Voronoi Diagrams, Delaunay Triangulations and Polytopes

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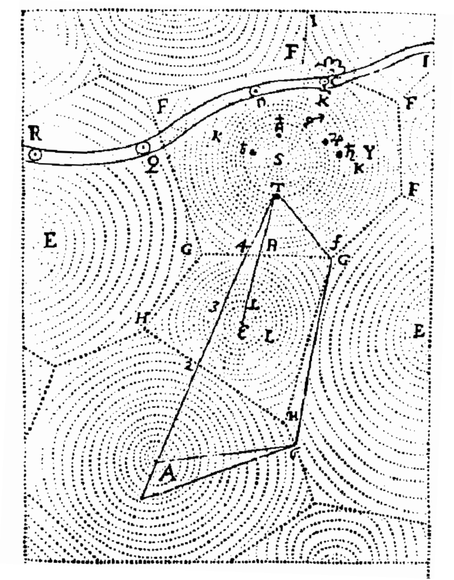
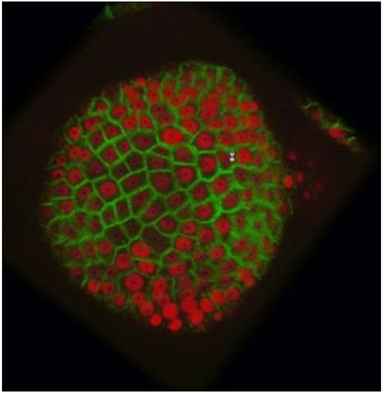
Winter School, University of Nice Sophia Antipolis
January 26-30, 2015

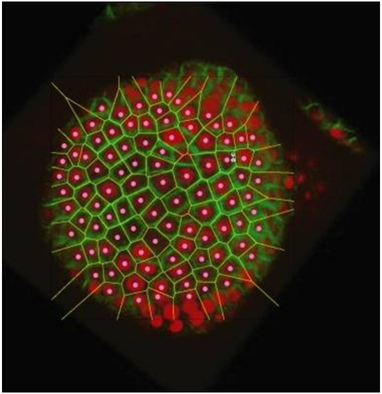
Voronoi diagrams in nature

The solar system (Descartes)

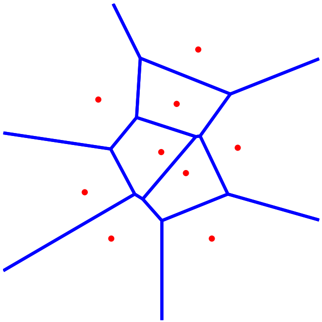


Growth of merystem





Euclidean Voronoi diagrams



Voronoi cell $V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram (P) $= \{$ collection of all cells $V(p_i)$, $p_i \in P$ $\}$

Voronoi diagrams and polytopes

Polytope

The intersection of a finite collection of half-spaces : $\mathcal{V} = \bigcap_{i \in I} h_i^+$

- Each Voronoi cell is a polytope
- The Voronoi diagram has the structure of a cell complex
- The Voronoi diagram of P is the projection of a polytope of $\mathbb{R}^{d+1}$

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Voronoi diagrams and polyhedra

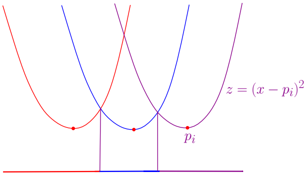
- Vor$(p_1, \ldots, p_n)$ is the minimization diagram of the n functions $\delta_i(x) = (x - p_i)^2$

- $\arg\min(\delta_i) = \arg\max(h_i)$ where $h_{p_i}(x) = 2\, p_i \cdot x - p_i^2$

- The minimization diagram of the δ_i is also the maximization diagram of the affine functions $h_{p_i}(x)$

- The faces of Vor(P) are the projections of the faces of $\mathcal{V}(P) = \bigcap_i h_{p_i}^+$

  $h_{p_i}^+ = \{x : x_{d+1} > 2p_i \cdot x - p_i^2\}$



Note !

the graph of $h_{p_i}(x)$ is the hyperplane tangent to $\mathcal{Q}$: $x_{d+1} = x^2$ at (x, x^2)

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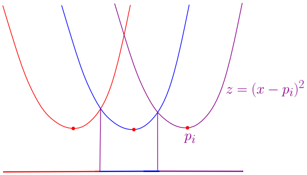

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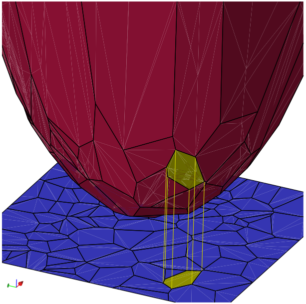

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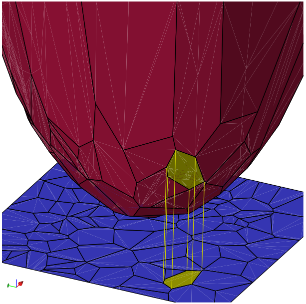

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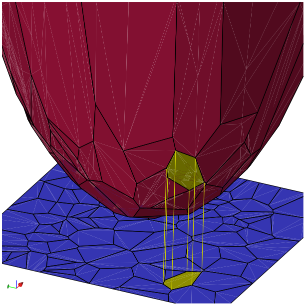
Voronoi diagrams and polyhedra



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Voronoi diagrams and polytopes

Lifting map

The faces of $\mathrm{Vor}(P)$ are the projection of the faces of the polytope

$$\mathcal{V}(P) = \bigcap_i \; h_{p_i}^+$$

where h_{p_i} is the hyperplane tangent to paraboloid $\mathcal{Q}$ at the lifted point (p_i, p_i^2)

Corollaries

- The size of $\mathrm{Vor}(\mathcal{P})$ is the same as the size of $\mathcal{V}(P)$
- Computing $\mathrm{Vor}(\mathcal{P})$ reduces to computing $\mathcal{V}(P)$

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Polytopes (convex polyhedra)

Two ways of defining polytopes

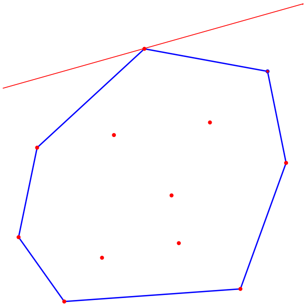
- Convex hull of a finite set of points : $\mathcal{V} = \mathrm{conv}(P)$
- Intersection of a finite set of half-spaces : $\mathcal{H} = \cap_{h \in H} h_i^+$

Facial structure of a polytope

Supporting hyperplane h :
$H \cap \mathcal{P} \neq \emptyset$
$\mathcal{P}$ on one side of h

Faces : $\mathcal{P} \cap h$, h supp. hyp.

Dimension of a face :
the dim. of its affine hull



General position

Points in general position

- P is in general position iff no subset of $k+2$ points lie in a k-flat

⇒ If P is in general position, all faces of $\mathrm{conv}(P)$ are simplices

Hyperplanes in general position

- H is in general position iff the intersection of any subset of $d-k$ hyperplanes intersect in a k-flat

⇒ any k-face is the intersection of $d-k$ hyperplanes

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Duality between points and hyperplanes

hyperplane of $\mathbb{R}^d$ $h : x_d = a \cdot x' - b$ $\longrightarrow$ point $h^* = (a,b) \in \mathbb{R}^d$

point $p = (p', p_d) \in \mathbb{R}^d$ $\longrightarrow$ hyperplane $p^* \subset \mathbb{R}^d$
$= \{(a,b) \in \mathbb{R}^d : b = p' \cdot a - p_d\}$

Duality

- preserves incidences :

$$p \in h \iff p_d = a \cdot p' - b \iff b = p' \cdot a - p_d \iff h^* \in p^*$$
$$p \in h^+ \iff p_d > a \cdot p' - b \iff b > p' \cdot a - p_d \iff h^* \in p^{*+}$$

- is an involution and thus is bijective : $h^{**} = h$ and $p^{**} = p$

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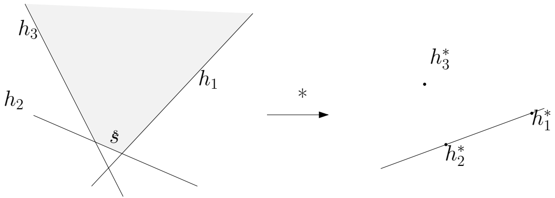
Duality between polytopes

Let $h_1, \ldots, h_n$ be n hyperplanes of $\mathbb{R}^d$ and let $\mathcal{H} = \cap h_i^+$



A vertex s of $\mathcal{H}$ is the intersection of $k \geq d$ hyperplanes $h_1, \ldots, h_k$ lying above all the other hyperplanes

$\Longrightarrow s^*$ is a hyperplane that 1. contains $h_1^*, \ldots, h_k^*$
2. supports $\mathcal{H}^*$= conv$^-(h_1^*, \ldots, h_k^*)$

General position
s is the intersection of d hyperplanes $\Rightarrow$ s^* supports a $(d-1)$-simplex de $\mathcal{H}^*$

More generally and under the general position assumption,

Let f be a $(d-k)$-face of $\mathcal{H}$ and $\mathrm{aff}(f) = \cap_{i=1}^{k} h_i$

$$p \in f \quad \Leftrightarrow \quad \begin{array}{l} h_i^* \in p^* \ \text{ for } \ i = 1, \ldots, k \\ h_i^* \in p^{*+} \ \text{ for } \ i = k+1, \ldots, n \end{array}$$

$$\Leftrightarrow \quad \begin{array}{l} p^* \text{support. hyp. of } \mathcal{H}^* = \mathrm{conv}(h_1^*, \ldots, h_n^*) \\ p^* \ni h_1^*, \ldots, h_k^* \end{array}$$

$$\Leftrightarrow \quad f^* = \mathrm{conv}(h_1^*, \ldots, h_k^*) \ \text{ is a } \ (k-1) - \text{face of } \ \mathcal{H}^*$$

Duality between $\mathcal{H}$ and $\mathcal{H}^*$

- The correspondence between the faces of $\mathcal{H}$ and $\mathcal{H}^*$ is involutive and therefore bijective
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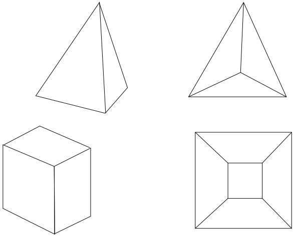
Algorithmic consequences

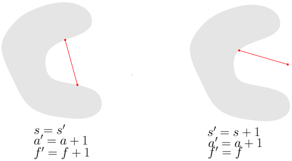
- Depending on the application, the primal or the dual setting may be more appropriate
- We will bound the combinatorial complexity of the intersection of n upper half-spaces
- We will compute the convex hull of n points
- By duality, the results extend to the dual case

The numbers of vertices s, edges a and facets f of a polytope of $\mathbb{R}^3$ satisfy

$$s - a + f = 2$$

Schlegel diagram





Euler formula for 3-polytopes : $s - a + f = 2$

Incidences edges-facets

$$2a \geq 3f \quad \Longrightarrow \quad \begin{array}{l} a \leq 3s - 6 \\ f \leq 2s - 4 \end{array}$$

with equality when all facets are triangles

Beyond the 3rd dimension

Upper bound theorem [McMullen 1970]

If $\mathcal{H}$ is the intersection of n half-spaces of $\mathbb{R}^d$

$$\text{nb faces of } \mathcal{H} = \Theta(n^{\lfloor \frac{d}{2} \rfloor})$$

Hyperplanes in general position

- any k-face is the intersection of $d-k$ hyperplanes defining $\mathcal{H}$
- all vertices of $\mathcal{H}$ are incident to d edges and have distinct x_d
- the convex hull of $k < d$ edges incident to a vertex p is a k-face of $\mathcal{H}$

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Proof of the upper bound theorem

Bounding the number of vertices

1. $\geq \lceil \frac{d}{2} \rceil$ edges incident to a vertex p are in $h_p^+ : x_d \geq x_d(p)$ or in h_p^-
   $\Rightarrow$ p is a x_d-max or x_d-min vertex of at least one $\lceil \frac{d}{2} \rceil$-face of $\mathcal{H}$
   $\Rightarrow$ # vertices of $\mathcal{H} \leq 2 \times$# $\lceil \frac{d}{2} \rceil$-faces of $\mathcal{H}$

2. A k-face is the intersection of $d-k$ hyperplanes defining $\mathcal{H}$

   $\Rightarrow$ # k-faces $= \binom{n}{d-k} = O(n^{d-k})$

   $\Rightarrow$ # $\lceil \frac{d}{2} \rceil$-faces $= O(n^{\lfloor \frac{d}{2} \rfloor})$

Bounding the total number of faces

The number of faces incident to p depends on d but not on n

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Representation of a convex hull

Adjacency graph (AG) of the facets

In general position, all the facets are $(d-1)$-simplexes

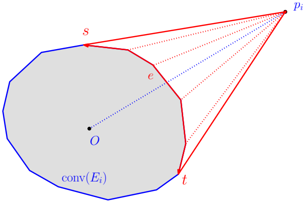
Adjacency graph (V, E)

- $V =$ set of $(d-1)$-faces (facets)
- $(f, f') \in E$ iff $f \cap f'$ share a $(d-2)$-face

Incremental algorithm

$\mathcal{P}_i$: set of the i points that have been inserted first

$\mathrm{conv}(\mathcal{P}_i)$: convex hull at step i



$f = [p_1, ..., p_d]$ is a red facet iff its supporting hyperplane separates p_i from $\mathrm{conv}(\mathcal{P}_i)$

$$\iff \mathtt{orient}(p_1, ..., p_d, p_i) \times \mathtt{orient}(p_1, ..., p_d, O) < 0$$

$$\mathrm{orient}(p_0, p_1, ..., p_d) = \begin{vmatrix} 1 & 1 & ... & 1 \\ p_0 & p_1 & ... & p_d \end{vmatrix} = \begin{vmatrix} 1 & 1 & ... & 1 \\ x_{01} & x_{11} & ... & x_{d1} \\ \vdots & \vdots & ... & \vdots \\ x_{0d} & x_{1d} & ... & x_{dd} \end{vmatrix}$$

Update of $\mathrm{conv}(\mathcal{P}_i)$

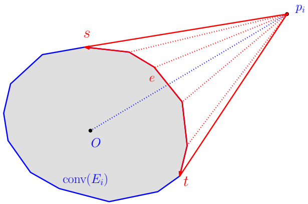
red facet = facet whose supporting hyperplane separates o and p_{i+1}

horizon : $(d-2)$-faces shared by a blue and a red facet

Update $\mathrm{conv}(\mathcal{P}_i)$:

1. find the red facets
2. remove them and create the new facets

   $[p_{i+1}, g]$, $\forall g \in$ horizon



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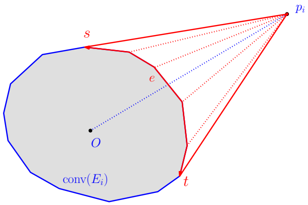
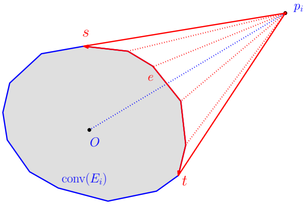


Complexity

proportional to the number of red facets

Complexity analysis

- update proportional to the number of red facets
- # new facets $= |\text{conv}(i, d-1)|$
  $= O(i^{\lfloor \frac{d-1}{2} \rfloor})$
- fast locate : insert the points in lexicographic order and search a 1st red facet in $\text{star}(p_{i-1})$
  (which necessarily exists)



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Worst-case optimal in even dimensions

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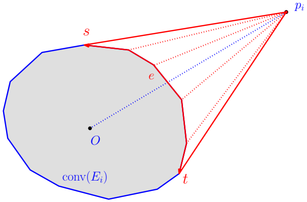

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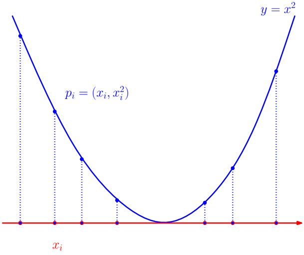
Lower bound



$\text{conv}(\{p_i\}) \Longrightarrow \text{tri}(\{x_i\})$

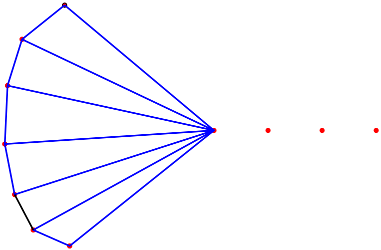
the orientation test reduces to 3 comparisons

$$\text{orient}(p_i, p_j, p_k) = \begin{vmatrix} x_i - x_j & x_i - x_k \\ x_i^2 - x_j^2 & x_i^2 - x_k^2 \end{vmatrix}$$

$$= (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

$\Longrightarrow$ Lower bound : $\Omega(n \log n)$

Lower bound for the incremental algorithm



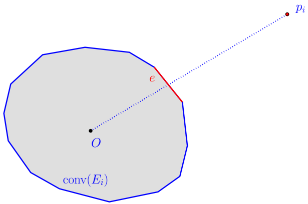
No incremental algorithm can compute the convex hull of n points of $\mathbb{R}^3$ in less than $\Omega(n^2)$

Randomized incremental algorithm

o : a point inside $\text{conv}(\mathcal{P})$

$\mathcal{P}_i$: the set of the first i inserted points

$\text{conv}(\mathcal{P}_i)$: convex hull at step i



Conflict graph

bipartite graph $\{p_j\} \times \{$facets of $\text{conv}(\mathcal{P}_i)\}$

$$p_j \dagger f \iff j > i \quad (p_j \text{ not yet inserted}) \quad \wedge \quad f \cap op_j \neq \emptyset$$

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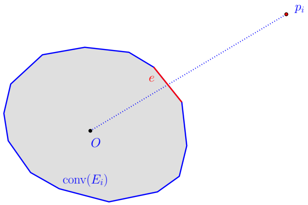

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Randomized analysis

Hyp. : points are inserted in random order

Conflict : †

Notations R : random sample of size r of $\mathcal{P}$

$F(R) = \{$ subsets of d points of $R\}$

$F_0(R) = \{$ elements of $F(R)$ with 0 conflict in $R\}$

(i.e. $\in \text{conv}(R)$)

$F_1(R) = \{$ elements of $F(R)$ with 1 conflict in $R\}$

$C_i(r, \mathcal{P}) = E(|F_i(R)|)$

(expectation over all random samples $R \subset \mathcal{P}$ of size r)

Lemma

$$C_i(r, \mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor}), \quad i = 1, 2$$

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$$C_i(r,\mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor}), \quad i = 1, 2$$

Proof of the lemma : $C_1(r,\mathcal{P}) = C_0(r,\mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor})$

$R' = R \setminus \{p\}$

$f \in F_0(R')$ if $f \in F_1(R)$ and $p \dagger f$ (proba $= \frac{1}{r}$)
or $f \in F_0(R)$ and $R' \ni$ the d vertices of f (proba $= \frac{r-d}{r}$)

Taking the expectation,

$$\begin{aligned}
C_0(r-1,R) &= \frac{1}{r}\,|F_1(R)| + \frac{r-d}{r}\,|F_0(R)| \\
C_0(r-1,\mathcal{P}) &= \frac{1}{r}\,C_1(r,\mathcal{P}) + \frac{r-d}{r}\,C_0(r,\mathcal{P}) \\
C_1(r,\mathcal{P}) &= d\,C_0(r,\mathcal{P}) - r\,(C_0(r,\mathcal{P}) - C_0(r-1,\mathcal{P})) \\
&\leq d\,C_0(r,\mathcal{P}) \\
&= O(r^{\lfloor \frac{d}{2} \rfloor})
\end{aligned}$$

Randomized analysis 1

Updating the convex hull + memory space

Expected number $N(i)$ of facets created at step i

$$\begin{aligned} N(i) &= \sum_{f \in F(\mathcal{P})} \text{proba}(f \in F_0(\mathcal{P}_i)) \times \frac{d}{i} \\ &= \frac{d}{i}\, O\left(i^{\lfloor \frac{d}{2} \rfloor}\right) \\ &= O(n^{\lfloor \frac{d}{2} \rfloor - 1}) \end{aligned}$$

Expected total number of created facets = $O(n^{\lfloor \frac{d}{2} \rfloor})$

$O(n)$ if $d = 2,\ 3$

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Randomized analysis2

Updating the conflict graph

Cost proportional to the number of faces of $\mathrm{conv}(\mathcal{P}_i)$ in conflict with p_{i+1} and some p_j, $j > i$

$N(i,j)$ = expected number of faces of $\mathrm{conv}(\mathcal{P}_i)$ in conflict with p_{i+1} and p_j, $j > i$

$\mathcal{P}_i^+ = \mathcal{P}_i \cup \{p_{i+1}\} \cup \{p_j\}$: a random subset of $i+2$ points of $\mathcal{P}$

$$N(i,j) = \sum_{f \in F(\mathcal{P})} \mathrm{proba}(f \in F_2(\mathcal{P}_i^+)) \times \begin{pmatrix} i+2 \\ 2 \end{pmatrix}^{-1} = C_2(i+1)\,\frac{2}{(i+1)(i+2)} = O(i^{\lfloor \frac{d}{2} \rfloor - 2})$$

Expected total cost of updating the conflict graph

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} N(i,j) = \sum_{i=1}^{n} (n-i)\, O(i^{\lfloor \frac{d}{2} \rfloor - 2}) = O(n \log n + n^{\lfloor \frac{d}{2} \rfloor})$$

Theorem

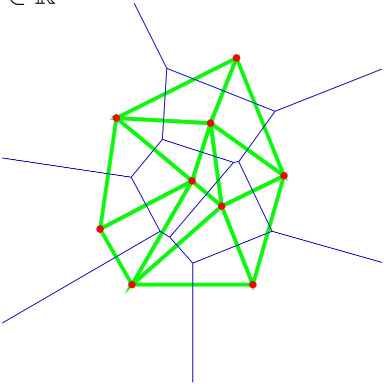
- The convex hull of n points of $\mathbb{R}^d$ can be computed in time $O(n\log n + n^{\lfloor \frac{d}{2} \rfloor})$ using $O(n^{\lfloor \frac{d}{2} \rfloor})$ space
- The same bounds hold for computing the intersection of n half-spaces of $\mathbb{R}^d$
- The randomized algorithm can be derandomized

  [Chazelle 1992]

- The same results hold for Voronoi diagrams provided that $d \to d+1$

Voronoi diagram and Delaunay triangulation

Finite set of points $P \in \mathbb{R}^d$



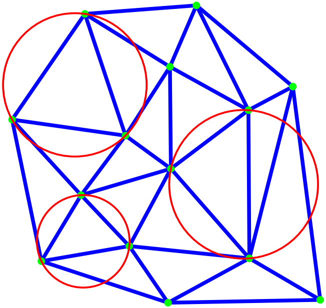
- The Delaunay complex is the nerve of the Voronoi diagram
- It is not always embedded in $\mathbb{R}^d$

Empty circumballs

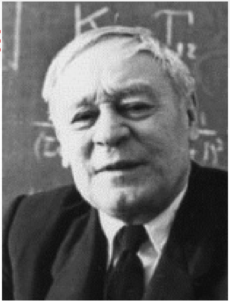
An (open) d-ball B circumscribing a simplex $\sigma \subset \mathcal{P}$ is called empty if

1. $\mathrm{vert}(\sigma) \subset \partial B$
2. $B \cap \mathcal{P} = \emptyset$

$\mathrm{Del}(\mathcal{P})$ is the collection of simplices admitting an empty circumball



Point sets in general position wrt spheres



$P = \{p_1, p_2 \ldots p_n\}$ is said to be in general position wrt spheres if $\nexists\, d+2$ points of P lying on a same $(d-1)$-sphere

Theorem [Delaunay 1936]

If P is in general position wrt spheres, the simplicial map

$$f : \mathrm{vert}(\mathrm{Del} P) \rightarrow P$$

provides a realization of $\mathrm{Del}(P)$ called the Delaunay triangulation of P.

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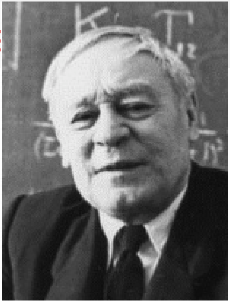

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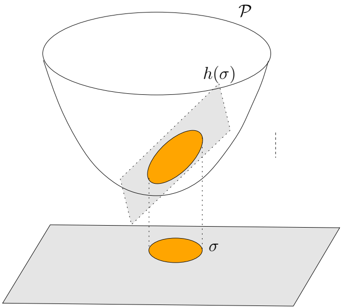
Proof of Delaunay's theorem 1



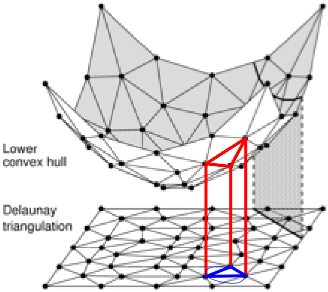
Linearization

$$S(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2$$

$$S(x) < 0 \Leftrightarrow \begin{cases} z < 2c \cdot x - s & (h_S^-) \\ z = x^2 & (\mathcal{P}) \end{cases}$$

$$\Leftrightarrow \hat{x} = (x, x^2) \in h_S^-$$

Proof of Delaunay's theorem 2



Proof of Delaunay's th.

P general position wrt spheres
$\Leftrightarrow \hat{P}$ in general position

σ a simplex, S_σ its circumscribing sphere

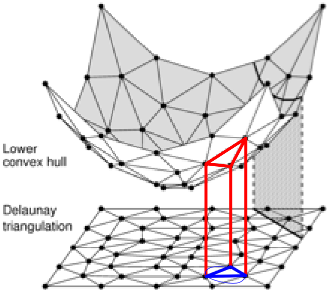
$\sigma \in \mathrm{Del}(P) \Leftrightarrow S_\sigma$ empty

$\Leftrightarrow \forall i,\ \hat{p}_i \in h^+_{S_\sigma}$

$\Leftrightarrow \hat{\sigma}$ is a face of $\mathrm{conv}^-(\hat{P})$

$$\mathrm{Del}(P) = \mathrm{proj}(\mathrm{conv}^-(\hat{P}))$$

Proof of Delaunay's theorem 2



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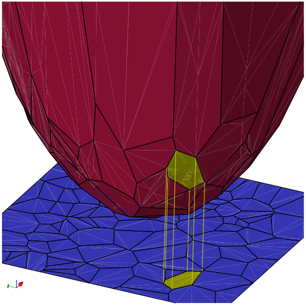
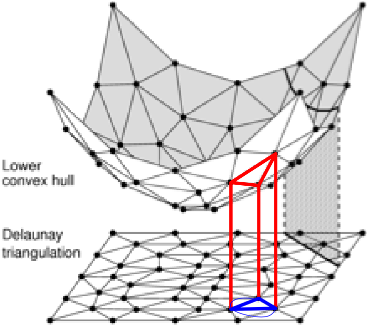
Duality





$\mathcal{V}(P) = \cap_i h_{p_i}^+$

$\mathcal{D}(P) = \text{conv}^-(\hat{P})$

Voronoi diagrams, Delaunay triangulations and polytopes

If P is in general position wrt spheres :

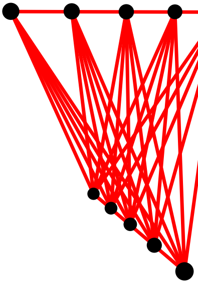
$$\mathcal{V}(P) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ \quad \stackrel{\text{duality}}{\longrightarrow} \quad \mathcal{D}(P) = \text{conv}^-(\{\hat{p}_1, \ldots, \hat{p}_n\})$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

Voronoi Diagram of P $\stackrel{\text{nerve}}{\longrightarrow}$ Delaunay Complex of P

Combinatorial complexity

The combinatorial complexity of the Delaunay triangulation diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of a convex hull of n points of $\mathbb{R}^{d+1}$



$\Theta(n^{\lceil \frac{d}{2} \rceil})$

Quadratic in $\mathbb{R}^3$

Constructing $\mathrm{Del}(P), \quad P = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

Algorithm

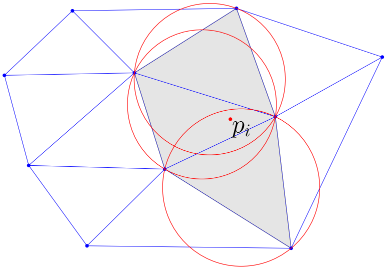
1. Lift the points of P onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$:
   $p_i \rightarrow \hat{p}_i = (p_i, p_i^2)$
2. Compute $\mathrm{conv}(\{\hat{p}_i\})$
3. Project the lower hull $\mathrm{conv}^-(\{\hat{p}_i\})$ onto $\mathbb{R}^d$

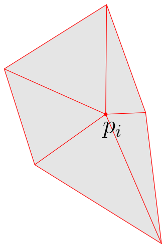
Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$

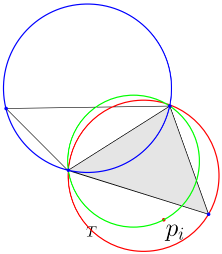
Direct algorithm : insertion of a new point p_i

1. Location : find all the d-simplices that conflict with p_i
   i.e. whose circumscribing ball contains p_i

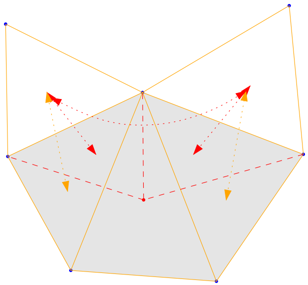
2. Update : construct the new d-simplices







Updating the adjacency graph



We look at the d-simplices to be removed and at their neighbors

Each d-simplex is considered $\leq \frac{d(d+1)}{2}$ times

Update cost $= O(\#$ created and deleted simplices $)$
$= O(\#$ created simplices$)$

Exercise : computing the DT of an ε-net

Definition Let Ω be a bounded subset of $\mathbb{R}^d$ and P a finite point set in Ω. P is called an (ε, η)-net of Ω if

1. Covering : $\forall p \in \Omega,\ \exists p \in P,\ \|p - x\| \leq \varepsilon$
2. Packing : $\forall p, q \in P, \|p - q\| \geq \eta$

Questions

1. Show that $(\varepsilon, \varepsilon)$-nets exist
2. Show that any simplex with all its vertices at distance $> \varepsilon$ from $\partial\Omega$ has a circumradius $\leq \varepsilon$
3. Show that the complexity of $\mathrm{Del}(P)$ is $O(n)$ for fixed d
4. Improve the construction of $\mathrm{Del}(P)$

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