to cells with level \( \in [k, k+d] \)

Cells of level \( k \) have incident vertices with level \( \in [k-d, k] \)

\[
c_{\leq k}(n) = O(c'_{\leq k}(n))
\]
Regions, conflicts and the sampling theorem

\( O \) a set of \( n \) objects.

\( \mathcal{F}(O) \) set of configurations defined by \( O \)
- each configuration is defined by a subset of \( b \) objects
- each configuration is in conflict with a subset of \( O \)

\( \mathcal{F}_j(O) \) set of configurations in conflict with \( j \) objects
\( |\mathcal{F}_{\leq k}(O)| \) number of configurations defined by \( O \)
in conflict with at most \( k \) objects of \( O \)

\( f_0(r) = \text{Exp}(|\mathcal{F}_0(R)|) \) expected number of configurations defined and without conflict on a random \( r \)-sample of \( O \).

**The sampling theorem** [Clarkson & Shor 1992]

For \( 2 \leq k \leq \frac{n}{b+1} \),

\[
|\mathcal{F}_{\leq k}(O)| \leq 4 (b + 1)^b k^b f_0\left(\left\lfloor \frac{n}{k} \right\rfloor \right)
\]
Regions, conflicts and the sampling theorem

$O$ a set of $n$ objects.

$\mathcal{F}(O)$ set of configurations defined by $O$

- each configuration is defined by a subset of $b$ objects
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$|\mathcal{F}_{\leq k}(O)|$ number of configurations defined by $O$

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The sampling theorem [Clarkson & Shor 1992]

For $2 \leq k \leq \frac{n}{b+1}$, $|\mathcal{F}_{\leq k}(O)| \leq 4 (b + 1)^b k^b f_0\left(\left\lfloor \frac{n}{k} \right\rfloor \right)$
Proof of the sampling theorem

\[ f_0(r) = \sum_j |\mathcal{F}_j(O)| \frac{\binom{n - b - j}{r - b}}{\binom{n}{r}} \geq |\mathcal{F}_{\leq k}(O)| \frac{\binom{n - b - k}{r - b}}{\binom{n}{r}} \]

then, we prove that

for \( r = \frac{n}{k} \)

\[ \frac{\binom{n - b - k}{r - b}}{\binom{n}{r}} \geq \frac{1}{4(b + 1)^b k^b} \]

\[ \frac{r!}{(r - b)!} \frac{(n - b)!}{n!} \frac{(n - r)!}{(n - r - k)!} \frac{(n - b - k)!}{(n - b)!} \geq \frac{1}{4} \]
Proof of the sampling theorem

\[
\frac{(n - r)!}{(n - r - k)!} \frac{(n - b - k)!}{(n - b)!} = \prod_{j=1}^{k} \frac{n - r - k + j}{n - b - k + j} \geq \left(\frac{n - r - k + 1}{n - b - k + 1}\right)^k \\
\geq \left(\frac{n - n/k - k + 1}{n - k}\right)^k \\
\geq (1 - 1/k)^k \geq 1/4 \text{ pour } (2 \leq k),
\]

\[
\frac{r!}{(r - b)!} \frac{(n - b)!}{n!} = \prod_{l=0}^{b-1} \frac{r - l}{n - l} \geq \prod_{l=1}^{b} \frac{r + 1 - b}{n} \\
\geq \prod_{l=1}^{b} \frac{n/k - b}{n} \\
\geq 1/k^b (1 - b/k)^b \geq \frac{1}{k^b(b + 1)^b} \text{ pour } (k \leq \frac{n}{b + 1}).
\]
Bounding the number of \(k\)-sets

\[ c_k(P) : \text{Number of } k\text{-sets of } P = \text{Number of cells of level } k \text{ in } A(P^*). \]

\[ c_{\leq k}(P) = \sum_{l \leq k} c_l(P) \]

\[ c'_{\leq k}(P) : \text{Number of vertices of } A(P^*) \text{ with level at most } k. \]

**Objects** \(O\): \(n\) hyperplanes of \(\mathbb{R}^d\)

**Configurations** : vertices in \(A(O)\), \(b = d\)

**Conflict** between \(v\) and \(h\) : \(v \in h^+\)

**Sampling th:** \(c'_{\leq k}(P) \leq 4(d + 1)^d k^d f_0 \left( \left\lfloor \frac{n}{k} \right\rfloor \right) \)

**Upper bound th:** \(f_0(\left\lfloor \frac{n}{k} \right\rfloor) = O \left( \frac{n \left\lfloor \frac{d}{2} \right\rfloor}{k \left\lfloor \frac{d}{2} \right\rfloor} \right) \)

\[ \Rightarrow c'_{\leq k}(n) = O \left( k \left\lceil \frac{d}{2} \right \rceil n \left\lfloor \frac{d}{2} \right\rfloor \right) \]
Combinatorial complexities

- Number of vertices of level \( \leq k \) in an arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \)

- Number of cells of level \( \leq k \) in an arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \)

- Total number of \( j \leq k \) sets for a set of \( n \) points in \( \mathbb{R}^d \)
  \[
  \left( k \left\lceil \frac{d}{2} \right\rceil n \left\lfloor \frac{d}{2} \right\rfloor \right)
  \]

- Total number of faces in the Voronoi diagrams of order \( j \leq k \) for a set of \( n \) points in \( \mathbb{R}^d \)
  \[
  \left( k \left\lceil \frac{d+1}{2} \right\rceil n \left\lfloor \frac{d+1}{2} \right\rfloor \right)
  \]
Combinatorial complexities

- Number of vertices of level $\leq k$ in an arrangement of $n$ hyperplanes in $\mathbb{R}^d$

- Number of cells of level $\leq k$ in an arrangement of $n$ hyperplanes in $\mathbb{R}^d$

- Total number of $j \leq k$ sets for a set of $n$ points in $\mathbb{R}^d$
  \[
  \left( k \left\lceil \frac{d}{2} \right\rceil n \left\lfloor \frac{d}{2} \right\rfloor \right)
  \]

- Total number of faces in the Voronoi diagrams of order $j \leq k$ for a set of $n$ points in $\mathbb{R}^d$
  \[
  \left( k \left\lceil \frac{d+1}{2} \right\rceil n \left\lfloor \frac{d+1}{2} \right\rfloor \right)
  \]
Let $\Omega \subseteq \mathbb{R}^d$ and $P \in \mathbb{R}^d$ a finite set of points.

$\text{Vor}(E) \cap \Omega$ is a cover of $\Omega$. Its nerve is called the Delaunay triangulation of $E$ restricted to $\Omega$, noted $\text{Del}|_{\Omega}(P)$.

If $\text{Vor}(E) \cap \Omega$ is a good cover of $\Omega$, $\text{Del}|_{\Omega}(P)$ is homotopy equivalent to $\Omega$ (Nerve theorem).
Union of balls

- What is the combinatorial complexity of the boundary of the union $U$ of $n$ balls of $\mathbb{R}^d$?
- Compare with the complexity of the arrangement of the bounding hyperspheres
- How can we compute $U$?
- What is the image of $U$ in the space of spheres?
Restriction of $\text{Del}(B)$ to $U = \bigcup_{b \in B} b$

- $U = \bigcup_{b \in B} b \cap V(b)$ and $\partial U \cap \partial b = V(b) \cap \partial b$.
- The nerve of $\mathcal{C}$ is the restriction of $\text{Del}(B)$ to $U$, i.e. the subcomplex $\text{Del}_{|U}(B)$ of $\text{Del}(B)$ whose faces have a circumcenter in $U$.
- $\forall b$, $b \cap V(b)$ is convex and thus contractible.
- $\mathcal{C} = \{b \cap V(b), b \in B\}$ is a good cover of $U$.
- The nerve of $\mathcal{C}$ is a deformation retract of $U$.
- Homotopy equivalent (Nerve theorem)
Restriction of $\text{Del}(B)$ to $U = \bigcup_{b \in B} b$

- $U = \bigcup_{b \in B} b \cap V(b)$ and $\partial U \cap \partial b = V(b) \cap \partial b$.
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- $\forall b, \ b \cap V(b)$ is convex and thus contractible.
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The nerve of $C$ is the restriction of $\text{Del}(B)$ to $U$, i.e. the subcomplex $\text{Del}|_U(B)$ of $\text{Del}(B)$ whose faces have a circumcenter in $U$

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Restriction of $\text{Del}(B)$ to $U = \bigcup_{b \in B} b$

- $U = \bigcup_{b \in B} b \cap V(b)$ and $\partial U \cap \partial b = V(b) \cap \partial b$.
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- The nerve of $\mathcal{C}$ is a deformation retract of $U$.
- homotopy equivalent (Nerve theorem)
Cech complex versus $\text{Del}_{U}(B)$

- Both complexes are homotopy equivalent to $U$
- The size of $\text{Cech}(B)$ is $\Theta(n^d)$
- The size of $\text{Del}_{U}(B)$ is $\Theta(n^\lceil \frac{d}{2} \rceil)$
Filtration of a simplicial complex

1. A filtration of $K$ is a sequence of subcomplexes of $K$

$$\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$$

such that:  $K^{i+1} = K^i \cup \sigma^{i+1}$, where $\sigma^{i+1}$ is a simplex of $K$

2. Alternatively a filtration of $K$ can be seen as an ordering $\sigma_1, \ldots \sigma_m$ of the simplices of $K$ such that the set $K^i$ of the first $i$ simplices is a subcomplex of $K$

The ordering should be consistent with the dimension of the simplices

*Filtration plays a central role in topological persistence*
A filtration of $K$ is a sequence of subcomplexes of $K$

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Alternatively a filtration of $K$ can be seen as an ordering $\sigma_1, \ldots, \sigma_m$ of the simplices of $K$ such that the set $K^i$ of the first $i$ simplices is a subcomplex of $K$.

The ordering should be consistent with the dimension of the simplices.

Filtration plays a central role in topological persistence.
\( P \) a finite set of points of \( \mathbb{R}^d \)

\[ U(\alpha) = \bigcup_{p \in P} B(p, \alpha) \]

\( \alpha \)-complex \( = \) \( \text{Del}_{|U(\alpha)}(P) \)

The filtration \( \{\text{Del}_{|U(\alpha)}(P), \ \alpha \in \mathbb{R}^+\} \) is called the \( \alpha \)-filtration of \( \text{Del}(P) \)
Alpha Controls the desired level of detail.

$\alpha = \infty$
Shape reconstruction using $\alpha$-complexes (3d)
Constructing the $\alpha$-filtration of $\text{Del}(P)$

$\sigma \in \text{Del}(P)$ is said to be Gabriel iff $\sigma \cap \sigma^* \neq \emptyset$

![A Gabriel edge](image1)

![A non Gabriel edge](image2)

Algorithm

for each $d$-simplex $\sigma \in \text{Del}(P)$:

$\alpha_{\text{min}}(\sigma) = r(\sigma)$

for $k = d - 1, \ldots, 0$,

for each $k$-face $\sigma \in \text{Del}(P)$

$\alpha_{\text{med}}(\sigma) = \min_{\sigma \in \text{coface}(\sigma)} \alpha_{\text{min}}(\sigma)$

if $\sigma$ is Gabriel then $\alpha_{\text{min}}(\sigma) = r(\sigma)$

else $\alpha_{\text{min}}(\sigma) = \alpha_{\text{med}}(\sigma)$
Constructing the $\alpha$-filtration of $\text{Del}(P)$

$\sigma \in \text{Del}(P)$ is said to be Gabriel iff $\sigma \cap \sigma^* \neq \emptyset$

Algorithm

for each $d$-simplex $\sigma \in \text{Del}(P)$ : $\alpha_{\text{min}}(\sigma) = r(\sigma)$

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for each $k$-face $\sigma \in \text{Del}(P)$

$\alpha_{\text{med}}(\sigma) = \min_{\sigma' \in \text{coface}(\sigma)} \alpha_{\text{min}}(\sigma')$

if $\sigma$ is Gabriel then $\alpha_{\text{min}}(\sigma) = r(\sigma)$

else $\alpha_{\text{min}}(\sigma) = \alpha_{\text{med}}(\sigma)$
\( \alpha \)-filtration of weighted Delaunay complexes

\[
B = \{ b_i = (p_i, r_i) \}_{i=1,\ldots,n} \quad W(\alpha) = \bigcup_{i=1}^{n} B \left( p_i, \sqrt{r_i^2 + \alpha^2} \right)
\]

\(\alpha\)-complex = \( \text{Del}_{W(\alpha)}(B) \)

Filtration: \( \{ \text{Del}_{W(\alpha)}(B), \ \alpha \in \mathbb{R}^+ \} \)